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FOOTPRINT AND MINIMUM DISTANCE FUNCTIONS

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Abstract. Let S be a polynomial ring over a field K, with a monomial order $\prec$, and let I be an unmixed graded ideal of S. In this paper we study two functions associated to I: The minimum distance function δ_I and the footprint function fp_I. It is shown that δ_I is positive and that fp_I is positive if the initial ideal of I is unmixed. Then we show that if I is radical and its associated primes are generated by linear forms, then δ_I is strictly decreasing until it reaches the asymptotic value 1. If I is the edge ideal of a Cohen–Macaulay bipartite graph, we show that $\delta_I(d) = 1$ for d greater than or equal to the regularity of S/I. For a graded ideal of dimension ≥ 1, whose initial ideal is a complete intersection, we give an exact sharp lower bound for the corresponding minimum distance function.

1. Introduction

Let $S = K[t_1, \ldots, t_s] = \oplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading and let $I \neq (0)$ be a graded ideal of S. The *degree* or *multiplicity* of S/I is denoted by $\deg(S/I)$.

Given an integer $d \geq 1$, let $\mathcal{F}_d$ be the set of all zero-divisors of S/I not in I of degree $d \geq 1$:

$$\mathcal{F}_d := \{\, f \in S_d \mid f \notin I,\ (I\colon f) \neq I\},$$

where $(I\colon f) := \{h \in S \mid hf \in I\}$ is the *quotient* ideal or *colon* ideal of I with respect to f. The *minimum distance function* of I is the function $\delta_I\colon \mathbb{N}_+ \to \mathbb{Z}$ given by

$$\delta_I(d) := \begin{cases} \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in \mathcal{F}_d\} & \text{if } \mathcal{F}_d \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{F}_d = \emptyset. \end{cases}$$

Fix a graded monomial order $\prec$ on S. The initial ideal of I is denoted by $\mathrm{in}_\prec(I)$. Let $\Delta_\prec(I)$ be the *footprint* or *Gröbner éscalier* of S/I consisting of all

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the *standard monomials* of S/I, that is, all the monomials of S not in the ideal $\mathrm{in}_\prec(I)$.

Let $\mathcal{M}_{\prec,d}$ be the set of all zero-divisors of $S/\mathrm{in}_\prec(I)$ of degree $d \geq 1$ that are in $\Delta_\prec(I)$:

$$\mathcal{M}_{\prec,d} := \{t^a \,|\, t^a \in \Delta_\prec(I) \cap S_d,\ (\mathrm{in}_\prec(I)\colon t^a) \neq \mathrm{in}_\prec(I)\}.$$

The *footprint function* of I, denoted fp_I, is the function $\mathrm{fp}_I\colon \mathbb{N}_+ \to \mathbb{Z}$ given by

$$\mathrm{fp}_I(d) := \begin{cases} \deg(S/I) - \max\{\deg(S/(\mathrm{in}_\prec(I), t^a)) \,|\, t^a \in \mathcal{M}_{\prec,d}\} & \text{if } \mathcal{M}_{\prec,d} \neq \emptyset, \\ \deg(S/I) & \text{if } \mathcal{M}_{\prec,d} = \emptyset. \end{cases}$$

In this paper we study δ_I and fp_I from a theoretical point of view. The functions δ_I and fp_I were introduced in [18] and [17], respectively. The interest in these functions is essentially due to the following two facts: The minimum distance function is related to the minimum distance in coding theory [18, Theorem 4.7] and the footprint function is much easier to compute. There are significant cases in which either the footprint function is a lower bound for the minimum distance function [17, Lemma 3.10(a)] or the two functions coincide [17, Corollary 4.4].

The footprint lower bound was used in the works of Geil [7] and Carvalho [3] to study affine Reed-Muller-type codes. Long before these two papers appeared the footprint was used by Geil in connection with all kinds of codes (including one-point algebraic geometric codes); see [8–10] and the references therein.

The contents of this paper are as follows. In Section 2 we present some of the results and terminology that will be needed in the paper.

Our first result shows that δ_I is positive if I is unmixed, and that fp_I is also positive if $\mathrm{in}_\prec(I)$ is unmixed (Theorem 3.6). This improves the correlated non-negativity of the functions δ_I and fp_I that was shown in [17, Lemma 3.10]. We show that if I is a radical unmixed ideal whose associated primes are generated by linear forms, then δ_I is strictly decreasing until it reaches the asymptotic value 1 (Theorem 3.8). This gives a wide generalization of [18, Theorem 4.5(vi)]. Then we conjecture that $\delta_I(d) = 1$ for $d \geq \mathrm{reg}(S/I)$, where $\mathrm{reg}(S/I)$ is the regularity of S/I (Conjecture 4.2). We show this conjecture when I is the edge ideal of a Cohen–Macaulay bipartite graph without isolated vertices (Proposition 4.7).

If I is a complete intersection monomial ideal of dimension ≥ 1, we present an explicit formula for $\mathrm{fp}_I(d)$ (Theorem 5.5). In this case $\mathrm{fp}_I(d) = \delta_I(d)$ (Proposition 2.14). For a graded ideal of dimension ≥ 1, whose initial ideal is a complete intersection, we give an exact sharp lower bound for the corresponding minimum distance function (Theorem 5.6). As a particular case we recover [17, Theorem 3.14]; as is seen in [17] this result has interesting applications to coding theory and to packing and covering in combinatorics.

For all unexplained terminology and additional information, we refer to [2,5] (for the theory of Gröbner bases, commutative algebra, and Hilbert functions).

2. Preliminaries

All results of this section are well-known. To avoid repetitions we continue to employ the notations and definitions used in Section 1.

Let $S = K[t_1,\ldots,t_s] = \oplus_{d=0}^{\infty} S_d$ be a polynomial ring over a field K with the standard grading and let $I \neq (0)$ be a graded ideal of S of dimension k. By the dimension of I we mean the Krull dimension of S/I. The *Hilbert function* of S/I, denoted H_I, is given by:

$$H_I(d) := \dim_K(S_d/I_d), \quad d = 0,1,2,\ldots,$$

where $I_d = I \cap S_d$. By a theorem of Hilbert [2, Theorem 4.1.3] there is a unique polynomial $h_I(t) \in \mathbb{Q}[t]$ of degree $k-1$ such that $h_I(d) = H_I(d)$ for $d \gg 0$. By convention the degree of the zero polynomial is -1.

The *degree* or *multiplicity* of S/I, denoted $\deg(S/I)$, is the positive integer

$$\deg(S/I) := \begin{cases} (k-1)! \lim\limits_{d\to\infty} H_I(d)/d^{k-1} & \text{if } k \geq 1, \\ \dim_K(S/I) & \text{if } k = 0. \end{cases}$$

Definition 2.1. If I is a graded ideal of S, the *Hilbert series* of S/I, denoted $F_I(x)$, is given by

$$F_I(x) = \sum_{d=0}^{\infty} H_I(d)x^d, \text{ where } x \text{ is a variable.}$$

Theorem 2.2 (Hilbert–Serre [22, p. 58])**.** *Let $I \subset S$ be a graded ideal of dimension k. Then there is a unique polynomial $h(x) \in \mathbb{Z}[x]$ such that*

$$F_I(x) = \frac{h(x)}{(1-x)^k} \quad \textit{and} \quad h(1) > 0.$$

Remark 2.3. The leading coefficient of the Hilbert polynomial $h_I(x)$ is equal to $h(1)/(k-1)!$. Thus $h(1)$ is equal to $\deg(S/I)$.

Definition 2.4. Let $I \subset S$ be a graded ideal. The a-invariant of S/I, denoted $a(S/I)$, is the degree of $F_I(x)$ as a rational function, that is, $a(S/I) = \deg(h(x)) - k$.

Definition 2.5. Let $I \subset S$ be a graded ideal and let $\mathbb{F}_\star$ be the minimal graded free resolution of S/I as an S-module:

$$\mathbb{F}_\star: \quad 0 \to \bigoplus_j S(-j)^{b_{gj}} \to \cdots \to \bigoplus_j S(-j)^{b_{1j}} \to S \to S/I \to 0.$$

The *Castelnuovo–Mumford regularity* of S/I (*regularity* of S/I for short) is defined as

$$\operatorname{reg}(S/I) = \max\{j - i \,|\, b_{ij} \neq 0\}.$$

An excellent reference for the regularity of graded ideals is the book of Eisenbud [6]. The a-invariant, the regularity, and the depth of S/I are closely related.

Theorem 2.6 ([23, Corollary B.4.1]). *$a(S/I) \leq \operatorname{reg}(S/I) - \operatorname{depth}(S/I)$, with equality if S/I is Cohen–Macaulay.*

Definition 2.7. The *regularity index* of the Hilbert function of S/I, or simply the *regularity index* of S/I, denoted $\operatorname{ri}(S/I)$, is the least integer $n \geq 0$ such that $H_I(d) = h_I(d)$ for $d \geq n$.

If I is a graded Cohen-Macaulay ideal of S of dimension 1, then $\operatorname{reg}(S/I)$, the regularity of S/I is equal to $\operatorname{ri}(S/I)$, the regularity index of S/I. This follows from Theorem 2.6.

Definition 2.8. An ideal $I \subset S$ is called a *complete intersection* if there exist $g_1, \ldots, g_r$ in S such that $I = (g_1, \ldots, g_r)$, where $r = \operatorname{ht}(I)$ is the height of I.

Remark 2.9. (a) A graded ideal I is a complete intersection if and only if I is generated by a homogeneous regular sequence with $\operatorname{ht}(I)$ elements (see [14, Chapter 3]).

(b) A monomial ideal I is a complete intersection if and only if I is minimally generated by a regular sequence of monomials with $\operatorname{ht}(I)$ elements.

Lemma 2.10 ([22, Corollary 3.3]). *If $I \subset S$ is an ideal generated by homogeneous polynomials $f_1, \ldots, f_r$, with $r = \operatorname{ht}(I)$ and $\delta_i = \deg(f_i)$, then the Hilbert series of S/I is given by*

$$F_I(x) = \frac{\prod_{i=1}^{r} \left(1 - x^{\delta_i}\right)}{(1-x)^s}.$$

Lemma 2.11 ([19, Example 1.5.1], [4, Lemma 3.5]). *If $I \subset S$ is a complete intersection ideal generated by homogeneous polynomials $f_1, \ldots, f_r$, with $r = \operatorname{ht}(I)$ and $\delta_i = \deg(f_i)$, then the degree and regularity of S/I are given by $\deg(S/I) = \delta_1 \cdots \delta_r$ and $\operatorname{reg}(S/I) = \sum_{i=1}^{r}(\delta_i - 1)$.*

Proof. The formula for the degree follows from Remark 2.3 and Lemma 2.10. As S/I is Cohen–Macaulay, the formula for the regularity follows from Lemma 2.10 and Theorem 2.6. $\square$

If f is a non-zero polynomial in S and $\prec$ is a monomial order on S, we denote the *leading monomial* of f by $\operatorname{in}_\prec(f)$. For $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, we set $t^a = t_1^{a_1} \cdots t_s^{a_s}$. Let $I \subset S$ be an ideal. A monomial t^a is called a *standard monomial* of S/I, with respect to $\prec$, if t^a is not the leading monomial of any polynomial in I, that is, t^a is not in the ideal $\operatorname{in}_\prec(I)$. A polynomial f is called *standard* if $f \neq 0$ and f is a K-linear combination of standard monomials. The set of standard monomials, denoted $\Delta_\prec(I)$, is called the *footprint* of S/I or *Gröbner éscalier* of I. A subset $\mathcal{G} = \{g_1, \ldots, g_r\}$ of I is called a *Gröbner basis* of I if

$$\operatorname{in}_\prec(I) = (\operatorname{in}_\prec(g_1), \ldots, \operatorname{in}_\prec(g_r)).$$

An element $f \in S$ is called a *zero-divisor* of S/I—as an S-module—if there is $\overline{0} \neq \overline{a} \in S/I$ such that $f\overline{a} = \overline{0}$, and f is called *regular* on S/I otherwise. Notice that f is a zero-divisor if and only if $(I\colon f) \neq I$.

Lemma 2.12 ([17, Lemma 2.8])**.** *Let $\prec$ be a monomial order, let $I \subset S$ be an ideal, and let f be a polynomial of S of positive degree. If $\mathrm{in}_\prec(f)$ is regular on $S/\mathrm{in}_\prec(I)$, then f is regular on S/I.*

An *associated prime* of I is a prime ideal $\mathfrak{p}$ of S of the form $\mathfrak{p} = (I\colon f)$ for some f in S. An ideal $I \subset S$ is called *unmixed* if all its associated primes have the same height and I is called *radical* if I is equal to its radical.

Definition 2.13. If $\mathrm{fp}_I(d) = \delta_I(d)$ for $d \geq 1$, we say that I is a *Geil–Carvalho ideal*.

Proposition 2.14 ([17, Proposition 3.11])**.** *If I is an unmixed monomial ideal and $\prec$ is any monomial order, then $\delta_I(d) = \mathrm{fp}_I(d)$ for $d \geq 1$, that is, I is a Geil–Carvalho ideal.*

Proposition 2.15. *Let $I \subset S$ be an unmixed graded ideal, let $\prec$ be a monomial order on S, and let $d \geq 1$ be an integer. The following hold.*
- (a) [17, Lemma 3.10(a)] $\delta_I(d) \geq \mathrm{fp}_I(d)$.
- (b) [18, Theorem 4.5(iv)] *If t_i is a zero-divisor of S/I for all i, then* $\mathrm{fp}_I(d) \geq 0$.

The lower bound of Proposition 2.15(b) is sharp. In Example 6.3 we show an unmixed graded ideal I of dimension 1 such that t_i is a zero-divisor for all i and $\mathrm{fp}_I(d) = 0$ for $d = 1$.

Proposition 2.16 (Additivity of the degree [20, Proposition 2.5])**.** *If I is an ideal of S and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ is an irredundant primary decomposition, then*

$$\deg(S/I) = \sum_{\mathrm{ht}(\mathfrak{q}_i)=\mathrm{ht}(I)} \deg(S/\mathfrak{q}_i).$$

The additivity is one of the most useful and well-known facts about the degree.

3. Minimum and footprint functions

In this section we study the footprint and minimum distance functions of unmixed graded ideals over an arbitrary field.

Lemma 3.1. *Let $I \subset S$ be an unmixed graded ideal and let $\prec$ be a monomial order. If $f \in S$ is homogeneous and $(I\colon f) \neq I$, then*
- (i) [18, Lemma 4.1] $\deg(S/(I, f)) \leq \deg(S/(\mathrm{in}_\prec(I), \mathrm{in}_\prec(f))) \leq \deg(S/I)$,
- (ii) $\deg(S/I) = \deg(S/(I\colon f)) + \deg(S/(I, f))$ *if $f \notin I$, and*
- (iii) $\deg(S/(I, f)) < \deg(S/I)$ *if $f \notin I$.*

Proof. (ii) Using that I is unmixed, it is not hard to see that S/I, $S/(I\colon f)$, and $S/(I, f)$ have the same Krull dimension. There is an exact sequence

$$0 \longrightarrow S/(I\colon f)[-d] \xrightarrow{f} S/I \longrightarrow S/(I, f) \longrightarrow 0.$$

Hence, by the additivity of Hilbert functions [25, Lemma 5.1.1], we get

$$H_I(i) = H_{(I\colon f)}(i-d) + H_{(I,f)}(i) \quad \text{for } i \geq 0. \tag{3.1}$$

If $\dim S/I = 0$, then using Eq. (3.1) one has

$$\sum_{i\geq 0} H_I(i) = \sum_{i\geq 0} H_{(I\colon f)}(i) + \sum_{i\geq 0} H_{(I,f)}(i).$$

Therefore, using the definition of degree, the required equality follows. If $k = \dim S/I - 1$ and $k \geq 1$, by the Hilbert theorem [2, Theorem 4.1.3], H_I, $H_{(I,f)}$, and $H_{(I\colon f)}$ are polynomial functions of degree k. Then dividing Eq. (3.1) by i^k and taking limits as i goes to infinity, the required equality follows.

(iii) This part follows at once from part (ii). □

The next alternative formula for δ_I is valid for unmixed graded ideals. This expression for δ_I will be used to show some of our results.

Corollary 3.2 ([18, Theorem 4.4])**.** *Let $I \subset S$ be an unmixed graded ideal. If $\mathfrak{m} = (t_1, \ldots, t_s)$ and $d \geq 1$ is an integer such that $\mathfrak{m}^d \not\subset I$, then*

$$\delta_I(d) = \min\{\deg(S/(I\colon f)) \mid f \in S_d \setminus I\}.$$

Proof. If $\mathcal{F}_d = \emptyset$, then $\delta_I(d) = \deg(S/I)$, and for any $f \in S_d \setminus I$ one has that $(I\colon f)$ is equal to I. Thus equality holds. Assume that $\mathcal{F}_d \neq \emptyset$. Take $f \in S_d \setminus I$. If $(I\colon f) = I$, then $\deg(S/(I\colon f))$ is equal to $\deg(S/I)$. On the other hand if $(I\colon f) \neq I$, that is, $f \in \mathcal{F}_d$, then by Lemma 3.1(ii) one has the equality:

$$\deg(S/(I\colon f)) = \deg(S/I) - \deg(S/(I,f)).$$

Notice that in this case $\deg(S/(I\colon f)) \leq \deg(S/I)$. Therefore

$$\begin{aligned} \delta_I(d) &= \deg(S/I) - \max\{\deg(S/(I,f)) \mid f \in \mathcal{F}_d\} \\ &= \min\{\deg(S/(I\colon f)) \mid f \in \mathcal{F}_d\} \\ &= \min\{\deg(S/(I\colon f)) \mid f \in S_d \setminus I\}. \end{aligned}$$ □

Definition 3.3. Let $I \subset S$ be a non-zero proper graded ideal. The *Vasconcelos function* of I is the function $\vartheta_I \colon \mathbb{N}_+ \to \mathbb{N}_+$ given by

$$\vartheta_I(d) = \begin{cases} \min\{\deg(S/(I\colon f)) \mid f \in S_d \setminus I\} & \text{if } \mathfrak{m}^d \not\subset I, \\ \deg(S/I) & \text{if } \mathfrak{m}^d \subset I. \end{cases}$$

Very little is known about the Vasconcelos function when I is not an unmixed graded ideal. Next we show that in certain cases the footprint function can be expressed in terms of the degree of colon ideals.

Corollary 3.4. *Let I be a graded ideal and let $\prec$ be a monomial order. If $\mathrm{in}_\prec(I)$ is an unmixed ideal and $\mathcal{M}_{\prec,d} \neq \emptyset$, then*

$$\mathrm{fp}_I(d) = \min\{\deg(S/(\mathrm{in}_\prec(I)\colon t^a)) \mid t^a \in S_d \setminus \mathrm{in}_\prec(I)\}.$$

Proof. Take $t^a\in\mathcal{M}_{\prec,d}$. By Lemma 3.1(ii) one has the equality:

$$\deg(S/(\mathrm{in}_\prec(I)\colon t^a))=\deg(S/\mathrm{in}_\prec(I))-\deg(S/(\mathrm{in}_\prec(I),t^a)).$$

In this case $\deg(S/(\mathrm{in}_\prec(I)\colon t^a))\leq\deg(S/\mathrm{in}_\prec(I))$. Therefore, noticing that $\deg(S/\mathrm{in}_\prec(I))$ is equal to $\deg(S/I)$, we get

$$\begin{aligned}\mathrm{fp}_I(d)&=\deg(S/I)-\max\{\deg(S/(\mathrm{in}_\prec(I),t^a))\,|\,t^a\in\mathcal{M}_{\prec,d}\}\\&=\min\{\deg(S/(\mathrm{in}_\prec(I)\colon t^a))\,|\,t^a\in\mathcal{M}_{\prec,d}\}\\&=\min\{\deg(S/(\mathrm{in}_\prec(I)\colon t^a))\,|\,t^a\in S_d\setminus\mathrm{in}_\prec(I)\}.\end{aligned}$$

□

One can apply the corollary to graded lattice ideals of dimension 1.

Proposition 3.5. *Let $I\subset S$ be a graded lattice ideal of dimension* 1 *and let $\prec$ be a graded monomial order with $t_1\succ\cdots\succ t_s$. The following hold.*

(a) *If $\mathrm{in}_\prec(I)$ is not prime, then $\mathrm{in}_\prec(I)$ is unmixed and $\mathcal{M}_{\prec,d}\neq\emptyset$ for $d\geq 1$.*
(b) *If $\mathrm{in}_\prec(I)$ is prime, then $I=(t_1-t_s,\ldots,t_{s-1}-t_s)$ and $\mathcal{M}_{\prec,d}=\emptyset$ for $d\geq 1$.*

Proof. The reduced Gröbner basis of I consists of binomials of the form $t^{a_+}-t^{a_-}$ (see [25, Proposition 8.2.7]). It follows that t_s is a regular element on both S/I and $S/\mathrm{in}_\prec(I)$. Hence I and $\mathrm{in}_\prec(I)$ are Cohen–Macaulay ideals. In particular these ideals are unmixed.

(a) Assume that $\mathrm{in}_\prec(I)$ is not prime. Then there is an associated prime $\mathfrak{p}$ of $S/\mathrm{in}_\prec(I)$ such that $\mathrm{in}_\prec(I)\subsetneq\mathfrak{p}$. Pick a variable t_i in $\mathfrak{p}\setminus\mathrm{in}_\prec(I)$. Then $t_it_s^{d-1}$ is in $\mathfrak{p}$ and is not in $\mathrm{in}_\prec(I)$ for $d\geq 1$. Thus $t_it_s^{d-1}$ is in $\mathcal{M}_{\prec,d}$ for $d\geq 1$.

(b) Assume that $\mathrm{in}_\prec(I)$ is prime. This part follows by noticing that $\mathrm{in}_\prec(I)$, being a face ideal generated by variables, is equal to $(t_1,\ldots,t_{s-1})$. □

The next result is a broad generalization of [17, Lemma 3.10].

Theorem 3.6. *Let $I\subset S$ be an unmixed graded ideal, let $\prec$ be a monomial order on S, and let $d\geq 1$ be an integer. The following hold.*

(a) $\delta_I(d)\geq 1$.
(b) $\mathrm{fp}_I(d)\geq 1$ *if $\mathrm{in}_\prec(I)$ is unmixed.*
(c) *If $\dim(S/I)\geq 1$ and $\mathcal{F}_d\neq\emptyset$ for $d\geq 1$, then $\delta_I(d)\geq\delta_I(d+1)\geq 1$ for $d\geq 1$.*

Proof. (a) If $\mathcal{F}_d=\emptyset$, then $\delta_I(d)=\deg(S/I)\geq 1$, and if $\mathcal{F}_d\neq\emptyset$, then using Lemma 3.1(iii) it follows that $\delta_I(d)\geq 1$.

(b) If $\mathcal{M}_{\prec,d}=\emptyset$, then $\mathrm{fp}_I(d)=\deg(S/I)\geq 1$. Next assume that $\mathcal{M}_{\prec,d}\neq\emptyset$. As $\mathrm{in}_\prec(I)$ is unmixed, by Corollary 3.4, $\mathrm{fp}_I(d)\geq 1$.

(c) By part (a), one has $\delta_I(d)\geq 1$. The set $\mathcal{F}_d$ is not empty for $d\geq 1$. Thus, by Corollary 3.2, $\delta_I(d)=\deg(S/(I\colon f))$ for some $f\in\mathcal{F}_d$. As I is unmixed and $\dim(S/I)\geq 1$, $\mathfrak{m}$ is not an associated prime of S/I. Thus, since $(I\colon f)$ is a graded ideal, one has $(I\colon f)\subsetneq\mathfrak{m}$. Pick a linear form $h\in S_1$ such that $hf\notin I$.

As f is a zero-divisor of S/I, so is hf. The ideals $(I\colon f)$ and $(I\colon hf)$ have height equal to $\mathrm{ht}(I)$. Therefore taking Hilbert functions in the exact sequence

$$0 \longrightarrow (I\colon hf)/(I\colon f) \longrightarrow S/(I\colon f) \longrightarrow S/(I\colon hf) \longrightarrow 0$$

it follows that $\deg(S/(I\colon f)) \geq \deg(S/(I\colon hf))$. Therefore, applying Corollary 3.2, we get the inequality $\delta_I(d) \geq \delta_I(d+1)$. □

Lemma 3.7. *Let $I \subset S$ be a radical unmixed graded ideal and let $\mathfrak{p}_1, \dots \mathfrak{p}_m$ be its associated primes. If $f \in \mathcal{F}_d$ for some $d \geq 1$, then*

$$\deg(S/(I\colon f)) = \sum_{f \notin \mathfrak{p}_i} \deg(S/\mathfrak{p}_i).$$

Proof. Since I is a radical ideal, we get that $I = \cap_{i=1}^m \mathfrak{p}_i$. From the equalities

$$(I\colon f) = \cap_{i=1}^m (\mathfrak{p}_i\colon f) = \cap_{f \notin \mathfrak{p}_i} \mathfrak{p}_i,$$

and using the additivity of the degree (see Proposition 2.16), the required equality follows. □

We come to the main result of this section—about the asymptotic behavior of the minimum distance function–which gives a wide generalization of [18, Theorem 4.5(vi)].

Theorem 3.8. *Let $I \subset S$ be an unmixed radical graded ideal. If all the associated primes of I are generated by linear forms, then there is an integer $r_0 \geq 1$ such that*

$$\delta_I(1) > \cdots > \delta_I(r_0) = \delta_I(d) = 1 \quad \textit{for} \quad d \geq r_0.$$

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the associated primes of I. As $\mathfrak{p}_i$ is generated by linear forms, then $\deg(S/\mathfrak{p}_i) = 1$ for all i. Indeed if $\mathfrak{p}_i = \mathfrak{m}$, then $\deg(S/\mathfrak{p}_i)$ is $\dim_K(S/\mathfrak{p}_i) = 1$, and if $\mathfrak{p}_i \subsetneq \mathfrak{m}$, then the initial ideal of $\mathfrak{p}_i$, with respect to the GRevLex order $\prec$, is generated by a subset of $t_1, \dots, t_s$ and $\deg(S/\mathfrak{p}_i)$ is equal to $\deg(S/\mathrm{in}_\prec(\mathfrak{p}_i)) = 1$. The last equality follows noticing that $S/\mathrm{in}_\prec(\mathfrak{p}_i)$ is a polynomial ring.

If I is prime, then $I = \mathfrak{p}_i$ for some i and $\mathcal{F}_d = \emptyset$ for $d \geq 1$. Thus $\delta_I(d) = \deg(S/\mathfrak{p}_i) = 1$ for $d \geq 1$, and we can take $r_0 = 1$. We may now assume that I has at least two associated primes, that is, $m \geq 2$. As $I \subsetneq \mathfrak{p}_1$, there is a form h of degree 1 in $\mathfrak{p}_1 \setminus I$. Hence, as I is a radical ideal, we get that h^d is in $\mathfrak{p}_1 \setminus I$. Thus $\mathcal{F}_d \neq \emptyset$ for $d \geq 1$. Therefore, by Theorem 3.6(c), one has that $\delta_I(d) \geq \delta_I(d+1) \geq 1$ for $d \geq 1$. Hence, assuming that $\delta_I(d) > 1$, it suffices to show that $\delta_I(d) > \delta_I(d+1)$. By Corollary 3.2, there is $f \in \mathcal{F}_d$ such that $\delta_I(d) = \deg(S/(I\colon f))$. Then, by Lemma 3.7, one has

$$\delta_I(d) = \deg(S/(I\colon f)) = \sum_{f \notin \mathfrak{p}_i} \deg(S/\mathfrak{p}_i) \geq 2.$$

Hence there are $\mathfrak{p}_k \neq \mathfrak{p}_j$ such that f is not in $\mathfrak{p}_k \cup \mathfrak{p}_j$. Pick a linear form h in $\mathfrak{p}_k \setminus \mathfrak{p}_j$. Then $hf \notin I$ because $hf \notin \mathfrak{p}_j$, and hf is a zero-divisor of S/I because

$(I\colon f)\neq I$. Noticing that $f\notin\mathfrak{p}_k$ and $hf\in\mathfrak{p}_k$, one obtains the strict inclusion

$$\{\mathfrak{p}_i|\, hf\notin\mathfrak{p}_i\}\subsetneq\{\mathfrak{p}_i|\, f\notin\mathfrak{p}_i\}.$$

Therefore, by Lemma 3.7, we get

$$\deg(S/(I\colon f))=\sum_{f\notin\mathfrak{p}_i}\deg(S/\mathfrak{p}_i)>\sum_{hf\notin\mathfrak{p}_i}\deg(S/\mathfrak{p}_i)=\deg(S/(I\colon hf)).$$

Hence, by Corollary 3.2, we get $\delta_I(d)>\delta_I(d+1)$. □

4. Asymptotic behavior of the minimum distance

Let $I\subset S$ be an unmixed radical graded ideal whose associated primes are generated by linear forms. According to Theorem 3.8, there is an integer $r_0\geq 1$ such that

$$\delta_I(1)>\cdots>\delta_I(r_0)=\delta_I(d)=1 \quad\text{for}\quad d\geq r_0.$$

Definition 4.1. The integer r_0 is called the *regularity index* of δ_I.

If I is the graded vanishing ideal of a set of points in a projective space over a finite field, then $r_0\leq\operatorname{reg}(S/I)$ [11, 21], but we do not know whether this holds in general. The regularity of S/I can be computed using *Macaulay* [12], but r_0 is very difficult to compute.

Conjecture 4.2. *Let $I\subset S$ be an unmixed radical graded ideal. If all the associated primes of I are generated by linear forms, then $\delta_I(d)=1$ for $d\geq\operatorname{reg}(S/I)$, that is, $r_0\leq\operatorname{reg}(S/I)$.*

In this section we give some support for this conjecture. In what follows we focus in the case that I is an unmixed ideal generated by square-free monomial ideals of degree 2.

Definition 4.3 ([24])**.** Let G be a graph with vertex set $V(G)=\{t_1,\ldots,t_s\}$ and edge set $E(G)$. The *edge ideal* of G, denoted by $I(G)$, is the ideal of S generated by all monomials $x_e=\prod_{t_i\in e}t_i$ such that $e\in E(G)$.

Let G be a graph. A subset F of $V(G)$ is called *stable* if $e\not\subset F$ for any $e\in E(G)$, and a subset C of $V(G)$ is a *vertex cover* if and only if $V(G)\setminus C$ is a stable vertex set. A *minimal vertex cover* is a vertex cover which is minimal with respect to inclusion. A graph is called *unmixed* if all its minimal vertex covers have the same cardinality.

Conjecture 4.2 is open even in the case that I is the edge ideal of an unmixed bipartite graph. Below we prove the conjecture for edge ideals of Cohen-Macaulay graphs.

Definition 4.4. Let A be a set of vertices of a graph G. The *induced subgraph* on A, denoted by $G[A]$, is the maximal subgraph of G with vertex set A. A graph of the form $G[A]$ for some $A\subset V(G)$ is called an *induced subgraph* of G.

Notice that $G[A]$ may have isolated vertices, i.e., vertices that do not belong to any edge of $G[A]$. If G is a discrete graph, i.e., all the vertices of G are isolated, we set $I(G) = 0$.

Definition 4.5. An *induced matching* in a graph G is a set of pairwise disjoint edges $f_1, \ldots, f_r$ such that the only edges of G contained in $\cup_{i=1}^{r} f_i$ are $f_1, \ldots, f_r$. The *induced matching number*, denoted by $\mathrm{im}(G)$, is the number of edges in the largest induced matching.

Proposition 4.6 ([15, Lemma 2.2])**.** *If G is a graph, then* $\mathrm{reg}(R/I(G)) \geq \mathrm{im}(G)$.

Next we prove Conjecture 4.2 for edge ideals of Cohen–Macaulay bipartite graphs. A graph G is called *Cohen–Macaulay* if $S/I(G)$ is Cohen–Macaulay.

Proposition 4.7. *If $I = I(G)$ is the edge ideal of a Cohen–Macaulay bipartite graph without isolated vertices, then $\delta_I(d) = 1$ for $d \geq \mathrm{reg}(S/I)$.*

Proof. By [16, Theorem 1.1], $\mathrm{reg}(S/I) = \mathrm{im}(G)$. Thus, by Theorem 3.8, it suffices to show that $\delta_I(d) = 1$ for some $d \leq \mathrm{im}(G)$. According to [13, Theorem 3.4], there is a bipartition $V_1 = \{x_1, \ldots, x_g\}$, $V_2 = \{y_1, \ldots, y_g\}$ of G such that:

(a) $e_i = \{x_i, y_i\} \in E(G)$ for all i,
(b) if $\{x_i, y_j\} \in E(G)$, then $i \leq j$, and
(c) if $\{x_i, y_j\}$, $\{x_j, y_k\}$ are in $E(G)$ and $i < j < k$, then $\{x_i, y_k\} \in E(G)$.

Next we construct a sequence $x_{i_1}, \ldots, x_{i_d}$ such that $e_{i_1}, \ldots, e_{i_d}$ form an induced matching and V_2 is a pairwise disjoint union

$$V_2 = N_G(x_{i_1}) \cup \cdots \cup N_G(x_{i_d}), \tag{4.1}$$

where $N_G(x_{i_j}) \cap N_G(x_{i_k}) = \emptyset$ for $j \neq k$ and $N_G(x_{i_j})$ is the neighbor set of x_{i_j}, that is, $N_G(x_{i_j})$ is the set of vertices of G adjacent to x_{i_j}. We set $i_1 = 1$. If $N_G(x_{i_1}) \subsetneq V_2$, pick y_{i_2} in $V_2 \setminus N_G(x_{i_1})$. By condition (b), e_{i_1}, e_{i_2} is an induced matching and $N_G(x_{i_1}) \cap N_G(x_{i_2}) = \emptyset$. If $N_G(x_{i_1}) \cup N_G(x_{i_2}) \subsetneq V_2$, pick y_{i_3} in $V_2 \setminus (N_G(x_{i_1}) \cup N_G(x_{i_2}))$. By condition (b), $e_{i_1}, e_{i_2}, e_{i_3}$ form an induced matching and $N_G(x_{i_j}) \cap N_G(x_{i_k}) = \emptyset$ for $j \neq k$. Thus one can continue this process until we get a sequence $x_{i_1}, \ldots, x_{i_d}$ such that V_2 is the disjoint union of the $N_G(x_{i_j})$'s and the e_{i_j}'s form an induced matching.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the associated primes of I. There are minimal vertex covers $C_1, \ldots, C_m$ of G such that $\mathfrak{p}_i$ is generated by C_i for $i = 1, \ldots, m$ (see [24, p. 279]). We may assume that $C_m = V_2$. Setting $x^a = x_{i_1} \cdots x_{i_d}$, by Corollary 3.2, it suffices to show that x^a is in $\cap_{i=1}^{m-1} \mathfrak{p}_i \setminus \mathfrak{p}_m$ and that $\deg(S/(I\colon x^a)) = 1$, where $S = K[V(G)]$. If $i \neq m$, there is $y_\ell \notin C_i$. From Eq. (4.1), there is x_{i_j} such that $y_\ell \in N_G(x_{i_j})$ for some i_j. Hence, as C_i covers the edge $\{x_{i_j}, y_\ell\}$, one has that x_{i_j} is in $\mathfrak{p}_i$. Thus x^a is in $\cap_{i=1}^{m-1} \mathfrak{p}_i$ and x^a is not in $\mathfrak{p}_m$ because $\mathfrak{p}_m = (y_1, \ldots, y_g)$. Therefore

$$(I\colon x^a) = (\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m\colon x^a) = (\mathfrak{p}_1\colon x^a) \cap \cdots \cap (\mathfrak{p}_m\colon x^a) = \mathfrak{p}_m.$$

Hence $\deg(S/(I\colon x^a))=1$, as required. $\square$

5. Complete intersections

Let $S=K[t_1,\ldots,t_s]=\oplus_{d=0}^{\infty}S_d$ be a polynomial ring over a field K with the standard grading and let $\prec$ be a graded monomial order.

Proposition 5.1. *Let $I\subset S$ be a graded ideal and let $\prec$ be a monomial order. Suppose that $\mathrm{in}_\prec(I)$ is a complete intersection of height r generated by $t^{\alpha_1},\ldots,t^{\alpha_r}$ with $d_i=\deg(t^{\alpha_i})$ and $d_i\geq 1$ for all i. The following hold.*

(a) [19, Example 1.5.1] *I is a complete intersection and* $\dim(S/I)=s-r$.
(b) $\deg(S/I)=d_1\cdots d_r$ *and* $\operatorname{reg} S/I=\sum_{i=1}^{r}(d_i-1)$.
(c) $1\leq \mathrm{fp}_I(d)\leq \delta_I(d)$ *for* $d\geq 1$.

Proof. (a) The rings S/I and $S/\mathrm{init}_\prec(I)$ have the same dimension. Thus $\dim(S/I)=s-r$. As $\prec$ is a graded order, there are $f_1,\ldots,f_r$ homogeneous polynomials in I with $\mathrm{in}_\prec(f_i)=t^{\alpha_i}$ for $i\geq 1$. Since

$$\mathrm{in}_\prec(I)=(\mathrm{in}_\prec(f_1),\ldots,\mathrm{in}_\prec(f_r)),$$

the polynomials $f_1,\ldots,f_r$ form a Gröbner basis of I, and in particular they generated I. Hence I is a graded ideal of height r generated by r polynomials, that is, I is a complete intersection.

(b) This follows at once from part (a) and Lemma 2.11.

(c) By part (a), I is a complete intersection. In particular I is a Cohen–Macaulay unmixed ideal. Hence this part follows from Proposition 2.15 and Theorem 3.6. $\square$

Lemma 5.2. *Let $I\subset S$ be a complete intersection ideal minimally generated by $t^{\alpha_1},\ldots,t^{\alpha_r}$ and let $t^a=t_1^{a_1}\cdots t_s^{a_s}$ be a zero-divisor of S/I not in I. The following hold.*

(a) *t^{α_i} and t^{α_j} have no common variable for $i\neq j$.*
(b) *If $t_j^{a_j}$ is regular on S/I and $t^c=t^a/t_j^{a_j}$, then $(I\colon t^a)=(I\colon t^c)$.*
(c) *If t_j is a zero-divisor of S/I, then there is a unique $\alpha_i=(\alpha_{i,1},\ldots,\alpha_{i,s})$ such that $\alpha_{i,j}>0$, that is, t_j occurs in exactly one t^{α_i}. If $a_j>\alpha_{i,j}$ and $t^c=t^a/t_j$, then $(I\colon t^a)=(I\colon t^c)$.*
(d) *For each i there is t^{β_i} dividing t^{α_i} such that* $\deg(t^{\beta_i})<\deg(t^{\alpha_i})$ *and* $(I\colon t^a)=(I\colon t^\beta)$, *where $t^\beta=t^{\beta_1}\cdots t^{\beta_r}$.*

Proof. (a) This follows readily from the Krull principal ideal theorem [25, Theorem 2.3.16].

(b) The inclusion "$\supset$" is clear. To show the reverse inclusion take t^δ in $(I\colon t^a)$, that is, $t^\delta t^a=t^\delta t_j^{a_j}t^c$ is in I. Hence $t^\delta t^c$ is in I because $t_j^{a_j}$ is regular on S/I. Thus t^δ is in $(I\colon t^c)$.

(c) If t_j is a zero-divisor of S/I, then t_j is in some associated prime of S/I. Hence, by part (a), t_j must occur in a unique t^{α_i} for some i. Thus one has $\alpha_{i,j}>0$. We claim that $((t^{\alpha_k})\colon t^a)=((t^{\alpha_k})\colon t^c)$ for all k. If $k\neq i$, by part (a),

t_j is regular on $S/(t^{\alpha_k})$. Thus, as in the proof of part (b), we get the asserted equality. Next we assume that $k = i$. The inclusion "$\supset$" is clear. To show the reverse inclusion take t^δ in $((t^{\alpha_i})\colon t^a)$, that is, $t^\delta t^a = t^\gamma t^{\alpha_i}$ for some t^γ. Since $a_j > \alpha_{i,j} > 0$, t_j must divide t^γ. Then we can write $t^\delta t^c = t^\omega t^{\alpha_i}$, where $t^\omega = t^\gamma/t_j$. Thus t^δ is in $((t^{\alpha_i})\colon t^c)$. This completes the proof of the claim. Therefore one has

$$\begin{aligned}(I\colon t^a) &= ((t^{\alpha_1})\colon t^a) + \cdots + ((t^{\alpha_r})\colon t^a)\\ &= ((t^{\alpha_1})\colon t^c) + \cdots + ((t^{\alpha_r})\colon t^c) = (I\colon t^c).\end{aligned}$$

(d) Using part (a) and successively applying parts (b) and (c) to t^a, we get a monomial t^β that divides t^a such that the following conditions are satisfied: (i) all variables that occur in t^β are zero-divisors of S/I, (ii) if $t^\beta = t_1^{\gamma_1}\cdots t_s^{\gamma_s}$ and $\gamma_j > 0$, then $\alpha_{i,j} \geq \gamma_j$, where t^{α_i} is the unique monomial, among $t^{\alpha_1}, \ldots, t^{\alpha_r}$, containing t_j, and (iii) $(I\colon t^a) = (I\colon t^\beta)$. We let t^{β_i} be the product of all $t_j^{\gamma_j}$ such that t_j occurs in t^{α_i}. Clearly t^{β_i} divides t^{α_i}, and $\deg(t^{\alpha_i}) > \deg(t^{\beta_i})$ because t^a is not in I by hypothesis. $\square$

The next result gives some additional support to Conjecture 4.2.

Proposition 5.3. *Let $I \subset S$ be a complete intersection monomial ideal of dimension ≥ 1 minimally generated by $t^{\alpha_1}, \ldots, t^{\alpha_r}$. If $d_i = \deg(t^{\alpha_i})$ for $i = 1, \ldots, r$. The following hold.*

(a) $\operatorname{reg}(S/I) = \sum_{i=1}^r (d_i - 1)$,
(b) $\delta_I(d) = 1$ *if* $d \geq \operatorname{reg}(S/I)$,
(c) $\delta_I(d) \leq (d_{k+1} - \ell)\, d_{k+2} \cdots d_r$ *if* $d < \operatorname{reg}(S/I)$, *where* $0 \leq k \leq r-1$ *and* ℓ *are integers such that* $d = \sum_{i=1}^k (d_i - 1) + \ell$ *and* $1 \leq \ell \leq d_{k+1} - 1$.

Proof. (a) This follows at once from Lemma 2.11.

(b) By Lemma 5.2(a) the monomials t^{α_i} and t^{α_j} have no common variables for $i \neq j$. For each i pick t_{j_i} in t^{α_i}. If I is prime, then $I = (t_{j_1}, \ldots, t_{j_r})$, $\operatorname{reg}(S/I) = 0$, $\mathcal{F}_d = \emptyset$ and $\delta_I(d) = 1$ for $d \geq 1$. Thus we may assume that I is not prime. We claim that $\mathcal{F}_d \neq \emptyset$ for $d \geq 1$. As I is not prime, there is m such that t_{j_m} a zero-divisor of S/I not in I. If a variable t_n is not in t^{α_i} for any i, then t_n is a regular element on S/I, and $\mathcal{F}_d \neq \emptyset$ because $t_{j_m} t_n^{d-1}$ is in $\mathcal{F}_d$. If any variable t_n is in t^{α_i} for some i, then any monomial of degree d is a zero-divisor of S/I because any variable t_n belongs to at least one associated prime of S/I. As $\dim(S/I) \geq 1$, one has $\mathfrak{m}^d \not\subset I$. Pick a monomial t^a of degree d not in I. Then $\mathcal{F}_d \neq \emptyset$ because t^a is in $\mathcal{F}_d$. This completes the proof of the claim. We set $t^{c_i} = t^{\alpha_i}/t_{j_i}$ for $i = 1, \ldots, r$ and $t^c = t^{c_1} \cdots t^{c_r}$. Then it is seen that $(I\colon t^c) = (t_{j_1}, \ldots, t_{j_r})$ and $\deg S/(I\colon t^c) = 1$. Notice that t^c is a zero-divisor of S/I, $t^c \notin I$ and $\deg(t^c) = \operatorname{reg}(S/I)$. Hence, by Corollary 3.2, we get that $\delta_I(d) = 1$ for $d = \operatorname{reg}(S/I)$. Thus, by Theorem 3.6(c), we get $\delta_I(d) = 1$ for $d \geq \operatorname{reg}(S/I)$.

(c) There is a monomial t^a of degree ℓ that divides $t^{\alpha_{k+1}}$ because ℓ is a positive integer less than or equal to $d_{k+1} - 1$. Setting $t^c = t^{c_1} \cdots t^{c_k} t^a$ and

$t^\gamma = t^{\alpha_{k+1}}/t^a$, one has

$$(I\colon t^c) = (t_{j_1}, \ldots, t_{j_k}, t^\gamma, t^{\alpha_{k+2}}, \ldots, t^{\alpha_r}).$$

Hence, by Lemma 2.11, we get $\deg S/(I\colon t^c) = (d_{k+1} - \ell)d_{k+2} \cdots d_r$ because $(I\colon t^c)$ is a complete intersection. Since $\deg(t^c) = d = \sum_{i=1}^{k}(d_i - 1) + \ell$, t^c is not in I, and t^c is a zero-divisor of S/I, by Corollary 3.2 we get that $\deg S/(I\colon t^c) \geq \delta_I(d)$, as required. $\square$

Proposition 5.4 ([18, Proposition 5.7]). *Let $1 \leq e_1 \leq \cdots \leq e_m$ and $0 \leq b_i \leq e_i - 1$ for $i = 1, \ldots, m$ be integers. If $b_0 \geq 1$, then*

$$\prod_{i=1}^{m}(e_i - b_i) \geq \left(\sum_{i=1}^{k+1}(e_i - b_i) - (k-1) - b_0 - \sum_{i=k+2}^{m} b_i\right) e_{k+2} \cdots e_m \tag{5.1}$$

for $k = 0, \ldots, m-1$, where $e_{k+2} \cdots e_m = 1$ and $\sum_{i=k+2}^{m} b_i = 0$ if $k = m-1$.

We come to the main result of this section.

Theorem 5.5. *Let $I \subset S$ be a complete intersection monomial ideal of dimension ≥ 1 minimally generated by $t^{\alpha_1}, \ldots, t^{\alpha_r}$ and let $d \geq 1$ be an integer. If $d_i = \deg(t^{\alpha_i})$ for $i = 1, \ldots, r$ and $d_1 \leq \cdots \leq d_r$, then*

$$\delta_I(d) = \mathrm{fp}_I(d) = \begin{cases} (d_{k+1} - \ell)\, d_{k+2} \cdots d_r & \text{if } d < \sum\limits_{i=1}^{r}(d_i - 1), \\ 1 & \text{if } d \geq \sum\limits_{i=1}^{r}(d_i - 1), \end{cases}$$

where $0 \leq k \leq r-1$ and ℓ are integers such that $d = \sum_{i=1}^{k}(d_i - 1) + \ell$ and $1 \leq \ell \leq d_{k+1} - 1$.

Proof. The ideal I is unmixed because I is Cohen–Macaulay. Hence, by Proposition 2.14, I is Geil–Carvalho, that is, $\delta_I(d) = \mathrm{fp}_I(d)$ for $d \geq 1$. Therefore, by Proposition 5.3, it suffices to show that

$$\mathrm{fp}_I(d) \geq (d_{k+1} - \ell)d_{k+2} \cdots d_r \quad \text{for } d < \mathrm{reg}(S/I).$$

Let t^a be a monomial of degree d such that $t^a \notin I$ and $(I\colon t^a) \neq I$. By Lemma 5.2(d), for each i there is a monomial t^{β_i} dividing t^{α_i} such that $\deg(t^{\beta_i}) < \deg(t^{\alpha_i})$ and $(I\colon t^a) = (I\colon t^\beta)$, where $t^\beta = t^{\beta_1} \cdots t^{\beta_r}$. One can write

$$t^{\alpha_i} = t_1^{\alpha_{i,1}} \cdots t_s^{\alpha_{i,s}} \quad \text{and} \quad t^{\beta_i} = t_1^{\beta_{i,1}} \cdots t_s^{\beta_{i,s}}$$

for $i = 1, \ldots, r$. According to Lemma 5.2(a) the monomials t^{α_i} and t^{α_j} have no common variables for $i \neq j$. As $(I\colon t^\beta)$ is a monomial ideal, it follows that

$$(I\colon t^a) = (I\colon t^\beta) = (\{t_1^{\alpha_{i,1}-\beta_{i,1}} \cdots t_s^{\alpha_{i,s}-\beta_{i,s}}\}_{i=1}^{r}).$$

Hence, setting $g_i = t_1^{\alpha_{i,1}-\beta_{i,1}} \cdots t_s^{\alpha_{i,s}-\beta_{i,s}}$ for $i = 1, \ldots, r$ and observing that g_i and g_j have no common variables for $i \neq j$, we get that $g_1, \ldots, g_r$ form a regular sequence, that is, $(I\colon t^a)$ is again a complete intersection. Thus, by Lemma 2.11, we obtain

$$\deg(S/(I\colon t^a)) = \prod_{i=1}^{r}\left[\sum_{j=1}^{s}(\alpha_{i,j}-\beta_{i,j})\right] = \prod_{i=1}^{r}\left[\deg(t^{\alpha_i})-\deg(t^{\beta_i})\right].$$

Therefore, setting $b_i=\deg(t^{\beta_i})$ for $i=1,\ldots,r$, we get

$$\deg(S/(I\colon t^a)) = \prod_{i=1}^{r}(d_i-b_i).$$

Thus, by Corollary 3.2, it suffices to show the inequality

$$\deg(S/(I\colon t^a)) = \prod_{i=1}^{r}(d_i-b_i) \geq (d_{k+1}-\ell)d_{k+2}\cdots d_r.$$

Noticing that $d=\deg(t^a)=\sum_{i=1}^{k}(d_i-1)+\ell\geq \deg(t^\beta)=\sum_{i=1}^{r}b_i$, one has

$$\left(d_{k+1}+\sum_{i=1}^{k}(d_i-1)-\sum_{i=1}^{r}b_i\right)d_{k+2}\cdots d_r \geq (d_{k+1}-\ell)d_{k+2}\cdots d_r.$$

Hence, we need only show the inequality

$$\prod_{i=1}^{r}(d_i-b_i)\geq\left(\sum_{i=1}^{k+1}(d_i-b_i)-k-\sum_{i=k+2}^{r}b_i\right)d_{k+2}\cdots d_r,$$

which follows making $b_0=1$ and $m=r$ in Proposition 5.4. □

Theorem 5.6. *Let $I\subset S$ be a graded ideal of dimension ≥ 1 and let $\prec$ be a monomial order. If $\mathrm{in}_\prec(I)$ is a complete intersection of height r generated by $t^{\alpha_1},\ldots,t^{\alpha_r}$ with $d_i=\deg(t^{\alpha_i})$ and $1\leq d_i\leq d_{i+1}$ for $i\geq 1$, then $\delta_I(d)\geq \mathrm{fp}_I(d)\geq 1$ and the footprint function is given by*

$$\mathrm{fp}_I(d)=\begin{cases}(d_{k+1}-\ell)d_{k+2}\cdots d_r & \text{if } 1\leq d\leq \sum\limits_{i=1}^{r}(d_i-1)-1,\\ 1 & \text{if } d\geq \sum\limits_{i=1}^{r}(d_i-1),\end{cases}$$

where $0\leq k\leq r-1$ and ℓ are integers such that $d=\sum_{i=1}^{k}(d_i-1)+\ell$ and $1\leq \ell\leq d_{k+1}-1$.

Proof. By Proposition 5.1 one has $\delta_I(d)\geq \mathrm{fp}_I(d)\geq 1$. Since $\mathrm{fp}_I(d)$ is equal to $\mathrm{fp}_{\mathrm{in}_\prec(I)}(d)$ for $d\geq 1$, the formula for $\mathrm{fp}_I(d)$ follows directly from Theorem 5.5. □

It is an open question whether in Theorem 5.6 one has the equality $\delta_I(d)=\mathrm{fp}_I(d)$ for $d\geq 1$. If we make $r=s-1$ in Theorem 5.6, we recover [17, Theorem 3.14]. The reader is referred to [17] for some interesting applications of this result to algebraic coding theory. As is seen in [17, Corollary 4.5] this result can also be used to extend a result of Alon and Füredi [1, Theorem 1] about coverings of the cube $\{0,1\}^n$ by affine hyperplanes.

6. Computing the minimum distance function

Let $I \subset S$ be a graded ideal and let $\prec$ be a monomial order. The minimum distance function of I can be expressed as follows.

Theorem 6.1. *If $\Delta_\prec(I) \cap S_d = \{t^{a_1}, \ldots, t^{a_n}\}$ is the set of all standard monomials of S/I of degree $d \geq 1$ and*

$$\mathcal{F}_{\prec,d} = \left\{ f = \sum_i \lambda_i t^{a_i} \;\middle|\; f \neq 0,\ \lambda_i \in K,\ (I\colon f) \neq I \right\},$$

then

$$\delta_I(d) = \deg(S/I) - \max\{\deg(S/(I,f))\,|\, f \in \mathcal{F}_{\prec,d}\}.$$

Proof. Let f be any element of $\mathcal{F}_d$. Pick a Gröbner basis $g_1, \ldots, g_r$ of I. Then, by the division algorithm [5, Theorem 3, p. 63], we can write $f = \sum_{i=1}^r a_i g_i + h$, where h is a homogeneous standard polynomial of S/I of degree d. Since $(I\colon f) = (I\colon h)$, we get that h is in $\mathcal{F}_{\prec,d}$. Hence, as $(I,f) = (I,h)$, we get the equalities:

$$\begin{aligned}\delta_I(d) &= \deg(S/I) - \max\{\deg(S/(I,f))\,|\, f \in \mathcal{F}_d\}\\ &= \deg(S/I) - \max\{\deg(S/(I,f))\,|\, f \in \mathcal{F}_{\prec,d}\}. \qquad \square\end{aligned}$$

Notice that $\mathcal{F}_d \neq \emptyset$ if and only if $\mathcal{F}_{\prec,d} \neq \emptyset$. If $K = \mathbb{F}_q$ is a finite field, then the number of standard polynomials of degree d is $q^n - 1$, where n is the number of standard monomials of degree d. Hence, we can compute $\delta_I(d)$ for small values of d, n, and q. To compute $\mathrm{fp}_I(d)$ is much easier even if the field is infinite because $\mathcal{M}_{\prec,d}$ has at most n elements.

Example 6.2. Let K be the field $\mathbb{F}_2$ and let I be the ideal of $S = \mathbb{F}_2[t_1, t_2, t_3]$ generated by the binomials $t_1t_2^2 - t_1^2t_2$, $t_1t_3^2 - t_1^2t_3$, $t_2^2t_3 - t_2t_3^2$. If S has the GRevLex order $\prec$, then using Theorem 6.1 and the procedure below for *Macaulay* [12] we get

d	1	2	3	$\cdots$
$\deg(S/I)$	7	7	7	$\cdots$
$H_I(d)$	3	6	7	$\cdots$
$\delta_I(d)$	4	2	1	$\cdots$
$\mathrm{fp}_I(d)$	4	1	1	$\cdots$

```
q=2
S=ZZ/q[t1,t2,t3]
I=ideal(t1*t2^q-t1^q*t2,t1*t3^q-t1^q*t3,t2^q*t3-t2*t3^q)
M=coker gens gb I, degree M, regularity M
h=(d)->degree M - max apply(apply(apply(apply(
toList (set(0..q-1))^**(hilbertFunction(d,M))-
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```
(set{0})^**(hilbertFunction(d,M)),toList),
x->basis(d,M)*vector x),
z->ideal(flatten entries z)),x-> if not
quotient(I,x)==I then degree ideal(I,x) else 0)--The function
h(d)--gives the minimum distance in degree d
init=ideal(leadTerm gens gb I)
hilbertFunction(1,M),hilbertFunction(2,M),hilbertFunction(3,M)
f=(x)-> if not quotient(init,x)==init then degree ideal(init,x)
else 0
fp=(d) ->degree M -max apply(flatten entries basis(d,M),f)--The
--function fp(d) gives the footprint in degree d
h(1), h(2), fp(1), fp(2)
```

Example 6.3. Let $S = \mathbb{F}_3[t_1, t_2, t_3, t_4]$ be a polynomial ring over the field $\mathbb{F}_3$ with the GRevLex order $\prec$, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_5$ be the prime ideals

$$\begin{array}{lll} \mathfrak{p}_1 = (t_3 + t_4, t_2 + t_4, t_1 + t_4), & \mathfrak{p}_2 = (t_3 + t_4, t_2, t_1 - t_4), & \mathfrak{p}_3 = (t_4, t_2, t_1), \\ \mathfrak{p}_4 = (t_4, t_3, t_1), & \mathfrak{p}_5 = (t_4, t_2 - t_3, t_1), & \end{array}$$

and let $I = \cap_{i=1}^5 \mathfrak{p}_i$ be the intersection of these prime ideals. Then, using *Macaulay*2 [12], we get $\mathrm{reg}(S/I) = 2$, $\deg(S/I) = 5$, the initial ideal of I is

$$\mathrm{in}_\prec(I) = (t_3t_4,\, t_1t_4,\, t_1t_3,\, t_1t_2,\, t_1^2,\, t_2^2t_4,\, t_2^2t_3),$$

$\mathrm{in}_\prec(I)$ is a monomial ideal of height 3, $\mathfrak{m}$ is an associated prime of $\mathrm{in}_\prec(I)$, and $\mathrm{fp}_I(1) = 0$. Thus the lower bound for the footprint $\mathrm{fp}_I(d)$ given in Proposition 2.15(b) is sharp.

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