Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 73–83
https://doi.org/10.4134/CKMS.c170127
pISSN: 1225-1763 / eISSN: 2234-3024

REMARKS ON GENERALIZED JORDAN $(\alpha,\beta)^*$-DERIVATIONS OF SEMIPRIME RINGS WITH INVOLUTION

Motoshi Hongan and Nadeem ur Rehman

Dedicated to Professor Atsushi Nakajima on his 75th Birthday

Abstract. Let R be an associative ring with involution $*$ and $\alpha,\beta : R \to R$ ring homomorphisms. An additive mapping $d : R \to R$ is called an $(\alpha,\beta)^*$-derivation of R if $d(xy) = d(x)\alpha(y^*) + \beta(x)d(y)$ is fulfilled for any $x, y \in R$, and an additive mapping $F : R \to R$ is called a generalized $(\alpha,\beta)^*$-derivation of R associated with an $(\alpha,\beta)^*$-derivation d if $F(xy) = F(x)\alpha(y^*) + \beta(x)d(y)$ is fulfilled for all $x, y \in R$. In this note, we intend to generalize a theorem of Vukman [12], and a theorem of Daif and El-Sayiad [6], moreover, we generalize a theorem of Ali et al. [4] and a theorem of Huang and Koc [9] related to generalized Jordan triple $(\alpha,\beta)^*$-derivations.

1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $n = 0$. An additive mapping $d : R \to R$ is called an (α,β)-derivation of R if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for any $x, y \in R$, and an additive mapping $F : R \to R$ is called a generalized (α,β)-derivation of R associated with an (α,β)-derivation d if $F(xy) = F(x)\alpha(y)+\beta(x)d(y)$ is fulfilled for all $x, y \in R$, we denote this generalized (α,β)-derivation as (F, d). If $\alpha = \beta$ is an identity map of R, then we call a $(1,1)$-derivation d, we call a generalized $(1,1)$-derivation F a generalized derivation associated with a derivation d.

Let R be a $*$-ring. An additive mapping $d : R \to R$ is said to be a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$. Note that the mapping $x \mapsto ax^* - xa$, where a is a fixed element in R, is a Jordan $*$-derivation. Such a Jordan $*$- derivation is said to be inner. The study of Jordan $*$-derivations has been motivated by the

Received March 27, 2017; Revised April 25, 2017; Accepted May 24, 2017.
2010 *Mathematics Subject Classification.* 16W25, 16N60, 16U80.
Key words and phrases. semiprime rings, Lie ideals, $(\alpha,\beta)^*$-derivations, generalized $(\alpha,\beta)^*$-derivations, Jordan $(\alpha,\beta)^*$-derivations, generalized Jordan $(\alpha,\beta)^*$-derivations.

problem of the representatively of quadratic forms by bilinear forms. It turns out that the question, whether each quadratic form can be represented by some bilinear form, is connected with the question, whether every Jordan $*$-derivation is inner, as shown by Šemrl [11]. In [5], Brešar and Vukman studied some algebraic properties of Jordan $*$-derivations. An additive mapping $F : R \to R$ is called a generalized $*$-derivation if there exists a $*$-derivation $d : R \to R$ such that $F(xy) = F(x)y^* + xd(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized Jordan $*$-derivation if there exists a Jordan $*$-derivation $d : R \to R$ such that $F(x^2) = F(x)x^* + xd(x)$ holds for all $x \in R$. An additive mapping d of a $*$-ring R into itself is called a Jordan triple $*$-derivation if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ is fulfilled for all $x, y \in R$. One can easily prove that every Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation. However, the converse of this statement is not true in general. In [13], Vukman showed that the converse holds if R is 6-torsion free semiprime $*$-ring. Recently, the M. Fošner and Iliševič [7] proved that every Jordan triple $*$-derivation on a 2-torsion free semiprime $*$-ring is a Jordan $*$-derivation.

Let R be a $*$-ring, and let $\alpha, \beta : R \to R$ ring homomorphisms. An additive mapping $d : R \to R$ is called an $(\alpha, \beta)^*$-derivation (resp. Jordan $(\alpha, \beta)^*$- derivation) if $d(xy) = d(x)\alpha(y^*) + \beta(x)d(y)$ (resp. $d(x^2) = d(x)\alpha(x^*) + \beta(x)d(x)$) holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a Jordan triple $(\alpha, \beta)^*$-derivation if $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ for all $x, y \in R$. Obviously, every $(\alpha, \beta)^*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $(\alpha, \beta)^*$-derivation (see the proof of [4, Lemma 2.3]) but the converse is in general not true [3, Example 2.1]. An additive mapping $d : R \to R$ is called a reverse $(\alpha, \beta)^*$-derivation if $d(xy) = d(y)\alpha(x^*) + \beta(y)d(x)$ for all $x, y \in R$ and $F : R \to R$ is called a generalized reverse $(\alpha, \beta)^*$-derivation associated with some reverse $(\alpha, \beta)^*$-derivation d if $F(xy) = F(y)\alpha(x^*) + \beta(y)d(x)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is called a generalized Jordan triple $(\alpha, \beta)^*$-derivation if there exists a Jordan triple $(\alpha, \beta)^*$-derivation d such that $F(xyx) = F(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ holds for all $x, y \in R$. Note that a generalized Jordan triple $(I, I)^*$-derivation is just a generalized Jordan triple $*$-derivation. Clearly, this notion includes the definitions of Jordan triple $(\alpha, \beta)^*$-derivations when $F = d$, of Jordan triple $*$- derivations when $F = d$ and $\alpha = \beta = I$, and of generalized Jordan triple $*$-derivations when $\alpha = \beta = I$, where I is the identity map on R. It is easy to see that for a 2-torsion free $*$-ring any generalized Jordan $*$-derivation is a generalized Jordan triple $*$-derivation. But the converse is in general not true. Thus, the concept of generalized Jordan triple $\alpha = \beta = I$-derivation covers both the concepts of a Jordan triple $\alpha = \beta = I$-derivation and of a Jordan triple left α^*-centralizer, i.e., an additive mapping $T : R \to R$ satisfying $T(xyx) = T(x)\alpha(y^*x^*)$ for all $x, y \in R$.

In [12], J. Vukman introduced an additive mappings $F : R \to R$ such that $F(xyx) = F(xy)x + xyF(x)$ for all $x, y \in R$, and $G : R \to R$ such that

$G(xyx) = G(x)yx + xG(yx)$ for all $x, y \in R$, and he proved the following results:

Theorem 1.1 ([12, Theorem 2]). *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that either $D(xyx) = D(xy)x + xyD(x)$ or $D(xyx) = D(x)yx + xD(yx)$ holds for all pairs $x, y \in R$. In both cases D is a derivation.*

In [6], Daif and Tammam El-Sayiad obtained the following result:

Theorem 1.2 ([6, Theorem 2.1]). *Let R be a 2-torsion free semiprime ring and let $G : R \to R$ be an additive mapping. If $G(xyx) = G(x)yx + xD(yx)$ for all $x, y \in R$ for some derivation D of R. Then G is a generalized Jordan derivation.*

Recently, Ali et al. [3], obtained some results related to generalized Jordan $(\alpha,\beta)^*$-derivations. In particular they proved the following:

Theorem 1.3 ([4, Theorem 3.1]). *Let R be a 2-torsion free semiprime $*$-ring, α, β surjective endomorphisms of R. Then every generalized Jordan triple $(\alpha,\beta)^*$-derivation is a generalized Jordan $(\alpha,\beta)^*$-derivation.*

On the other hand, Huang and Koc [9], proved the following:

Theorem 1.4 ([9, Theorem 2.1]). *Let R be a 6-torsion free semiprime $*$-ring, α an endomorphism of R, β an epimorphism of R and $F : R \to R$ an additive mapping. Then F is a generalized Jordan $(\alpha,\beta)^*$-derivation if and only if F is a generalized Jordan triple $(\alpha,\beta)^*$-derivation.*

In this note, first, we intend to generalize above theorem of Vukman [12], and a theorem of Daif and El-Sayiad [6] to the results in case of rings with involution. Secondly, we will generalize a theorem of Ali, et al. [4] and a theorem of Huang and Koc [9].

2. Results

We begin with the following results which will be extensively to prove our theorems.

Lemma 2.1. *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L \nsubseteq Z(R)$, $a \in L$ and let $A, B : L \times L \to L$ be biadditive mappings.*

(i) *If $aLa = 0$, then $a = 0$.*
(ii) *If $A(x,y)zB(x,y) = 0$ for all $x, y, z \in L$, then $A(x,y)zB(u,v) = 0$ for all $x, y, z, u, v \in L$.*
(iii) *Let $a, b \in L$ be fixed elements. If $az[x,y] = 0$ for all $x, y, z \in L$, then $a, ab, ba \in Z(L)$ (the center of L).*

Proof. (i) The proof is the same as [8, Corollary 2.1(1)].
(ii) is a special case of [10, Lemma 2.7].

(iii) If $az[x,y]=0$ for all $y\in L$, then we have $xaz[x,a]=0$ and $axz[x,a]=0$, and so we have $[x,a]z[x,a]=0$ for all $z\in L$. This implies $[x,a]=0$ for all $x\in L$ by (i), so $a\in Z(L)$. By a similar method, we get $ab, ba\in Z(L)$. □

Lemma 2.2. *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L\nsubseteq Z(R)$, α,β endomorphisms of R such that $\alpha(L)=L$ and $\beta(L)=L$. If there exists an element $x\in L$ such that $\alpha(y)x\beta(y^*)=0$ for all $y\in L$, then $x=0$.*

Proof. This lemma is proved by a using the similar method to the proof of [1, Lemma 1] and Lemma 2.1(i). □

Proposition 2.1. *Let R be a 2-torsion free semiprime ring, L a square-closed Lie ideal such that $L\nsubseteq Z(L)$, and $\alpha: R\to R$ an endomorphism such that $\alpha(L)=L$. And let $T: L\to L$ be an additive mapping such that $T(x^2)=\alpha(x)T(x)$ for all $x\in L$, then $T(xy)=\alpha(x)T(y)$ holds for all $x,y\in L$.*

Proof. Using the similar technique as used in [14, Proposition 1.4] and Lemma 2.1, the result follows. □

Now, we note the result with respect to reverse centralizers as follows.

Proposition 2.2. *Let R be a 2-torsion free semiprime $*$-ring, L a square-closed Lie ideal such that $L\nsubseteq Z(L)$ and $\alpha: R\to R$ an endomorphism such that $\alpha(L)=L$ and $(\alpha(x))^*=\alpha(x^*)$. And let $T: R\to R$ be an additive mapping such that $T(L)\subseteq L$ which satisfies $T(x^2)=T(x)\alpha(x^*)$ for all $x\in L$. Then $T(xy)=T(y)\alpha(x^*)$ for all $x,y\in L$.*

Proof. Applying similar techniques as used in the proof of Proposition [2, Proposition 2.3] and Proposition 2.1, we get the required result. □

Proposition 2.3. *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L\nsubseteq Z(R)$, α,β endomorphisms such that $\alpha(L)=L$ and $\beta(L)=L$. And let $F: R\to R$ be an additive mapping such that $F(L)\subseteq L$. Then the following conditions are equivalent*:

(i) *F is a Jordan $(\alpha,\beta)^*$-derivation on L.*
(ii) *F is a Jordan triple $(\alpha,\beta)^*$-derivation on L.*

Proof. This proposition is proved by a similar method to [1, Theorem 1] and using Lemma 2.2 yields the required result. □

Now, we state the following theorem which is motivated by Vukman, and by Daif and Sayiad (Theorem 1.2).

Theorem 2.1. *Let R be a 2-torsion free semiprime $*$-ring and $L(\nsubseteq Z(R))$ be a square-closed Lie ideal of R such that $L^*\subseteq L$. Let $F,D: R\to R$ be additive mappings such that $F(L)\subseteq L$ and $D(L)\subseteq L$, and let α,β be ring homomorphisms of R such that $\alpha(L)\subseteq L$ and $\beta(L)\subseteq L$.*

(i) *If $F(xyx) = F(xy)\alpha(x^*)+\beta(xy)D(x)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a Jordan $(\alpha,\beta)^*$-derivation on L.*
(ii) *If $F(xyx) = F(x)\alpha(y^*x^*) + \beta(x)D(yx)$ and $D(xyx) = D(x)\alpha(y^*x^*) + \beta(x)D(yx)$ hold for all $x, y \in L$ and $\alpha(L) = L$, then D is a Jordan $(\alpha,\beta)^*$-derivation on L and F is a generalized Jordan $(\alpha,\beta)^*$-derivation associated with D.*
(iii) *If $F(xyx) = \alpha(x)F(yx) + D(x)\beta(y^*x^*)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a right Jordan $(\alpha,\beta)^*$-derivation on L.*
(iv) *If $F(xyx) = \alpha(xy)F(x) + D(xy)\beta(x^*)$ and $D(xyx) = \alpha(xy)D(x) + D(xy)\beta(x^*)$ hold for all $x, y \in L$ and $\alpha(L) = L$, then D is a right Jordan $(\alpha,\beta)^*$-derivation L and F is a generalized right Jordan $(\alpha,\beta)^*$-derivation associated with D.*
(v) *If $F(xyx) = \alpha(x^*)F(yx)+D(x)\beta(yx)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a left Jordan $(\alpha,\beta)^*$-derivation on L.*
(vi) *If $F(xyx) = \alpha(x^*y^*)F(x) + D(xy)\beta(x)$ and $D(xyx) = \alpha(x^*y^*)D(x) + D(xy)\beta(x)$ hold for all $x, y \in L$, then D is a left Jordan $(\alpha,\beta)^*$-derivation on L and F is a generalized left Jordan $(\alpha,\beta)^*$-derivation on L associated with D.*

Proof. (i) Assume that

$$F(xyx) = F(xy)\alpha(x^*) + \beta(xy)D(x) \text{ for all } x, y \in L. \tag{2.1}$$

By linearization, we have

$$F(xyz + zyx) = F(xy)\alpha(z^*) + F(zy)\alpha(x^*) + \beta(xy)D(z) + \beta(zy)D(x) \tag{2.2}$$

for all $x, y, z \in L$. Putting $z = x^2$, we get

$$\begin{aligned} &F(xyx^2 + x^2yx) \\ &= F(xy)\alpha((x^*)^2) + F(x^2y)\alpha(x^*) + \beta(xy)D(x^2) + \beta(x^2y)D(x). \end{aligned} \tag{2.3}$$

On the other hand, in (2.1), substituting $xy + yx$ for y, we obtain

$$\begin{aligned} F(x^2yx + xyx^2) &= F(x^2y + xyx)\alpha(x^*) + \beta(x^2y + xyx)D(x) \\ &= F(x^2y)\alpha(x^*) + F(xy)\alpha((x^*)^2) + \beta(xy)D(x)\alpha(x^*) \\ &\quad + \beta(x^2y + xyx)D(x) \text{ for all } x, y \in L. \end{aligned} \tag{2.4}$$

Comparing (2.3) with (2.4), we have

$$\beta(x)\beta(y)\{D(x^2) - D(x)\alpha(x^*) - \beta(x)D(x)\} = 0 \text{ for all } x, y \in L.$$

Since $\beta(L) = L$,

$$\beta(x)z\{D(x^2) - D(x)\alpha(x^*) - \beta(x)D(x)\} = 0 \text{ for all } x, z \in L.$$

Now, we set

$$A(x) = D(x^2) - D(x)\alpha(x^*) - \beta(x)D(x).$$

Then we have

$$\beta(x)A(x)z\beta(x)A(x) = 0 \text{ for all } x, z \in L$$

and

$$A(x)\beta(x)zA(x)\beta(x) = 0 \text{ for all } x, z \in L.$$

Since R is semiprime, we have

$$A(x)\beta(x) = 0 \text{ for all } x \in L \tag{2.5}$$

and

$$\beta(x)A(x) = 0 \text{ for all } x \in L \tag{2.6}$$

by Lemma 2.1. In (2.5), substituting $x + z$ for x, we have

$$A(x)\beta(z) + A(z)\beta(x) + B(x, z)\beta(x) + B(x, z)\beta(z) = 0, \tag{2.7}$$

where

$$B(x, z) = D(xz + zx) - D(x)\alpha(z^*) - D(z)\alpha(x^*) - \beta(x)D(z) - \beta(z)D(x).$$

In (2.7), substituting $-x$ for x, we get

$$A(x)\beta(z) - A(z)\beta(x) + B(x, z)\beta(x) - B(x, z)\beta(z) = 0. \tag{2.8}$$

Comparing (2.7) and (2.8), we find that $2\{A(x)\beta(z) + B(x, z)\beta(x)\} = 0$. Since R is 2-torsion free, we have $A(x)\beta(z) + B(x, z)\beta(x) = 0$. Thus, by (2.6) we obtain

$$0 = A(x)\beta(z)A(x) + B(x, z)\beta(x)A(x) = A(x)\beta(z)A(x).$$

Since $\beta(L) = L$, we get $A(x)yA(x) = 0$ for all $x, y \in L$. By semiprimeness of R, we obtain that $A(x) = 0$ for all $x \in L$ by Lemma 2.1. And so, D is a Jordan $(\alpha, \beta)^*$-derivation on L.

(ii) Now, assume that

$$F(xyx) = F(x)\alpha(y^*x^*) + \beta(x)D(yx) \text{ for all } x, y \in L \tag{2.9}$$

and

$$D(xyx) = D(x)\alpha(y^*x^*) + \beta(x)D(yx) \text{ for all } x, y \in L.$$

In (2.9), by linearization, we have

$$F(xyz + zyx) = F(x)\alpha(y^*z^*) + F(z)\alpha(y^*x^*) + \beta(x)D(yz) + \beta(z)D(yx).$$

Now, substituting x^2 for z, we have

$$\begin{aligned} F(xyx^2 + x^2yx) = {} & F(x)\alpha(y^*(x^*)^2) + F(x^2)\alpha(y^*x^*) \\ & + \beta(x)D(yx^2) + \beta(x^2)D(yx). \end{aligned} \tag{2.10}$$

In (2.9), substituting $xy + yx$ for y, we have

$$\begin{aligned} F(x^2yx + xyx^2) &= F(x)\alpha(x^*y^*x^* + y^*(x^*)^2) + \beta(x)D(xyx + yx^2) \\ &= F(x)\alpha(x^*y^*x^*) + F(x)\alpha(y^*(x^*)^2) \\ &\quad + \beta(x)\{D(x)\alpha(y^*x^*) + \beta(x)D(yx) + D(yx^2)\}. \end{aligned} \tag{2.11}$$

By comparing (2.10) with (2.11), we get

$$\{F(x^2) - F(x)\alpha(x^*) - \beta(x)D(x)\}\alpha(y^*)\alpha(x^*) = 0 \text{ for all } x \in L.$$

Now, we set

$$E(x) = F(x^2) - F(x)\alpha(x^*) - \beta(x)D(x).$$

Since $\alpha(L) = L$, putting z for $\alpha(y^*)$, we have

$$E(x)z\alpha(x^*) = 0 \text{ for all } x, z \in L.$$

As a similar way to the proof of (i), we obtain $E(x) = 0$, that is

$$F(x^2) = F(x)\alpha(x^*) + \beta(x)D(x) \text{ for all } x \in L.$$

In the case of $D(xyx) = D(x)\alpha(y^*x^*) + \beta(x)D(yx)$, D is a Jordan $(\alpha,\beta)^*$-derivation on L by the similar arguments to the above arguments, and so F is a generalized Jordan $(\alpha,\beta)^*$-derivation on L associated with a Jordan $(\alpha,\beta)^*$-derivation D on L.

(iii) and (v) The proofs are similar to that of (i).

(iv) and (vi) The proofs are similar to that of (ii). □

Corollary 2.1. *Let R be a 2-torsion free semiprime $*$-ring, D an additive mapping, and let α, β be epimorphisms of R.*

(i) *If one of the following conditions is fulfilled, then D is a Jordan $(\alpha,\beta)^*$-derivation of R.*
   (1) $D(xyx) = D(xy)\alpha(x^*) + \beta(xy)D(x)$ *for all* $x, y \in R$.
   (2) $D(xyx) = D(x)\alpha(y^*x^*) + \beta(x)D(yx)$ *for all* $x, y \in R$.
   (3) $D(xyx) = D(xy)\alpha(x^*) + \beta(xy)D(x)$ *or* $D(xyx) = D(x)\alpha(y^*x^*) + \beta(x)D(yx)$ *for all* $x, y \in R$.

(ii) *If one of the following conditions is fulfilled, then D is a left Jordan $(\alpha,\beta)^*$-derivation of R.*
   (4) $D(xyx) = \alpha(x^*)D(yx) + D(x)\beta(yx)$ *for all* $x, y \in R$.
   (5) $D(xyx) = \alpha(x^*y^*)D(x) + D(xy)\beta(x)$ *for all* $x, y \in R$.
   (6) $D(xyx) = \alpha(x^*)D(yx) + D(x)\beta(yx)$ *or* $D(xyx) = \alpha(x^*y^*)D(x) + D(xy)\beta(x)$ *for all* $x, y \in R$.

Proof. (i) By Theorem 2.1, if condition (1) or (2) is fulfilled, then D is a Jordan $(\alpha,\beta)^*$-derivation.

Assume that condition (3) is fulfilled. We put

$$R_x = \{y \in R \mid D(xyx) = D(xy)\alpha(x^*) + \beta(xy)D(x) \text{ for all } x, y \in R\},$$

and

$$R_x^0 = \{y \in R \mid D(xyx) = D(x)\alpha(y^*x^*) + \beta(x)D(yx) \text{ for all } x, y \in R\}.$$

Then $R = R_x \cup R_x^0$, R_x and R_x^0 are additive groups. And so, we have $R = R_x$ or $R = R_x^0$ by Brauer's Trick. By the same method, we can see $R = \{x \in R \mid R = R_x\}$ or $R = \{x \in R \mid R = R_x^0\}$. Therefore, by (1) and (2), we have D is a Jordan $(\alpha,\beta)^*$-derivation.

(ii) is proved by the similar method to (i). □

We state another main theorem which is a generalization of a theorem of Ali et al. [4] and a theorem of Huang and Koc [9].

Theorem 2.2. *Let R be a 2-torsion free semiprime $*$-ring and $L(\nsubseteq Z(R))$ be a square-closed Lie ideal of R such that $L^* \subseteq L$. Let $F : R \to R$ be an additive mapping such that $F(L) \subseteq L$, and α, β ring homomorphisms of R such that $\alpha(L) = L$ and $\beta(L) \subseteq L$. Then the followings are equivalent*:

(i) *F is a generalized Jordan triple $(\alpha,\beta)^*$-derivation on L associated with a Jordan $(\alpha,\beta)^*$-derivation D on L.*

(ii) *F is a generalized Jordan $(\alpha,\beta)^*$-derivation on L associated with a Jordan $(\alpha,\beta)^*$-derivation D on L.*

Moreover, if $\beta(L) = L$ is fulfilled, the above conditions (i) *and* (ii) *are equivalent the next condition* (iii):

(iii) *F is a generalized Jordan triple $(\alpha,\beta)^*$-derivation on L associated with a Jordan triple $(\alpha,\beta)^*$-derivation D on L.*

Proof. (i)$\Rightarrow$(ii). Suppose that

$$F(xyx) = F(x)\alpha(y^*x^*) + \beta(x)D(y)\alpha(x^*) + \beta(xy)D(x) \text{ for all } x, y \in L. \tag{2.12}$$

Replacing x by $x + z$ $(z \in L)$ in (2.12), we have

$$\begin{aligned} &F(xyz + zyx) \\ &= F(x)\alpha(y^*z^*) + F(z)\alpha(y^*x^*) + \beta(x)D(y)\alpha(z^*) \\ &\quad + \beta(z)D(y)\alpha(x^*) + \beta(xy)D(z) + \beta(zy)D(x) \text{ for all } x, y, z \in L. \end{aligned} \tag{2.13}$$

Replacing z by x^2 in (2.13) and using (2.12), we have

$$\begin{aligned} &F(xyx^2 + x^2yx) \\ &= F(x)\alpha(y^*(x^*)^2) + F(x^2)\alpha(y^*x^*) + \beta(x)D(y)\alpha((x^*)^2) \\ &\quad + \beta(x^2)D(y)\alpha(x^*) + \beta(xy)D(x^2) + \beta(x^2y)D(x) \text{ for all } x, y \in L. \end{aligned} \tag{2.14}$$

On the other hand, substituting $xy + yx$ for y in (2.12), we have

$$\begin{aligned} &F(x^2yx + xyx^2) \\ &= F(x)\alpha(y^*(x^*)^2) + F(x)\alpha(x^*y^*x^*) \\ &\quad + \beta(x)D(xy + yx)\alpha(x^*) + \beta(xyx + x^2y)D(x) \text{ for all } x, y \in L. \end{aligned} \tag{2.15}$$

Since D is a Jordan $(\alpha,\beta)^*$-derivation, we have

$$D(xy + yx) = D(x)\alpha(y^*) + D(y)\alpha(x^*) + \beta(x)D(y) + \beta(y)D(x). \tag{2.16}$$

Using (2.16), (2.15) is rewritten as follows

$$\begin{aligned} &F(x^2yx + xyx^2) \\ &= F(x)\alpha(y^*(x^*)^2) + F(x)\alpha(x^*y^*x^*) + \beta(x)D(x)\alpha(y^*)\alpha(x^*) \\ &\quad + \beta(x)D(y)\alpha((x^*)^2) + \beta(x^2)D(y)\alpha(x^*) + \beta(xy)D(x)\alpha(x^*) \\ &\quad + \beta(xyx)D(x) + \beta(x^2y)D(x) \text{ for all } x, y \in L. \end{aligned} \tag{2.17}$$

Comparing (2.14) and (2.17), we get

$$F(x^2)\alpha(y^*x^*) = F(x)\alpha(x^*y^*x^*) + \beta(x)D(x)\alpha(y^*)\alpha(x^*)$$

for all $x, y \in L$. And so, we have $\{F(x^2) - F(x)\alpha(x^*) - \beta(x)D(x)\}\alpha(y^*x^*) = 0$ for all $x, y \in L$. Now, setting $A(x) = F(x^2) - F(x)\alpha(x^*) - \beta(x)D(x)$, we can write the above equation as follows:

$$A(x)\alpha(y^*)\alpha(x^*) = 0. \tag{2.18}$$

Since α is an endomorphism of R, substituting y for $\alpha(y^*)$ in (2.18), we have $A(x)y\alpha(x^*) = 0$. Since R is semiprime, using $\alpha(x^*)A(x)y\alpha(x^*)A(x) = 0$ for all $y \in L$, we get $\alpha(x^*)A(x) = 0$. On the other hand, replacing y by $\alpha(x^*)yA(x)$, we have $A(x)\alpha(x^*)yA(x)\alpha(x^*) = 0$ for all $y \in L$. Since R is semiprime, we get

$$A(x)\alpha(x^*) = 0. \tag{2.19}$$

Now, we set $B(x, y) = F(xy + yx) - F(x)\alpha(y^*) - \beta(x)D(y) - \beta(y)D(x)$ for all $x, y \in L$. Since $A(x + y) = A(x) + A(y) + B(x, y)$ for all $x, y \in L$, substituting $x + y$ for x in (2.19), we obtain

$$A(x)\alpha(y^*) + B(x, y)\alpha(x^*) + A(y)\alpha(x^*) + B(x, y)\alpha(y^*) = 0. \tag{2.20}$$

Substituting $-x$ for x in (2.20), we have

$$A(x)\alpha(y^*) + B(x, y)\alpha(x^*) - A(y)\alpha(x^*) - B(x, y)\alpha(y^*) = 0. \tag{2.21}$$

By (2.20) and (2.21), we get $2A(x)\alpha(y^*) + 2B(x, y)\alpha(x^*) = 0$. Since R is 2-torsion free, we obtain that

$$A(x)\alpha(y^*) + B(x, y)\alpha(x^*) = 0. \tag{2.22}$$

Right multiplication of (2.22) by $A(x)$ gives $A(x)\alpha(y^*)A(x) = 0$. Substituting y for $\alpha(y^*)$ we get $A(x)yA(x) = 0$ for all $y \in L$. Since R is semiprime, we obtain that $A(x) = 0$ for all $x \in R$, that is, $F(x^2) = F(x)\alpha(x^*) + \beta(x)D(x)$ for all $x \in L$. Therefore, F is a generalized Jordan $(\alpha,\beta)^*$-derivation associated with a Jordan $(\alpha,\beta)^*$-derivation D.

(ii)$\Rightarrow$(i). Suppose that

$$F(x^2) = F(x)\alpha(x^*) + \beta(x)D(x). \tag{2.23}$$

Replacing x by $x + y$ in (2.23), we have

$$\begin{aligned} F(xy + yx) = {} & F(x)\alpha(y^*) + F(y)\alpha(x^*) \\ & + \beta(x)D(y) + \beta(y)D(x) \text{ for all } x, y \in L. \end{aligned} \tag{2.24}$$

Replacing x by x^2 in (2.24), we have

$$\begin{aligned} & F(x^2y + yx^2) \\ & = F(x)\alpha(x^*y^*) + F(y)\alpha((x^*)^2) + \beta(x^2)D(y) + \beta(x)D(x)\alpha(y^*) \\ & \quad + \beta(y)D(x)\alpha(x^*) + \beta(yx)D(x) \text{ for all } x, y \in L. \end{aligned} \tag{2.25}$$

On the other hand, substituting $xy + yx$ for y in (2.24), we get

$$\begin{aligned} F(x^2y + yx^2 + 2xyx) = {} & F(x)\alpha(x^*y^* + y^*x^*) + F(xy + yx)\alpha(x^*) \\ & + \beta(x)D(xy + yx) + \beta(xy + yx)D(x). \end{aligned}$$

That is,

$$\begin{aligned}(2.26)\qquad & F(x^2y + yx^2 + 2xyx) \\ &= F(x)\alpha(x^*y^*) + 2F(x)\alpha(y^*x^*) + F(y)\alpha((x^*)^2) \\ &\quad + 2\beta(x)D(y)\alpha(x^*) + \beta(y)D(x)\alpha(x^*) + \beta(x)D(x)\alpha(y^*) \\ &\quad + \beta(x^2)D(y) + 2\beta(xy)D(x) + \beta(yx)D(x) \text{ for all } x, y \in L.\end{aligned}$$

Combining (2.25) and (2.26) and using the fact that R is 2-torsion free, we get

$$F(xyx) = F(x)\alpha(y^*x^*) + \beta(x)D(y)\alpha(x^*) + \beta(xy)D(x) \text{ for all } x, y \in L.$$

And so (ii)$\Rightarrow$(i) is proved.

(i) $\Longleftrightarrow$ (iii). By Proposition 2.3, we can easily see this. $\square$

Corollary 2.2 ([4, Theorem 3.1], [3, Theorem 2.1])**.** *Let R be a 2-torsion free semiprime $*$-ring and α, β ring epimorphisms of R. Let $F : R \to R$ be an additive mapping. Then the followings are equivalent*:

(i) *F is a generalized Jordan triple $(\alpha, \beta)^*$-derivation associated with a Jordan triple $(\alpha, \beta)^*$-derivation D.*

(ii) *F is a generalized Jordan $(\alpha, \beta)^*$-derivation associated with a Jordan $(\alpha, \beta)^*$-derivation D.*

Next Corollary is a generalization of ([2, Proposition 2.3] and [4, Theorem 4.1]).

Corollary 2.3. *Let R be a 2-torsion free semiprime $*$-ring and α a ring epimorphism such that $\alpha(L) = L$ and $(\alpha(x))^* = \alpha(x^*)$ for all $x \in L$. Let $F : R \to R$ be an additive mapping such that $F(L) \subseteq L$. Then the followings are equivalent*;

(i) *$F(x^2) = F(x)\alpha(x^*)$ for all $x \in L$.*

(ii) *$F(xyx) = F(x)\alpha(y^*x^*)$ for any $x, y \in R$.*

(iii) *$F(xy) = F(y)\alpha(x^*)$ for any $x \in R$.*

Proof. (i)$\Rightarrow$(iii). This is Proposition 2.2.

(iii)$\Rightarrow$(ii). $F(xyx) = F(x)\alpha(xy)^* = F(x)\alpha(y^*x^*)$ for all $x, y \in L$.

(ii)$\Rightarrow$(i). In Theorem 2.2, we can see by putting $\beta = 0$. $\square$

Acknowledgement. The authors are greatly indebted to the referee for careful reading of the paper and valuable comments.

References

[1] S. Ali, *Note on Jordan triple $(\alpha, \beta)^*$-derivations in H^*-algebras*, East-West J. Math. **13** (2011), no. 2, 200–206.

[2] S. Ali, N. A. Dar, and J. Vukman, *Jordan left $*$-centralizers of prime and semiprime rings with involution*, Beitr. Algebra Geom. **54** (2013), no. 2, 609–624.

[3] S. Ali and A. Fošner, *On Jordan $(\alpha, \beta)^*$-derivations in semiprime $*$-rings*, Int. J. Algebra **4** (2010), no. 1-4, 99–108.

[4] S. Ali, A. Fošner, M. Fošner, and M. S. Khan, *On generalized triple $(\alpha, \beta)^*$-derivations and related mappings*, Mediterr. J. Math. **10** (2013), no. 4, 1657–1668.

[5] M. Brešar and J. Vukman, *On some additive mappings in rings with involution*, Aequationes Math. **38** (1989), no. 2-3, 178–185.

[6] M. N. Daif and M. S. Tammam El-Sayiad, *An identity related to generalized derivations*, Int. J. Algebra **1** (2007), no. 11, 547–550.

[7] M. Fošner and D. Ilševič, *On Jordan derivations and related mappings*, Mediterr. J. Math. **5** (2008), no. 4, 415–427.

[8] M. Hongan, N. ur Rehman, and A. Omary, *Lie ideals and Jordan Triple derivations in rings*, Rend. Semin. Mat. Univ. Padova **125** (2011), 147–156.

[9] S. Huang and E. Koc, *Notes on generalized Jordan $(\sigma, \tau)^*$-derivations of semiprime rings with involution*, Bol. Soc. Parana. Mat. **32** (2014), no. 1, 257–264.

[10] N. Rehman and M. Hongan *Generalized Jordan derivations on Lie ideals associate with Hochschild 2-cocycles of rings*, Rend. Circ. Mat. Palermo **60** (2011), 437–444.

[11] P. Šemrl, *On Jordan $*$-derivations and an application*, Colloq. Math. **59** (1990), no. 2, 241–251.

[12] J. Vukman, *Some remarks on derivations in semiprime rings and standard operator algebras*, Glas. Mat. Ser. III **46** (2011), no. 1, 43–48.

[13] ———, *A note on Jordan $*$-derivations in semiprime rings with involution*, Int. Math. Forum **13** (2006), no. 13-16, 617–622.

[14] B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolin. **32** (1991), no. 4, 609–614.

Motoshi Hongan
Seki 772, Maniwa, Okayama 719-3156, Japan
Email address: hongan0061@gmail.com

Nadeem ur Rehman
Department of Mathematics
Aligarh Muslim University
Aligarh-202002, India
Email address: rehman100@gmail.com