Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 103–125
https://doi.org/10.4134/CKMS.c160157
pISSN: 1225-1763 / eISSN: 2234-3024

THE PROPERTIES OF JORDAN DERIVATIONS OF SEMIPRIME RINGS AND BANACH ALGEBRAS, I

Byung Do Kim

Abstract. Let R be a 5!-torsion free semiprime ring, and let $D : R \to R$ be a Jordan derivation on a semiprime ring R. Then $[D(x), x]D(x)^2 = 0$ if and only if $D(x)^2[D(x), x] = 0$ for every $x \in R$. In particular, let A be a Banach algebra with $\text{rad}(A)$ and if D is a continuous linear Jordan derivation on A, then we show that $[D(x), x]D(x)^2 \in \text{rad}(A)$ if and only if $D(x)^2[D(x), x] \in \text{rad}(A)$ for all $x \in A$ where $\text{rad}(A)$ is the Jacobson radical of A.

1. Introduction

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. $\text{rad}(A)$ denotes the *Jacobson radical* of a Banach algebra A. We say that A is *semisimple* if $\text{rad}(A) = (0)$ (see Bonsall and Duncan [1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Johnson and Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of Singer and Wermer [14] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra. Thomas [15] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

Vukman [17] has proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Received July 27, 2016; Revised June 1, 2017; Accepted August 9, 2017.
2010 *Mathematics Subject Classification.* 16N60, 16W25, 17B40.
Key words and phrases. Jordan derivation, derivation, semiprime ring, Banach algebra, the (Jacobson) radical.

Moreover, using the above result, he has proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that

$$[D(x), x]D(x) = 0$$

holds for all $x \in A$. In this case, $D = 0$. See [2], [6], [7], [8], [10], [12], [13] and [16] for further results.

In this paper, we generalize the statement of Theorem 3.1 in [11] except the torsion free condition, and we shall give a generalization of Theorem 3.2 and its application to the Banach algebra theory.

2. Preliminaries

The following lemma is due to Chung and Luh [4].

Lemma 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \ldots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t = 1, 2, \ldots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \le k \le n$.*

The following theorem is due to Brešar [3].

Theorem 2.2. *Let R be a 2-torsion free semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

The following theorems are due to Kim [11].

Theorem 2.3. *Let R be a $3!$-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. In this case, it follows that*

$$[D(x), x]D(x) = 0 \iff D(x)[D(x), x] = 0$$

for every $x \in R$.

Theorem 2.4. *Let A be a Banach algebra with rad(A). Let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. Then we obtain*

$$[D(x), x]D(x) \in rad(A) \iff D(x)[D(x), x] \in rad(A)$$

for every $x \in A$.

3. Main results

We need the following notations. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \le k \le m\}$, where m is a positive integer. When R is a ring, we shall denote the maps $B : R \times R \to R$, $f, g, h, F : R \to R$ by $B(x, y) \equiv [D(x), y] + [D(y), x]$, $f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$, $h(x) \equiv [g(x), x]$, $F(x) \equiv [D^2(x), x]$ for all $x, y \in R$ respectively. And in particular, for a Jordan derivation D on R, we shall denoted it by $D^2(x) = (D \circ D)(x) = D(D(x))$, and $D^n(x) = (D \circ D^{n-1})(x) = D(D^{n-1}(x))$ for all integers $n \ge 2$ and all $x \in R$. Moreover, we have the basic properties:

$$B(x, y) = B(y, x),\ B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z),$$

$$B(x, yD(x)) = B(x, y)D(x) + yF(x) + D(y)f(x) + [y, x]D^2(x),$$

$$B(x, D(x)y) = D(x)B(x, y) + F(x)y + f(x)D(y) + D^2(x)[y, x],$$
$$B(x, x) = 2f(x),\ B(x, x^2) = 2(f(x)x + xf(x)),\ x, y, z \in R.$$

Theorem 3.1. *Let R be a $3!$-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[[D(x), x], x]D(x) = 0$$

for all $x \in R$. Then we have $3f(x)D(x)^2 - D(x)f(x)D(x) = 0$ for all $x \in R$.

Proof. Let R be a commutative ring. Then $f(x) = [D(x), x] = 0$ for all $x \in R$. Hence it is clear that $3f(x)D(x)^2 - D(x)f(x)D(x) = 0$ holds for all $x \in R$. Thus it is sufficient to prove the above statement in the noncommutative case of R. By Theorem 2.2, we can see that D is a derivation on R.

Assume that

$$[[D(x), x], x]D(x) = [f(x), x]D(x) = g(x)D(x) = 0,\ x \in R. \tag{1}$$

Replacing $x + ty$ for x in (1), we have

$$\begin{aligned} &[[D(x + ty), x + ty], x + ty]D(x + ty)\\ &\equiv [[D(x), x], x]D(x) + t\{[B(x, y), x]D(x)\\ &\quad + [f(x), y]D(x) + g(x)D(y)\} + t^2G_1(x, y)\\ &\quad + t^3G_2(x, y) + t^4g(y)D(y) = 0,\ x, y \in R,\ t \in S_3, \end{aligned} \tag{2}$$

where G_1 and G_2 denote the term satisfying the identity (2).

From (1) and (2), we obtain

$$\begin{aligned} &t\{[B(x, y), x]D(x) + [f(x), y]D(x) + g(x)D(y)\} + t^2G_1(x, y)\\ &+ t^3G_2(x, y) = 0,\ x, y \in R, t \in S_3. \end{aligned} \tag{3}$$

Since R is $3!$-torsion free, by Lemma 2.1 the relation (3) yields

$$[B(x, y), x]D(x) + [f(x), y]D(x) + g(x)D(y) = 0,\ x, y \in R. \tag{4}$$

Let $y = x^2$ in (4). Then using (1), we get

$$\begin{aligned} 0 &= 2\{[f(x)x + xf(x), x]\}D(x) + (g(x)x + xg(x))D(x)\\ &\quad + g(x)(D(x)x + xD(x))\\ &= 2g(x)xD(x) + 2xg(x)D(x) + g(x)xD(x) + xg(x)D(x)\\ &\quad + g(x)D(x)x + g(x)xD(x)\\ &= 4g(x)xD(x) + 3xg(x)D(x) + g(x)D(x)x\\ &= 4g(x)xD(x) = 4h(x)D(x) = -4g(x)f(x) = 0,\ x \in R. \end{aligned} \tag{5}$$

Since R is $3!$-torsion free, it follows from (5) that

$$h(x)D(x) = g(x)f(x) = 0,\ x \in R. \tag{6}$$

Substituting $yD(x)$ for y in (4), we arrive at

$$0 = [B(x, yD(x)), x]D(x) + [f(x), yD(x)]D(x) + g(x)D(yD(x))$$

$$
\begin{aligned}
&= [B(x,y)D(x) + D(y)f(x) + yF(x) + [y,x]D^2(x), x]D(x) \\
&\quad + ([f(x),y]D(x)^2 + y[f(x),D(x)]D(x) + g(x)D(y)D(x) + g(x)yD^2(x) \\
&= [B(x,y)D(x),x]D(x) + [D(y)f(x),x]D(x) + [yF(x),x]D(x) \\
&\quad + [[y,x]D^2(x),x]D(x) \\
&= [B(x,y),x]D(x)^2 + B(x,y)f(x)D(x) + [D(y),x]f(x)D(x) \\
&\quad + D(y)g(x)D(x) + y[F(x),x]D(x) + [y,x]F(x)D(x) + [y,x]F(x)D(x) \\
&\quad + [[y,x],x]D^2(x)D(x) + [f(x),y]D(x)^2 + y[f(x),D(x)]D(x) \\
&\quad + g(x)D(y)D(x) + g(x)yD^2(x), \ x,y \in R. \qquad (7)
\end{aligned}
$$

Right multiplication of (4) by $D(x)$ leads to

$$
[B(x,y),x]D(x)^2 + [f(x),y]D(x)^2 + g(x)D(y)D(x) = 0, \ x,y \in R. \qquad (8)
$$

From (7) and (8),

$$
\begin{aligned}
&B(x,y)f(x)D(x) + [D(y),x]f(x)D(x) + D(y)g(x)D(x) \\
&+ y[F(x),x]D(x) + [y,x]F(x)D(x) + [y,x]F(x)D(x) \\
&+ [[y,x],x]D^2(x)D(x) + y[f(x),D(x)]D(x) + g(x)yD^2(x) = 0, \ x,y \in R. \qquad (9)
\end{aligned}
$$

Combining (1) with (9),

$$
\begin{aligned}
&B(x,y)f(x)D(x) + [D(y),x]f(x)D(x) + y[F(x),x]D(x) \\
&+ [y,x]F(x)D(x) + [y,x]F(x)D(x) + [[y,x],x]D^2(x)D(x) \\
&+ y[f(x),D(x)]D(x) + g(x)yD^2(x) = 0, \ x,y \in R. \qquad (10)
\end{aligned}
$$

Writing xy for y in (10), we arrive at

$$
\begin{aligned}
0 &= xB(x,y)f(x)D(x) + 2f(x)yf(x)D(x) + D(x)[y,x]f(x)D(x) \\
&\quad + [xD(y) + D(x)y, x]f(x)D(x) + xy[F(x),x]D(x) \\
&\quad + x[y,x]F(x)D(x) + x[y,x]F(x)D(x) + x[[y,x],x]D^2(x)D(x) \\
&\quad + xy[f(x),D(x)]D(x) + g(x)xyD^2(x) \\
&= xB(x,y)f(x)D(x) + 2f(x)yf(x)D(x) + D(x)[y,x]f(x)D(x) \\
&\quad + x[D(y),x]f(x)D(x) + D(x)[y,x]f(x)D(x) \\
&\quad + f(x)yf(x)D(x) + xy[F(x),x]D(x) \\
&\quad + x[y,x]F(x)D(x) + x[y,x]F(x)D(x) + x[[y,x],x]D^2(x)D(x) \\
&\quad + xy[f(x),D(x)]D(x) + g(x)xyD^2(x), \ x,y \in R. \qquad (11)
\end{aligned}
$$

Left multiplication of (10) by x leads to

$$
\begin{aligned}
&xB(x,y)f(x)D(x) + x[D(y),x]f(x)D(x) + xy[F(x),x]D(x) \\
&+ x[y,x]F(x)D(x) + x[y,x]F(x)D(x) + x[[y,x],x]D^2(x)D(x) \\
&+ xy[f(x),D(x)]D(x) + xg(x)yD^2(x) = 0, \ x,y \in R. \qquad (12)
\end{aligned}
$$

From (1), (11) and (12),

$$3f(x)yf(x)D(x) + 2D(x)[y,x]f(x)D(x) + h(x)yD^2(x) = 0,\ x,y \in R. \tag{13}$$

Substituting $D(x)y$ for y in (13), we arrive at

$$\begin{aligned} &3f(x)D(x)yf(x)D(x) + D(x)^2[y,x]f(x)D(x) + D(x)f(x)yf(x)D(x) \\ &+ D(x)^2[y,x]f(x)D(x) + D(x)f(x)yf(x)D(x) \\ &+ h(x)D(x)yD^2(x) = 0,\ x,y \in R. \end{aligned} \tag{14}$$

From (6) and (14),

$$\begin{aligned} &\{3f(x)D(x) + 2D(x)f(x)\}yf(x)D(x) + D(x)^2[y,x]f(x)D(x) \\ &+ D(x)^2[y,x]f(x)D(x) = 0,\ x,y \in R. \end{aligned} \tag{15}$$

Left multiplication of (13) by $D(x)$ leads to

$$\begin{aligned} &3D(x)f(x)yf(x)D(x) + D(x)^2[y,x]f(x)D(x) \\ &+ D(x)^2[y,x]f(x)D(x) + D(x)h(x)yD^2(x) = 0,\ x,y \in R. \end{aligned} \tag{16}$$

From (1), (15) and (16),

$$\{3f(x)D(x) - D(x)f(x)\}yf(x)D(x) - D(x)h(x)yD^2(x) = 0,\ x,y \in R. \tag{17}$$

Replacing $D(x)y$ for y in (17), we arrive at

$$\begin{aligned} &\{3f(x)D(x)^2 - D(x)f(x)D(x)\}yf(x)D(x) - D(x)h(x)D(x)yD^2(x) \\ &= 0,\ x,y \in R. \end{aligned} \tag{18}$$

From (6) and (18),

$$\{3f(x)D(x)^2 - D(x)f(x)D(x)\}yf(x)D(x) = 0,\ x,y \in R. \tag{19}$$

Writing $yD(x)$ for y in (19), we obtain

$$\{3f(x)D(x)^2 - D(x)f(x)D(x)\}yD(x)f(x)D(x) = 0, x,y \in R. \tag{20}$$

Right multiplication of (19) by $3D(x)$ leads to

$$\{3f(x)D(x)^2 - D(x)f(x)D(x)\}y(3f(x)D(x)^2) = 0,\ x,y \in R. \tag{21}$$

From (20) and (21),

$$\begin{aligned} &\{3f(x)D(x)^2 - D(x)f(x)D(x)\}y(3f(x)D(x)^2 - D(x)f(x)D(x)) \\ &= 0,\ x,y \in R. \end{aligned} \tag{22}$$

Since R is semiprime, it follows from (22) that

$$3f(x)D(x)^2 - D(x)f(x)D(x) = 0,\ x \in R. \tag{23}$$

□

Using the same technique with necessary variations one can prove the following Theorem and statement Theorem 3.2 without the proof.

Theorem 3.2. *Let R be a $3!$-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)[[D(x), x], x] = 0$$

for all $x \in R$. Then we have $3D(x)^2 f(x) - D(x)f(x)D(x) = 0$ for all $x \in R$.

Lemma 3.3. *Let R be a $5!$-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. And assume that*

$$D(x)g(x)yD^2(x)D(x) = D(x)[f(x), x]yD^2(x)D(x) = 0$$

for all $x, y \in R$. Then we have $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$.

Proof. When R is commutative, we see that if $f(x) = [D(x), x] = 0$ for all $x \in R$, then $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$.

Hence it is sufficient to prove the above statement in the noncommutative case of R. By Theorem 2.2, we can see that D is a derivation on R.

Suppose

$$D(x)g(x)yD^2(x)D(x) = [f(x), x]D(x)yD^2(x)D(x) = 0,\ x \in R. \tag{24}$$

Writing $x + tz$ for x in (24), we get

$$\begin{aligned} &D(x+tz)[[D(x+tz), x+tz], x+tz]yD^2(x+tz)D(x+tz) \\ &\equiv D(x)g(x)yD^2(x)D(x) + t\{(D(z)g(x) + D(x)([B(x,z), x] \\ &\quad + [f(x), z]))yD^2(x)D(x) + D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z))\} \\ &\quad + t^2 I_1(x,y,z) + t^3 I_2(x,y,z) + t^4 I_3(x,y,z) + t^5 I_4(x,y,z) \\ &\quad + t^6 D(z)g(z)yD^2(z)D(z) = 0,\ x,y,z \in R,\ t \in S_5, \end{aligned} \tag{25}$$

where I_1, I_2, I_3 and I_4 denote the term satisfying the identity (25).

From (24) and (25), we obtain

$$\begin{aligned} &t\{(D(z)g(x) + D(x)([B(x,z), x] + [f(x), z]))yD^2(x)D(x) \\ &+ D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z))\} + t^2 I_1(x,y,z) + t^3 I_2(x,y,z) \\ &+ t^4 I_3(x,y,z) + t^5 I_4(x,y,z) = 0,\ x,y,z \in R,\ t \in S_5. \end{aligned} \tag{26}$$

Since R is $5!$-torsion free, by Lemma 2.1 the relation (26) yields

$$\begin{aligned} &\{D(z)g(x) + D(x)([B(x,z), x] + [f(x), z])\}yD^2(x)D(x) \\ &+ D(x)g(x)y\{D^2(z)D(x) + D^2(x)D(z)\} = 0,\ x,y,z \in R. \end{aligned} \tag{27}$$

Writing $wD(x)g(x)y$ for y in (27), we get

$$\begin{aligned} &(D(z)g(x) + D(x)([B(x,z), x] + [f(x), z]))wD(x)g(x)yD^2(x)D(x) \\ &+ D(x)g(x)wD(x)g(x)y(D^2(z)D(x) + D^2(x)D(z)) = 0,\ x,y,z \in R. \end{aligned} \tag{28}$$

From (24) and (28),

$$D(x)g(x)wD(x)g(x)y(D^2(z)D(x) + D^2(x)D(z)) = 0,\ x,y,z \in R. \tag{29}$$

Replacing $y(D^2(z)D(x)+D^2(x)D(z))w$ for w in (29), we get

$$D(x)g(x)y(D^2(z)D(x)+D^2(x)D(z))wD(x)g(x)y\{D^2(z)D(x) + D^2(x)D(z)\}=0,\ x,y,z\in R. \tag{30}$$

Since R is semiprime, we get from (30)

$$D(x)g(x)y(D^2(z)D(x)+D^2(x)D(z))=0,\ x,y,z\in R. \tag{31}$$

Writing $x+tw$ for x in (31), we get

$$\begin{aligned} &D(x+tw)[f(x+tw),x+tw]y(D^2(z)D(x+tw)+D^2(x+tw)D(z))\\ &\equiv D(x)g(x)y(D^2(z)D(x)+D^2(x)D(z))+t\{D(w)g(x)\\ &\quad + D(x)([B(x,w),x]+[f(x),w]))y(D^2(z)D(x)+D^2(x)D(z))\\ &\quad + D(x)g(x)y(D^2(z)D(w)+D^2(w)D(z))\}+t^2K_1(x,y,z)\\ &\quad + t^3K_2(x,y,z)+t^4K_3(x,y,z)\\ &\quad + t^5D(w)g(w)y(D^2(z)D(w)+D(w)D^2(z))\\ &= 0,\ x,y,z\in R,\ t\in S_5, \end{aligned} \tag{32}$$

where K_1,K_2,K_3 denote the term satisfying the identity (32).

From (31) and (32), we obtain

$$\begin{aligned} &t\{D(w)g(x)+D(x)([B(x,w),x]+[f(x),w])y(D^2(z)D(x)+D^2(x)D(z))\\ &+D(x)g(x)y(D^2(z)D(w)+D^2(w)D(z))\}+t^2K_1(x,y,z)+t^3K_2(x,y,z)\\ &+t^4K_3(x,y,z)=0,\ w,x,y,z\in R,\ t\in S_5. \end{aligned} \tag{33}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (33) yields

$$\begin{aligned} &D(w)g(x)+D(x)([B(x,w),x]+[f(x),w]))y(D^2(z)D(x)+D^2(x)D(z))\\ &+D(x)g(x)y(D^2(z)D(w)+D^2(w)D(z))=0,\ w,x,y,z\in R. \end{aligned} \tag{34}$$

Writing $vD(x)g(x)y$ for y in (34), we get

$$\begin{aligned} &D(w)g(x)+D(x)([B(x,w),x]+[f(x),w]))vg(x)D(x)y\{D^2(z)D(x)\\ &+D^2(x)D(z)\}+D(x)g(x)vg(x)D(x)y(D^2(z)D(w)+D^2(w)D(z))\\ &=0,\ w,x,y,z\in R. \end{aligned} \tag{35}$$

From (31) and (35),

$$D(x)g(x)vD(x)g(x)y(D^2(z)D(w)+D^2(w)D(z))=0,\ v,w,x,y,z\in R. \tag{36}$$

Replacing $y(D^2(z)D(w)+D^2(w)D(z))v$ for w in (36), we get

$$D(x)g(x)y(D^2(z)D(w)+D^2(w)D(z))vD(x)g(x)y\{D^2(z)D(w) + D^2(w)D(z)\}=0,\ v,w,x,y,z\in R. \tag{37}$$

Since R is semiprime, we get from (37)

$$D(x)g(x)y(D^2(z)D(w)+D^2(w)D(z))=0,\ w,x,y,z\in R. \tag{38}$$

Putting $wD(z)$ instead of for w in (38), we have

$$D(x)g(x)y\{D^2(z)D(w)D(z) + D^2(z)wD^2(z) + D^2(w)D(z)^2)$$
$$+ 2D(w)D^2(z)D(z) + wD^3(z)D(z)\} = 0,\ w, x, y, z \in R. \tag{39}$$

From (24), (38) and (39), we obtain

$$D(x)g(x)y\{D^2(z)wD^2(z) + wD^3(z)D(z)) = 0,\ w, x, y, z \in R. \tag{40}$$

Writing $D(z)w$ for w in (40), we get

$$D(x)g(x)y\{D^2(z)D(z)wD^2(z) + D(z)wD^3(z)D(z)) = 0,\ w, x, y, z \in R. \tag{41}$$

From (24) and (41), we obtain

$$D(x)g(x)yD(z)wD^3(z)D(z) = 0,\ w, x, y, z \in R. \tag{42}$$

Replacing $yD^3(z)$ for y in (42), we get

$$D(x)g(x)yD^3(z)D(z)wD^3(z)D(z) = 0,\ w, x, y, z \in R. \tag{43}$$

Substituting $wD(x)g(x)y$ instead of for w in (43), we have

$$D(x)g(x)yD^3(z)D(z)wD(x)g(x)yD^3(z)D(z) = 0,\ w, x, y, z \in R. \tag{44}$$

Since R is semiprime, it follows from (44) that

$$D(x)g(x)yD^3(z)D(z) = 0,\ x, y, z \in R. \tag{45}$$

From (40) and (45), we obtain

$$D(x)g(x)yD^2(z)wD^2(z) = 0,\ w, x, y, z \in R. \tag{46}$$

Substituting $wD(x)g(x)y$ for w in (46), we get

$$D(x)g(x)yD^2(z)wD(x)g(x)yD^2(z) = 0,\ w, x, y, z \in R. \tag{47}$$

Since R is semiprime, it follows from (47) that

$$D(x)g(x)yD^2(z) = 0,\ x, y, z \in R. \tag{48}$$

Replacing zw for z in (48), we get

$$D(x)g(x)y(D^2(z)w + 2D(z)D(w) + zD^2(w)) = 0,\ w, x, y, z \in R. \tag{49}$$

From (48) and (49), we obtain

$$2D(x)g(x)yD(z)D(w) = 0,\ w, x, y, z \in R. \tag{50}$$

Since R is 3!-torsionfree, it follows from (50) that

$$D(x)g(x)yD(z)D(w) = 0,\ x, y, z \in R. \tag{51}$$

Writing wz for w in (51), we get

$$D(x)g(x)y\{D(z)D(w)z + D(z)wD(z)) = 0,\ w, x, y, z \in R. \tag{52}$$

From (51) and (52), we obtain

$$D(x)g(x)yD(z)wD(z) = 0,\ w, x, y, z \in R. \tag{53}$$

Substituting $wD(x)g(x)y$ for w in (53), we get

$$D(x)g(x)yD(z)wD(x)g(x)yD(z) = 0,\ w, x, y, z \in R. \tag{54}$$

Since R is semiprime, it follows from (54) that

$$D(x)g(x)yD(z) = 0,\ x, y, z \in R. \tag{55}$$

Right multiplication of (55) by $g(x)$ leads to

$$D(x)g(x)yD(z)g(z) = 0,\ x, y, z \in R. \tag{56}$$

Since R is semiprime, we obtain from (56)

$$D(x)g(x) = 0,\ x \in R. \tag{57}$$

□

Using the same technique with necessary variations one can prove the following lemmas and statements Lemmas 3.4 and 3.5 without the proofs.

Lemma 3.4. *Let R be a 5!-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. Assume that*

$$g(x)D(x)yD^2(x)D(x) = [f(x), x]D(x)yD^2(x)D(x) = 0$$

for all $x, y \in R$. Then we have $g(x)D(x) = [f(x), x]D(x) = 0$ for all $x \in R$.

Lemma 3.5. *Let R be a 3!-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. Assume that*

$$D(x)D^2(x)yD(x)g(x) = D(x)D^2(x)yD(x)[f(x), x] = 0$$

for all $x, y \in R$. Then we have $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$.

Lemma 3.6. *Let R be a 3!-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. Assume that*

$$D(x)D^2(x)yg(x)D(x) = D(x)D^2(x)y[f(x), x]D(x) = 0$$

for all $x, y \in R$. Then we have $g(x)D(x) = [f(x), x]D(x) = 0$ for all $x \in R$.

Proof. In the commutative case of R, we see that if $f(x) = [D(x), x] = 0$ for all $x \in R$. Hence it is clear that $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$. It is sufficient to prove the above statement in the noncommutative case of R.

By Theorem 2.2, we can see that D is a derivation on R.

In any semiprime ring, we see that $ayb = 0 \iff bya = 0$ for all $y \in R$. Thus it follows that

$$D(x)D^2(x)yg(x)D(x) = 0 \iff g(x)D(x)yD(x)D^2(x) = 0,\ x \in R.$$

Hence we may assume that

$$g(x)D(x)yD(x)D^2(x) = [f(x), x]D(x)yD(x)D^2(x) = 0,\ x \in R. \tag{58}$$

Writing $x + tz$ for x in (58), we get

$$\begin{aligned} &[[D(x+tz), x+tz], x+tz]D(x+tz)yD(x+tz)D^2(x+tz) \\ \equiv\ & g(x)D(x)yD(x)D^2(x) + t\{([B(x,z), x] + [f(x), z]))D(x) \end{aligned}$$

$$+ g(x)D(z))yD(x)D^2(x) + g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))\}$$
$$+ t^2N_1(x,y,z) + t^3N_2(x,y,z) + t^4N_3(x,y,z) + t^5N_4(x,y,z)$$
$$(59) \quad + t^6g(z)D(z)yD(z)D^2(z) = 0,\ x,y,z \in R,\ t \in S_5,$$

where N_1, N_2, N_3 and N_4 denote the term satisfying the identity (59).

From (58) and (59), we obtain

$$t\{([B(x,z),x] + [f(x),z]))D(x)$$
$$+ g(x)D(z))yD(x)D^2(x) + g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))\}$$
$$+ t^2N_1(x,y,z) + t^3N_2(x,y,z) + t^4N_3(x,y,z) + t^5N_4(x,y,z)$$
$$(60) \quad = 0,\ x,y,z \in R,\ t \in S_5.$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (60) yields

$$([B(x,z),x] + [f(x),z]))D(x) + g(x)D(z))yD(x)D^2(x)$$
$$(61) \quad + g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0,\ x,y,z \in R.$$

Writing $wg(x)D(x)y$ for y in (61), we get

$$([B(x,z),x] + [f(x),z]))D(x) + g(x)D(z))wg(x)D(x)yD(x)D^2(x)$$
$$(62) \quad + g(x)D(x)wg(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0,\ x,y,z \in R.$$

From (58) and (62),

$$(63) \quad g(x)D(x)wg(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0,\ x,y,z \in R.$$

Replacing $y(D(z)D^2(x) + D(x)D^2(z))w$ for w in (63), we get

$$g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))wg(x)D(x)y\{D(z)D^2(x)$$
$$(64) \quad + D(x)D^2(z)\} = 0,\ x,y,z \in R.$$

Since R is semiprime, we get from (64)

$$(65) \quad g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0,\ x,y,z \in R.$$

Writing $x + tw$ for x in (31), we get

$$[f(x+tw), x+tw]D(x+tw)y(D(z)D^2(x+tw) + D(x+tw)D^2(z))$$
$$\equiv g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) + t\{D(w)g(x) + ([B(x,w),x]$$
$$+ [f(x),w])D(x) + g(x)D(w))y(D(z)D^2(x) + D(x)D^2(z))$$
$$+ g(x)D(x)y(D(z)D^2(w)+D(w)D^2(z))\}+t^2P_1(x,y,z)+t^3P_2(x,y,z)$$
$$+ t^4P_3(x,y,z) + t^5g(w)D(w)y(D(z)D^2(w) + D(w)D^2(z))$$
$$(66) \quad = 0,\ x,y,z \in R,\ t \in S_5,$$

where P_1, P_2 and P_3 denote the term satisfying the identity (66).

From (65) and (66), we obtain

$$t\{([B(x,w),x] + [f(x),w])D(x)$$

$$+ g(x)D(w))y(D(z)D^2(x) + D(x)D^2(z)) + g(x)D(x)y(D(z)D^2(w)$$
$$+ D(w)D^2(z))\} + t^2P_1(x,y,z) + t^3P_2(x,y,z)$$
$$+ t^4P_3(x,y,z) = 0,\ w,x,y,z \in R,\ t \in S_5. \tag{67}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (67) yields

$$([B(x,w),x] + [f(x),w])D(x)$$
$$+ g(x)D(w))y(D(z)D^2(x) + D(x)D^2(z)) + g(x)D(x)y(D(z)D^2(w)$$
$$+ D(w)D^2(z)) = 0,\ w,x,y,z \in R. \tag{68}$$

Writing $vg(x)D(x)y$ for y in (68), we get

$$([B(x,w),x] + [f(x),w])D(x)$$
$$+ g(x)D(w))vg(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))$$
$$+ g(x)D(x)vg(x)D(x)y(D(z)D^2(w) + D(w)D^2(z)) = 0,\ w,x,y,z \in R. \tag{69}$$

From (65) and (69),

$$g(x)D(x)vg(x)D(x)y(D(z)D^2(w) + D(w)D^2(z)) = 0,\ v,w,x,y,z \in R. \tag{70}$$

Replacing $y(D(z)D^2(w) + D(w)D^2(z))v$ for v in (70), we get

$$g(x)D(x)y(D(z)D^2(w) + D(w)D^2(z))vg(x)D(x)y\{D(z)D^2(w)$$
$$+ D(w)D^2(z)\} = 0,\ v,w,x,y,z \in R. \tag{71}$$

Since R is semiprime, we get from (71)

$$g(x)D(x)y(D(z)D^2(w) + D(w)D^2(z)) = 0,\ w,x,y,z \in R. \tag{72}$$

Putting $D(z)w$ instead of for w in (72), we have

$$g(x)D(x)y\{D(z)^2D^2(w) + 2D(z)D^2(z)w + D(z)D^3(z)w$$
$$+ D(z)D(w)D^2(z) + D^2(z)wD^2(z)\} = 0,\ w,x,y,z \in R. \tag{73}$$

From (58), (72) and (73), we obtain

$$g(x)D(x)y\{D(z)D^3(z)w + D^2(z)wD^2(z)\} = 0,\ w,x,y,z \in R. \tag{74}$$

Writing $wD(z)$ for w in (74), we get

$$g(x)D(x)y\{D(z)D^3(z)wD(z) + D^2(z)wD(z)D^2(z)\} = 0,\ w,x,y,z \in R. \tag{75}$$

From (58) and (75), we obtain

$$g(x)D(x)yD(z)D^3(z)wD(z) = 0,\ w,x,y,z \in R. \tag{76}$$

Right multiplication of (76) by $g(x)$ leads to

$$g(x)D(x)yD(z)D^3(z)wD(z)D^3(z) = 0,\ w,x,y,z \in R. \tag{77}$$

Substituting $wg(x)D(x)y$ for w in (77), we have

$$g(x)D(x)yD(z)D^3(z)wg(x)D(x)yD(z)D^3(z) = 0,\ w,x,y,z \in R. \tag{78}$$

Since R is semiprime, it follows from (78) that

(79) $$g(x)D(x)yD(z)D^3(z) = 0,\ x, y, z \in R.$$

From (74) and (79), we obtain

(80) $$g(x)D(x)yD^2(z)wD^2(z) = 0,\ w, x, y, z \in R.$$

Substituting $wg(x)D(x)y$ for w in (80), we get

(81) $$g(x)D(x)yD^2(z)wg(x)D(x)yD^2(z) = 0,\ w, x, y, z \in R.$$

Since R is semiprime, it follows from (81) that

(82) $$g(x)D(x)yD^2(z) = 0,\ x, y, z \in R.$$

Replacing xz for z in (82), we get

(83) $$D(x)g(x)y(xD^2(z)w + 2D(x)D(z) + D^2(x)z) = 0,\ x, y, z \in R.$$

From (82) and (83), we obtain

(84) $$2g(x)D(x)yD(x)D(z) = 0,\ x, y, z \in R.$$

Since R is 3!-torsion free, it follows from (84) that

(85) $$g(x)D(x)yD(x)D(z) = 0,\ x, y, z \in R.$$

Writing zx for z in (85), we get

(86) $$g(x)D(x)y\{D(x)D(z)x + D(x)zD(x)) = 0,\ x, y, z \in R.$$

From (85) and (86), we obtain

(87) $$g(x)D(x)yD(x)zD(x) = 0,\ x, y, z \in R.$$

Substituting $zg(x)D(x)y$ for z in (87), we get

(88) $$g(x)D(x)yD(x)zg(x)D(x)yD(x) = 0,\ x, y, z \in R.$$

Since R is semiprime, it follows from (88) that

(89) $$g(x)D(x)yD(x) = 0,\ x, y, z \in R.$$

Putting $yg(x)$ for y in (89), we get

(90) $$g(x)D(x)yg(x)D(x) = 0,\ x, y \in R.$$

Since R is semiprime, we obtain from (90)

(91) $$g(x)D(x) = 0,\ x \in R.$$ □

Theorem 3.7. *Let R be a 5!-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. Then*

$$[D(x), x]D(x)^2 = 0 \ \Leftrightarrow\ D(x)^2[D(x), x] = 0$$

for every $x \in R$.

Proof. The proof of the commutative case is trivial. Thus it suffices to prove the case that R is noncommutative.

Necessity: Assume that

$$[D(x), x]D(x)^2 = f(x)D(x)^2 = 0,\ x \in R. \tag{92}$$

Replacing $x + ty$ for x in (92), we have

$$\begin{aligned}&[D(x+ty), x+ty]D(x+ty)^2\\ &\equiv f(x)D(x)^2 + t\{B(x,y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y)\}\\ &\quad + t^2H_1(x,y) + t^3H_2(x,y) + t^4 f(y)D(y)^2 = 0,\ x,y \in R,\ t \in S_3,\end{aligned} \tag{93}$$

where H_1, H_2 denote the term satisfying the identity (93).

From (92) and (93), we obtain

$$\begin{aligned}&t\{B(x,y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y)\}\\ &+ t^2H_1(x,y) + t^3H_2(x,y) = 0,\ x,y \in R,\ t \in S_5.\end{aligned} \tag{94}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (94) yields

$$B(x,y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y) = 0,\ x,y \in R. \tag{95}$$

Writing xy for y in (95), we have

$$\begin{aligned}&xB(x,y)D(x)^2 + 2f(x)yD(x)^2 + D(x)[y,x]D(x)^2\\ &+ f(x)xD(y)D(x) + f(x)D(x)yD(x) + f(x)D(x)xD(y)\\ &+ f(x)D(x)^2y = 0,\ x,y \in R.\end{aligned} \tag{96}$$

Left multiplication of (95) by x leads to

$$xB(x,y)D(x)^2 + xf(x)D(y)D(x) + xf(x)D(x)D(y) = 0,\ x,y \in R. \tag{97}$$

From (96) and (97), we obtain

$$\begin{aligned}&2f(x)yD(x)^2 + D(x)[y,x]D(x)^2 + g(x)D(y)D(x)\\ &+ f(x)D(x)yD(x) + \{g(x)D(x) + f(x)^2\}D(y)\\ &+ f(x)D(x)^2y = 0,\ x,y \in R.\end{aligned} \tag{98}$$

From (92) and (98), we obtain

$$\begin{aligned}&2f(x)yD(x)^2 + D(x)[y,x]D(x)^2 + g(x)D(y)D(x)\\ &+ f(x)D(x)yD(x) + \{g(x)D(x) + f(x)^2\}D(y) = 0,\ x,y \in R.\end{aligned} \tag{99}$$

Writing $yD(x)$ for y in (99), we have

$$\begin{aligned}&2f(x)yD(x)^3 + D(x)[y,x]D(x)^3 + D(x)yf(x)D(x)^2 + g(x)D(y)D(x)^2\\ &+ g(x)yD^2(x)D(x) + f(x)D(x)yD(x)^2 + \{g(x)D(x) + f(x)^2\}D(y)D(x)\\ &+ \{g(x)D(x) + f(x)^2\}yD^2(x) = 0,\ x,y \in R.\end{aligned} \tag{100}$$

Right multiplication of (99) by $D(x)$ leads to

$$\begin{aligned} &2f(x)yD(x)^3 + D(x)[y,x]D(x)^3 + g(x)D(y)D(x)^2 \\ (101) \quad &+ f(x)D(x)yD(x)^2 + \{g(x)D(x) + f(x)^2\}D(y)D(x) = 0,\ x,y \in R. \end{aligned}$$

From (100) and (101), we obtain

$$\begin{aligned} &D(x)yf(x)D(x)^2 + g(x)yD^2(x)D(x) \\ (102) \quad &+ \{g(x)D(x) + f(x)^2\}yD^2(x) = 0,\ x,y \in R. \end{aligned}$$

From (92) and (102), we obtain

$$(103) \quad g(x)yD^2(x)D(x) + \{g(x)D(x) + f(x)^2\}yD^2(x) = 0,\ x,y \in R.$$

On the other hand, we get from (92)

$$\begin{aligned} 0 &= [f(x)D(x)^2, x] \\ (104) \quad &= g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x),\ x \in R. \end{aligned}$$

On the one hand, let $y = x^2$ in (95). Then we obtain

$$\begin{aligned} 0 &= B(x,x^2)D(x)^2 + f(x)D(x^2)D(x) + f(x)D(x)D(x^2) \\ &= 2(f(x)x + xf(x))D(x)^2 + f(x)(D(x)x + xD(x))D(x) \\ &\quad + f(x)D(x)(D(x)x + xD(x)) \\ (105) \quad &= 3f(x)xD(x)^2 + 2xf(x)D(x)^2 + 2f(x)D(x)xD(x) + f(x)D(x)^2x,\ x \in R. \end{aligned}$$

From (92) and (105), we obtain

$$\begin{aligned} &3f(x)xD(x)^2 + 2xf(x)D(x)^2 + 2f(x)D(x)xD(x) + f(x)D(x)^2x \\ (106) \quad &= 0,\ x \in R. \end{aligned}$$

Since $3xf(x)D(x)^2 = 0, 2xf(x)D(x)^2 = 0$ holds for all $x \in R$ from (92), we get from (106)

$$\begin{aligned} 0 &= 3g(x)D(x)^2 + 2(g(x)D(x) + f(x)^2)D(x) \\ (107) \quad &= 5g(x)D(x)^2 + 2f(x)^2D(x),\ x \in R. \end{aligned}$$

Substituting $D(x)y$ for y in (107), we have

$$(108) \quad g(x)D(x)yD^2(x)D(x) + \{g(x)D(x)^2 + f(x)^2D(x)\}yD^2(x) = 0, x,y \in R.$$

From (104) and (108), we obtain

$$(109) \quad g(x)D(x)yD^2(x)D(x) - f(x)D(x)f(x)yD^2(x) = 0,\ x,y \in R.$$

Replacing $D(x)^2y$ for y in (109), we get

$$(110) \quad g(x)D(x)^3yD^2(x)D(x) - f(x)D(x)f(x)D(x)^2yD^2(x) = 0,\ x,y \in R.$$

From (92) and (110), it follows that

$$(111) \quad g(x)D(x)^3yD^2(x)D(x) = 0,\ x,y \in R.$$

Putting $D(x)y$ instead of y in (108), we arrive at

(112) $g(x)D(x)^2yD^2(x)D(x)+\{g(x)D(x)^3+f(x)^2D(x)^2\}yD^2(x)=0, x,y\in R.$

From (92), (111) and (112), one obtains

$$g(x)D(x)^2yD^2(x)D(x) = 0,\ x,y \in R. \tag{113}$$

Substituting $2yD^2(x)D(x)z$ for y in (108), we have

$$\begin{aligned}&2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x)\\&+\{2g(x)D(x)^2+2f(x)^2D(x)\}yD^2(x)D(x)zD^2(x)=0,\ x,y,z\in R.\end{aligned}\tag{114}$$

From (113) and (114), we get

$$\begin{aligned}&2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x)\\&+2f(x)^2D(x)yD^2(x)D(x)zD^2(x)=0,\ x,y,z\in R.\end{aligned}\tag{115}$$

From (107) and (115),

$$\begin{aligned}&2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x)\\&-5g(x)D(x)^2yD^2(x)D(x)zD^2(x)=0, x,y,z\in R.\end{aligned}\tag{116}$$

From (113) and (116), we arrive at

$$2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0,\ x,y,z \in R. \tag{117}$$

Since R is 3!-torsion free, it follows from (117) that

$$g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0,\ x,y,z \in R. \tag{118}$$

Replacing $zg(x)D(x)y$ for z in (118), we obtain

$$g(x)D(x)yD^2(x)D(x)zg(x)D(x)yD^2(x)D(x) = 0,\ x,y,z \in R. \tag{119}$$

By the semiprimeness of R, we get from (119)

$$g(x)D(x)yD^2(x)D(x) = 0, x,y \in R. \tag{120}$$

Substituting $D(x)y$ for y in (99), we have

$$\begin{aligned}&2f(x)D(x)yD(x)^2+D(x)^2[y,x]D(x)^2+D(x)f(x)yD(x)^2\\&+g(x)D(x)D(y)D(x)+g(x)D^2(x)yD(x)+f(x)D(x)^2yD(x)\\&+\{g(x)D(x)^2+f(x)^2D(x)\}D(y)+\{g(x)D(x)D^2(x)+f(x)^2D^2(x)\}y\\&=0,\ x,y\in R.\end{aligned}\tag{121}$$

Left multiplication of (99) by $D(x)$ leads to

$$\begin{aligned}&2D(x)f(x)yD(x)^2+D(x)^2[y,x]D(x)^2+D(x)g(x)D(y)D(x)\\&+D(x)f(x)D(x)yD(x)+\{D(x)g(x)D(x)+D(x)f(x)^2\}D(y)\\&=0,\ x,y\in R.\end{aligned}\tag{122}$$

From (121) and (122), we obtain

$$\{2f(x)D(x)-D(x)f(x)\}yD(x)^2$$

$$+\{g(x)D(x)-D(x)g(x)\}D(y)D(x)+g(x)D^2(x)yD(x)$$
$$+\{f(x)D(x)^2-D(x)f(x)D(x)\}yD(x)$$
$$+\{g(x)D(x)^2+f(x)^2D(x)-D(x)g(x)D(x)-D(x)f(x)^2\}D(y)$$
$$+\{g(x)D(x)D^2(x)+f(x)^2D^2(x)\}y=0,\ x,y\in R. \tag{123}$$

From (92) and (123), we obtain

$$\{2f(x)D(x)-D(x)f(x)\}yD(x)^2$$
$$+\{g(x)D(x)-D(x)g(x)\}D(y)D(x)+g(x)D^2(x)yD(x)$$
$$-D(x)f(x)D(x)yD(x)$$
$$+\{g(x)D(x)^2+f(x)^2D(x)-D(x)g(x)D(x)-D(x)f(x)^2\}D(y)$$
$$+\{g(x)D(x)D^2(x)+f(x)^2D^2(x)\}y=0,\ x,y\in R. \tag{124}$$

Writing $yD(x)$ for y in (124), we have

$$\{2f(x)D(x)-D(x)f(x)\}yD(x)^3$$
$$+\{g(x)D(x)-D(x)g(x)\}D(y)D(x)^2$$
$$+\{g(x)D(x)-D(x)g(x)\}yD^2(x)D(x)$$
$$+g(x)D^2(x)yD(x)^2-D(x)f(x)D(x)yD(x)^2$$
$$+\{g(x)D(x)^2+f(x)^2D(x)-D(x)g(x)D(x)-D(x)f(x)^2\}D(y)D(x)$$
$$+\{g(x)D(x)D^2(x)+f(x)^2D^2(x)\}yD(x)=0,\ x,y\in R. \tag{125}$$

Right multiplication of (124) by $D(x)$ leads to

$$\{2f(x)D(x)-D(x)f(x)\}yD(x)^3$$
$$+\{g(x)D(x)-D(x)g(x)\}D(y)D(x)^2+g(x)D^2(x)yD(x)^2$$
$$-D(x)f(x)D(x)yD(x)^2$$
$$+\{g(x)D(x)^2+f(x)^2D(x)-D(x)g(x)D(x)-D(x)f(x)^2\}D(y)D(x)$$
$$+\{g(x)D(x)D^2(x)+f(x)^2D^2(x)\}yD(x)=0,\ x,y\in R. \tag{126}$$

From (125) and (126), we obtain

$$\{g(x)D(x)-D(x)g(x)\}yD^2(x)D(x)=0,\ x,y\in R. \tag{127}$$

From (120) and (127), we obtain

$$D(x)g(x)yD^2(x)D(x)=0,\ x,y\in R. \tag{128}$$

By Lemma 3.4, we get from (128)

$$D(x)g(x)=0,\ x\in R. \tag{129}$$

Hence by Lemma 3.2, we obtain from (129), we get

$$3D(x)^2f(x)-D(x)f(x)D(x)=0,\ x\in R. \tag{130}$$

Right multiplication of (107) by $D(x)$ leads to

$$5g(x)D(x)^3 + 2f(x)^2D(x)^2 = 0, \ x \in R. \tag{131}$$

From (92) and (131), we obtain

$$5g(x)D(x)^3 = 0, \ x \in R. \tag{132}$$

Since R is 5!-torsion free, we get from (132)

$$g(x)D(x)^3 = 0, \ x, y \in R. \tag{133}$$

Replacing $D(x)^2y$ for y in (103), we have

$$g(x)D(x)^2yD^2(x)D(x) + \{g(x)D(x)^3 + f(x)^2D(x)^2\}yD^2(x) = 0, \ x, y \in R. \tag{134}$$

From (92), (133) and (134), we obtain

$$g(x)D(x)^2yD^2(x)D(x) = 0, \ x, y \in R. \tag{135}$$

Writing $2D(x)y$ for y in (103), we have

$$2g(x)D(x)yD^2(x)D(x) + \{2g(x)D(x)^2 + 2f(x)^2D(x)\}yD^2(x) = 0, \ x, y \in R. \tag{136}$$

From (107) and (136), we obtain

$$2g(x)D(x)yD^2(x)D(x) - 3g(x)D(x)^2yD^2(x) = 0, \ x, y \in R. \tag{137}$$

Replacing $yD^2(x)D(x)z$ for y in (137), we have

$$\begin{aligned} &2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) \\ &- 3g(x)D(x)^2yD^2(x)D(x)zD^2(x) = 0, \ x, y, z \in R. \end{aligned} \tag{138}$$

From (135) and (138), we obtain

$$2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0, \ x, y, z \in R. \tag{139}$$

Since R is 5!-torsion free, we get from (139)

$$g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0, \ x, y, z \in R. \tag{140}$$

Replacing $zg(x)D(x)y$ for z in (140), we have

$$g(x)D(x)yD^2(x)D(x)zg(x)D(x)yD^2(x)D(x) = 0, \ x, y, z \in R. \tag{141}$$

Since R is semiprime, it follows from (141) that

$$g(x)D(x)yD^2(x)D(x) = 0, \ x, y \in R. \tag{142}$$

By Lemma 3.3, we get from (142)

$$g(x)D(x) = 0, \ x \in R. \tag{143}$$

Hence by Lemma 3.1, we obtain from (143), we get

$$3f(x)D(x)^2 - D(x)f(x)D(x) = 0, \ x \in R. \tag{144}$$

Thus combining (130) with (144), we have

$$3(f(x)D(x)^2 - D(x)^2f(x)^2) = 0, \ x \in R. \tag{145}$$

Since R is 3!-torsion free, it follows from (145) that

$$f(x)D(x)^2 - D(x)^2 f(x) = 0,\ x \in R. \tag{146}$$

Thus from (92) and (146), we get

$$D(x)^2 f(x) = 0,\ x \in R.$$

Sufficiency: Assume that

$$D(x)^2[D(x), x] = D(x)^2 f(x) = 0,\ x \in R. \tag{147}$$

Replacing $x + ty$ for x in (147), we have

$$\begin{aligned} &D(x+ty)^2[D(x+ty), x+ty] \\ &\equiv D(x)^2 f(x) + t\{D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2 B(x,y)\} \\ &\quad + t^2 Q_1(x,y) + t^3 Q_2(x,y) + t^4 D(y)^2 f(y) = 0,\ x, y \in R,\ t \in S_3, \end{aligned} \tag{148}$$

where Q_1 and Q_2 denote the term satisfying the identity (148).

From (147) and (148), we obtain

$$\begin{aligned} &t\{D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2 B(x,y)\} \\ &\quad + t^2 P_1(x,y) + t^3 P_2(x,y) = 0,\ x, y \in R,\ t \in S_3. \end{aligned} \tag{149}$$

Since R is 3!-torsion free, by Lemma 2.1 the relation (149) yields

$$D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2 B(x,y) = 0,\ x, y \in R. \tag{150}$$

Right multiplication of (150) by x leads to

$$D(y)D(x)f(x)x + D(x)D(y)f(x)x + D(x)^2 B(x,y)x = 0,\ x, y \in R. \tag{151}$$

Substituting yx for y in (150), we have

$$\begin{aligned} &D(y)xD(x)f(x) + yD(x)^2 f(x) + D(x)D(y)xf(x) + D(x)yD(x)f(x) \\ &\quad + D(x)^2 B(x,y)x + 2D(x)^2 yf(x) + D(x)^2[y,x]D(x) = 0,\ x, y \in R. \end{aligned} \tag{152}$$

From (151) and (152), we obtain

$$\begin{aligned} &D(y)\{f(x)^2 + D(x)g(x)\} - yD(x)^2 f(x) - D(x)yD(x)f(x) \\ &\quad + D(x)D(y)g(x) - 2D(x)^2 yf(x) - D(x)^2[y,x]D(x) = 0,\ x, y \in R. \end{aligned} \tag{153}$$

From (147) and (153), we obtain

$$\begin{aligned} &D(y)\{f(x)^2 + D(x)g(x)\} - D(x)yD(x)f(x) + D(x)D(y)g(x) \\ &\quad - 2D(x)^2 yf(x) - D(x)^2[y,x]D(x) = 0,\ x, y \in R. \end{aligned} \tag{154}$$

On the one hand, let $y = x^2$ in (150). Then we arrive at

$$\begin{aligned} 0 &= D(x^2)D(x)f(x) + D(x)D(x^2)f(x) + D(x)^2 B(x, x^2) \\ &= \{D(x)x + xD(x)\}D(x)f(x) + D(x)\{D(x)x + xD(x)\}f(x) \\ &\quad + 2D(x)^2\{f(x)x + xf(x)\} \\ &= D(x)xD(x)f(x) + xD(x)^2 f(x) + D(x)^2 xf(x) + D(x)xD(x)f(x) \end{aligned}$$

$$+ 2D(x)^2\{f(x)x + xf(x)\}$$
$$= D(x)xD(x)f(x) + xD(x)^2 f(x) + D(x)^2 xf(x) + D(x)xD(x)f(x)$$
$$+ 2D(x)^2 f(x)x + 2D(x)^2 xf(x)$$
$$= 2D(x)xD(x)f(x)+xD(x)^2 f(x)+3D(x)^2 xf(x)+2D(x)^2 f(x)x,\ x \in R. \tag{155}$$

From (147) and (155), we obtain

$$2D(x)xD(x)f(x) + 3D(x)^2 xf(x) = 0,\ x \in R. \tag{156}$$

From (147) and (156), we have

$$0 = -\{2D(x)[x, D(x)f(x)] + 3D(x)^2[x, f(x)]\}$$
$$= 2D(x)f(x)^2 + 2D(x)^2 g(x) + 3D(x)^2 g(x)$$
$$= 2D(x)f(x)^2 + 5D(x)^2 g(x),\ x \in R. \tag{157}$$

Writing $D(x)y$ for y in (154), we have

$$D(x)D(y)\{f(x)^2 + D(x)g(x)\} + D^2(x)y\{f(x)^2 + D(x)g(x)\}$$
$$- D(x)^2 yD(x)f(x) + D(x)^2 D(y)g(x) + D(x)D^2(x)yg(x)$$
$$- 2D(x)^3 yf(x) - D(x)^3[y, x]D(x) - D(x)^2 f(x)yD(x) = 0,\ x, y \in R. \tag{158}$$

Left multiplication of (154) by $D(x)$ leads to

$$D(x)D(y)\{f(x)^2 + D(x)g(x)\} - D(x)^2 yD(x)f(x) + D(x)^2 D(y)g(x)$$
$$- 2D(x)^3 yf(x) - D(x)^3[y, x]D(x) = 0,\ x, y \in R. \tag{159}$$

From (158) and (159), we obtain

$$D^2(x)y\{f(x)^2 + D(x)g(x)\} + D(x)D^2(x)yg(x)$$
$$- D(x)^2 f(x)yD(x) = 0,\ x, y \in R. \tag{160}$$

From (147) and (160), we obtain

$$D^2(x)y\{f(x)^2 + D(x)g(x)\} + D(x)D^2(x)yg(x) = 0,\ x, y \in R. \tag{161}$$

Putting $2D(x)y$ instead of y in (161), we have

$$D^2(x)y\{2D(x)f(x)^2+2D(x)^2 g(x)\}+2D(x)D^2(x)yD(x)g(x) = 0,\ x, y \in R. \tag{162}$$

From (157) and (162), we obtain

$$-3D^2(x)yD(x)^2 g(x) + 2D(x)D^2(x)yD(x)g(x) = 0,\ x, y \in R. \tag{163}$$

Left multiplication of (157) by $D(x)$ yields

$$2D(x)^2 f(x)^2 + 5D(x)^3 g(x) = 0,\ x \in R. \tag{164}$$

From (147) and (164), we have

$$5D(x)^3 g(x) = 0,\ x \in R. \tag{165}$$

Since R is 5! torsion free, it follows from (165) that

$$D(x)^3 g(x) = 0,\ x \in R. \tag{166}$$

Replacing $yD(x)$ for y in (163), we have

$$-3D^2(x)yD(x)^3g(x) + 2D(x)D^2(x)yD(x)^2g(x) = 0, \ x, y \in R. \tag{167}$$

From (166) and (167), we have

$$2D(x)D^2(x)yD(x)^2g(x) = 0, \ x \in R. \tag{168}$$

Since R is 5! torsion free, it follows from (168) that

$$D(x)D^2(x)yD(x)^2g(x) = 0, \ x \in R. \tag{169}$$

Replacing $zD(x)D^2(x)y$ for y in (163), we obtain

$$\begin{aligned} &-3D^2(x)zD(x)D^2(x)yD(x)^2g(x) \\ &+2D(x)D^2(x)zD(x)D^2(x)yD(x)g(x) = 0, \ x, y \in R. \end{aligned} \tag{170}$$

From (169) and (170), we have

$$2D(x)D^2(x)zD(x)D^2(x)yD(x)g(x) = 0, \ x, y \in R. \tag{171}$$

Since R is 5! torsion free, we get from (171)

$$D(x)D^2(x)zD(x)D^2(x)yD(x)g(x) = 0, \ x, y \in R. \tag{172}$$

Substituting $yD(x)g(x)z$ for z in (172), we obtain

$$D(x)D^2(x)yD(x)g(x)zD(x)D^2(x)yD(x)g(x) = 0, \ x, y \in R. \tag{173}$$

By the semiprimeness of R, it follows from (173) that

$$D(x)D^2(x)yD(x)g(x) = 0, \ x, y \in R. \tag{174}$$

Thus by Lemma 3.4, we get from (174)

$$D(x)g(x) = 0, \ x \in R. \tag{175}$$

Hence by Lemma 3.2, we have from (175)

$$3D(x)^2f(x) - D(x)f(x)D(x) = 0, \ x \in R. \tag{176}$$

From (147) and (176), we get

$$D(x)f(x)D(x) = 0, \ x \in R. \tag{177}$$

From (177), we get

$$\begin{aligned} 0 &= [D(x)f(x)D(x), x] \\ &= f(x)^2D(x) + D(x)g(x)D(x) + D(x)f(x)^2, \ x \in R. \end{aligned} \tag{178}$$

From (175) and (178), we get

$$f(x)^2D(x) + D(x)f(x)^2 = 0, \ x \in R. \tag{179}$$

From (161) and (175), we obtain

$$D^2(x)yf(x)^2 + D(x)D^2(x)yg(x) = 0, \ x, y \in R. \tag{180}$$

From (157) and (175), we have

$$2D(x)f(x)^2 = 0, \ x \in R. \tag{181}$$

Since R is 5! torsion free, we get from (181)

$$D(x)f(x)^2 = 0,\ x \in R. \tag{182}$$

From (179) and (182), we have

$$f(x)^2 D(x) = 0,\ x \in R. \tag{183}$$

Right multiplication of (180) by $D(x)$ yields

$$D^2(x)yf(x)^2D(x) + D(x)D^2(x)yg(x)D(x) = 0,\ x, y \in R. \tag{184}$$

From (183) and (184), we have

$$D(x)D^2(x)yg(x)D(x) = 0,\ x \in R. \tag{185}$$

Thus Lemma 3.3, (185) yields

$$g(x)D(x) = 0,\ x \in R. \tag{186}$$

By Theorem 3.1, we obtain from (186)

$$3f(x)D(x)^2 - D(x)f(x)D(x) = 0,\ x \in R. \tag{187}$$

From (176) and (187), we obtain

$$3(f(x)D(x)^2 - D(x)^2f(x)) = 0,\ x \in R. \tag{188}$$

Since R is 5! torsion free, we get from (188)

$$f(x)D(x)^2 - D(x)^2f(x) = 0,\ x \in R. \tag{189}$$

From (147) and (189), we get

$$f(x)D(x)^2 = 0,\ x \in R.$$

□

Remark 3.8. Let R be a 3!-torsion free semiprime ring. Let $D : R \longrightarrow R$ be a Jordan derivation on R. In this case, by some calculations, it is checked that if $[D(x), x]D(x)^2 = 0$ for every $x \in R$, then $f(x) = [D(x), x] = 0$ for all $x \in R$.

The following theorem is nearly proved by the same arguments as in the proof of J. Vukman's theorem [17].

Theorem 3.9. *Let A be a Banach algebra with $rad(A)$. Let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. In this case, we show that*

$$[D(x), x]D(x)^2 \in rad(A) \iff D(x)^2[D(x), x] \in rad(A)$$

for every $x \in A$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B. E. Johnson and A. M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [9] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideals $P \subset A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. We see that if $[D(x), x]D(x)^2 \in \mathrm{rad}(A)$, we obtain $[D(x), x]D(x)^2 \in \text{rad}(A) \subset P$ for all primitive ideals of A, then $[D_P(\hat{x}), \hat{x}](D_P(\hat{x}))^2 = \hat{0}$. Then since A/P is a prime factor Banach algebra for all primitive ideals of A, by Theorem 3.7, we get $[D_P(\hat{x}), \hat{x}](D_P(\hat{x}))^2 = \hat{0} \iff (D_P(\hat{x}))^2[D_P(\hat{x}), \hat{x}] = \hat{0},\ \hat{x} \in A/P$ for all primitive ideals of A. Hence we conclude that $D(x)^2[D(x), x] \in P$ for all $x \in A$ and for all primitive ideals P of A. Therefore since $\text{rad}(A) = \cap\{P : P$ is any primitive ideals of $A\}$, it follows that

$$[D(x), x]D(x)^2 \in \text{rad}(A) \iff D(x)^2[D(x), x] \in \text{rad}(A)$$

for every $x \in A$. □

As a special case of Theorem 3.9 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 3.10. *Let A be a semisimple Banach algebra. Suppose*

$$[[y, x], x]][y, x]^2 = 0 \iff [y, x]^2[[y, x], x] = 0$$

for every $x, y \in A$.

Proof. Let $\delta_y(x) = [y, x], [[y, x], x] = [\delta_y(x), x], D = \delta_y$ for all $x, y \in R$. Hence we see that δ_y is a continuous (Jordan) derivation on A. Since A is semisimple, $\text{rad}(A) = (0)$. Thus all the conditions of Theorem 3.10 are fulfilled. □

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Byung Do Kim
Department of Mathematics
Gangneung-Wonju National University
Gangneung 210-702, Korea
Email address: bdkim@gwnu.ac.kr