Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 45–51
https://doi.org/10.4134/CKMS.c170055
pISSN: 1225-1763 / eISSN: 2234-3024

SOME REMARKS ON CAMILLO-KRAUSE CONJECTURE

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Abstract. This paper contains some results that grew out of an attempt to Camillo-Krause conjecture: Is a ring R right Noetherian if for each nonzero right ideal I of R, R/I is an Artinian right R-module?

1. Introduction

By a celebrated theorem which was proved independently by C. Hopkins and J. Levitzki in 1939, every (right) Artinian ring is (right) Noetherian (see for example [7, Theorem 18.13]). Motivated by this theorem, V. Camillo and G. Krause (in 1972) asked: Is a ring R right Noetherian if every proper cyclic R-module is Artinian? (see [4]). This question has been investigated by many authors and answered affirmatively in some special cases (see for example [1], [6] and [11]). A ring R is called a right *Camillo-Krause ring* if for each nonzero right ideal I of R, R/I is a right Artinian module. It can be easily seen that every commutative Camillo-Krause ring is Noetherian. A. N. Alahmadi [1] showed that this is also true for every right Camillo-Krause ring which satisfies in a polynomial identity. In fact if in a right Camillo-Krause ring R every right ideal contains a two-sided ideal, then R is right Noetherian. In particular this is the case for every right duo right Camillo-Krause ring.

A ring R is called a *right q.f.d ring* if each cyclic right R-module has finite Goldie dimension. A ring R is q.f.d if and only if all cyclic right R-modules have finitely generated (possibly zero) socle (See [2]). It can be easily seen that a right Camillo-Krause ring is a right q.f.d ring.

A *right V-ring* is a ring whose simple right modules are injective and a *right GV-ring* is a ring whose simple singular right modules are injective. It is not hard to show that every right Camillo-Krause right V-ring is right Noetherian.

In this paper, we state some facts about right Camillo-Krause rings and investigate the relationship between Camillo-Krause rings and injective and projective modules. It is known that if a right Camillo-Krause ring is not right Artinian, then it must be a right Ore domain (See [8, Lemma 3.1]). Therefore discussion about the Noetherianity of a right Camillo-Krause ring is meaningful

Received February 19, 2017; Revised June 16, 2017; Accepted July 26, 2017.
2010 *Mathematics Subject Classification.* 16P20, 16P40, 16E60.
Key words and phrases. Camillo-Krause ring, Artinian module, Noetherian module.

only in the case of a right Ore domain. It is shown in Proposition 2.2 that the injective hull of every proper cyclic right module over a right Camillo-Krause ring is semiArtinian and locally Artinian.

In the second section, we will state some results about Camillo-Krause rings using Jacobson radical. A ring R is called *right uniserial* if any two right ideals of R are linearly ordered by inclusion. Somsup et al. [13] have showed that a serial ring which satisfies both left and right restricted minimum condition must be Noetherian. As an immediate result it can be seen that every uniserial Camillo-Krause ring is Noetherian. Recently M. Behboodi and S. Roointan-Isfahani [3] have introduced and studied the class of almost right uniserial rings as straightforward common generalization of right uniserial rings and right principal ideal domains. We will see in Proposition 2.5 that an almost right uniserial right Camillo-Krause ring with nonzero Jacobson radical is right Noetherian. We prove in Theorem 2.6, a right Camillo-Krause ring with nonzero Jacobson radical J is hereditary right Noetherian if and only if J is finitely generated as a right module and every maximal right ideal is projective.

In Proposition 2.9 we give a new characterization of rings for which Camillo-Krause conjecture holds. It is presently not known whether a simple a ring with right Krull dimension 1 is Noetherian. In Theorem 2.10, it is shown that if R is a simple right Camillo-Krause ring whose non-simple local right R-modules are projective or injective, then R is right Noetherian.

2. Right Camillo-Krause ring with nonzero Jacobson radical

In this section we state some results about right Camillo-Krause rings using Jacobson radical. In fact, we mostly consider the class of right Camillo-Krause rings with nonzero Jacobson radical. Clearly every right Camillo-Krause domain with finitely many maximal right ideals that is not a division ring is in this class.

Lemma 2.1. *Let R be a right Camillo-Krause ring with Jacobson radical J. If R/J is an injective right R-module, then R is right Noetherian.*

Proof. If R is not right Artinian, then it is a domain. In this case if $J = 0$, then R is a division ring. If J is nonzero, then R/J is semisimple injective. Since every simple module is isomorphic to a factor module of R/J, R is a right V-ring and therefore R is right Noetherian. □

A module M is called semiArtinian if for every proper submodule N of M, $\mathrm{Soc}(M/N) \neq 0$. A ring R is called right semiArtinian if R_R is semiArtinian. Clearly a right Camillo-Krause ring R is right semiArtinian if and only if it has nonzero socle if and only if it is right Artinian. A right module is called *locally Artinian* if every its finitely generated submodule is Artinian.

Proposition 2.2. *Let R be a right Camillo-Krause ring. Then injective hull of every proper cyclic right module is semiArtinian and locally Artinian.*

Proof. Let I be a right ideal of R and R/I be nonisomorphic to R. We set $E = E(R/I)$. To show that E is semiArtinian, we prove that E/U has a simple submodule for every proper submodule U of E. Let $x \in E$ and $\overline{x}$ be the image of x in E/U. If $\overline{x}R$ is proper, then the proof is complete. Otherwise, the exact sequence $0 \to U \to xR+U \to (xR+U)/U \to 0$ splits and so E has a submodule K isomorphic to R. But then $K \cap (R/I) \neq 0$, so $Soc(K)$ (hence $Soc(R)$) is non zero. Thus R is a right Artinian ring. This shows that E is semiArtinian.

To see E is locally Artinian, it is enough to show that cyclic submodules of E are Artinian. Assume that $xR \leq E$ is a cyclic submodule. If xR is not isomorphic to R, then it is Artinian. So let xR be isomorphic to R_R. Since R/I has essential socle, $xR \cap \mathrm{Soc}(R/I) \neq 0$ and xR has essential socle. Therefore R_R has nonzero socle and so it is Artinian. This proves that E is locally Artinian. □

Let M_R be a module. The socle series $Soc_\alpha(M)$ is defined by transfinite induction as follows:

$$\begin{aligned} Soc_0(M) &= 0, \\ Soc_\alpha(M)/Soc_{\alpha-1}(M) &= Soc(M/Soc_{\alpha-1}(M)), \\ Soc_\alpha(M) &= \bigcup_{\beta<\alpha} Soc_\beta(M) \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The least α such that $Soc_\alpha(M) = Soc_{\alpha+1}(M)$ is called the *Loewy length* of M. It is well known that M is semiArtinian if for some ordinal α, $Soc_\alpha(M) = M$.

Corollary 2.3. *Let R be a right Camillo-Krause ring and let the Jacobson radical J of R be nonzero. Then $E(R/J)$ has Loewy length. Moreover, the module $E(R/J)$ is of Loewy length $\leq \omega$ if and only if R is right Noetherian.*

Proof. The first part follows from Proposition 2.2. Clearly every right Camillo-Krause ring is of right Krull dimension at most 1 and so the second part is by [11, Theorem 4.3]. □

The following lemma shows that a right Camillo-Krause ring satisfies the Jacobson conjecture.

Lemma 2.4. *Let R be a right Camillo-Krause ring. Then $\bigcap_{n=1}^{\infty} J^n = 0$.*

Proof. If R is Artinian, then J is nilpotent and hence the proof is complete. Otherwise, R is a domain with right Krull dimension 1. Suppose that $T = \bigcap_{n=1}^{\infty} J^n \neq 0$. Since R/T is Artinian, there exists a positive integer n such that $J^n = J^{2n}$ and hence J contains the nonzero idempotent ideal J^n, contradicting [10, Lemma 6.3.6]. □

Following M. Behboodi and S. Roointan-Isfahani [3], a ring R is called *almost right uniserial* if any two non-isomorphic right ideals of R are linearly ordered by inclusion. In the following proposition we show that an almost right uniserial right Camillo-Krause ring is right Noetherian.

Proposition 2.5. *Let R be a right Camillo-Krause ring with Jacobson radical J. Then*:

1) *If every principal right ideal contains a two sided ideal, then R is right Noetherian.*

2) *If R is an almost right uniserial ring with $J = J(R) \neq 0$, then R is right Noetherian.*

3) *If $J(R) \neq 0$ and R contains a nonzero two sided ideal I such that I_R is almost uniserial, then R is right Noetherian.*

Proof. 1) Take a right ideal F of R. We show that F is finitely generated. Let $x \in F$ be a nonzero element and $I \leq xR$ be a nonzero two sided ideal of R. Then R/xR is an R/I-module and so it is an Artinian R/I-module. Since R/I is right Noetherian, R/xR is a Noetherian R-module. Thus F/xR is Noetherian and so F is finitely generated.

2) Since R/J is a Noetherian ring, it is enough to show that J is right Noetherian. Set $F \leq J$. We show that F is finitely generated. Let $x \in F$ be a nonzero element. By Lemma 2.4, there exists a positive integer n such that $xR \not\leq J^n$. Because J is almost right uniserial, for every positive integer n, either $J^n \leq xR$, or $J^n \cong xR$. If $J^n \leq xR$, then R/xR is an R/J^n-module and so it is an Artinian R/J^n-module. Since R/J^n is right Noetherian, R/xR is Noetherian. Thus F/xR is Noetherian and so F is finitely generated. Otherwise J^n is finitely generated as a right R-module and since R/J^n is an Artinian ring, hence J is finitely generated as a right R-module. Now, by [11, Theorem 4.3], we conclude that R is right Noetherian.

3) If R is not Artinian, then it is an uniform domain and so $K := I \cap J \neq 0$. Now using a proof similar to part 2, we can show that R is right Noetherian. □

A ring R is called *right hereditary* if each right ideal of R is projective. A ring R is called *right semihereditary* if each finitely generated right ideal of R is projective.

In [9, Theorem 2.35], necessary and sufficient condition for right Artinian to be right hereditary is provided. In the following proposition, we state similar necessary and sufficient conditions for a right Camillo-Krause with nonzero Jacobson radical to be right hereditary and Noetherian.

Proposition 2.6. *Let R be a right Camillo-Krause ring with nonzero Jacobson radical J. Then the following statements are equivalent*:

(i) *R is hereditary right Noetherian.*

(ii) *J and each maximal right ideal $\mathfrak{m}$ of R are projective as right R-modules.*

(iii) *J is finitely generated as a right R-module and each maximal right ideal $\mathfrak{m}$ of R is projective as a right R-module.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (iii). By [11, the Corollary after Theorem 4.3], R is hereditary right Noetherian and the assertion holds.

(iii) $\Rightarrow$ (i). Since J is finitely generated, by [11, Theorem 4.3], R is right Noetherian. Thus R/I has finite length for any right ideal I of R. Now similar to the proof of [9, Theorem 2.35], the assertion holds. $\square$

Let I be a left ideal of a semiprime Goldie ring R, and Q the quotient ring of R. Define $I^* = \{q \in Q \,|\, Iq \subseteq R\}$ and $I^{**} = \{q \in Q \,|\, qI^* \subseteq R\}$. For a finitely generated projective left ideal I of R, we have $I^* \cong \mathrm{Hom}_R(I, R)$ and so $I = I^{**}$. With these notations, we are ready to prove the following proposition. Note that a right semihereditary ring of finite right Goldie dimension is left semihereditary (see [12]).

Proposition 2.7. *Let R be a semihereditary ring which satisfies both the left and right Camillo-Krause condition. Then R must be a Noetherian hereditary ring.*

Proof. An Artinian semihereditary is hereditary and so assume that R is not Artinian. Thus it is an Ore domain. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq R$ be an ascending chain of finitely generated left ideals. Choose a nonzero element a in I_1. Then we have a chain of right R-modules $a^{-1}R \supseteq I_1^* \supseteq I_2^* \supseteq \cdots \supseteq R$, so that we have $R \supseteq aI_1^* \supseteq aI_2^* \supseteq \cdots \supseteq aR$. This chain of right ideals terminates, because R/aR is Artinian. Therefore, $aI_n^* = aI_{n+1}^*$ for some $n \in \mathbb{N}$. Thus $I_n^* = I_{n+1}^*$ and hence $I_n = I_{n+1}$ by taking stars again and using the fact that each I_i is projective. So R satisfies a.c.c on finitely generated left ideals and so it is left Noetherian. Similarly we see R is right Noetherian and so R is a hereditary Noetherian ring. $\square$

Let M be an R-module. A factor module N/K, where $K \leq N$ are submodules of M, is called *proper subfactor* of M if N is a proper submodule of M. In [6], Er gave a necessary and sufficient condition on a right Camillo-Krause ring to be Noetherian in terms of special hollow subfactors of R_R. In the following proposition we prove another result for right Camillo-Krause rings with nonzero Jacobson radical.

Proposition 2.8. *Let R be a right Camillo-Krause ring with $J \neq 0$. Then R is right Noetherian if and only if R_R has no a non-finitely generated subfactor with all proper subfactors non-faithful.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose that R is not right Noetherian. Therefore it must be a domain, so that (nonzero) Jacobson radical is non-nilpotent. Then R_R contains a non-finitely generated submodule M. Let a be a nonzero element of M. Then, by assumption, $N = M/aR$ is non-finitely generated. N is Artinian R-module and so let C be minimal among non-finitely generated submodules of N. Now we claim that each proper subfactors of C is non-faithful. Let D be a proper subfactor of C. Thus D is finitely generated Artinian module and the descending chain $D \supseteq DJ \supseteq DJ^2 \supseteq \cdots$ terminates. Therefore, there exists an integer

n such that $DJ^n = DJ^{n+1}$, and hence $DJ^n = (DJ^n)J$. Now, by Nakayama's Lemma, we conclude $DJ^n = 0$. This proves our claim. □

In the following for a module M, by $\mathrm{codim}(M) = n$ we mean M is of dual Goldie (hollow) dimension n.

Proposition 2.9. *Let R be a right Camillo-Krause ring. Then R is right Noetherian if and only if every proper cyclic right R-module which is local is Noetherian.*

Proof. Assume that every proper local cyclic right R-module is a Noetherian module. To show R is Noetherian it is enough to prove for every nonzero right ideal I of R the module R/I is Noetherian. We will show this by induction on codim (R/I), which is finite, because the module R/I is Artinian. The assertion holds if codim $(R/I) = 0$, so assume that $R/I \neq 0$. It follows from [11, Lemma 3.1] that R/I has a local cyclic submodule C such that $\mathrm{codim}((R/I)/C) < \mathrm{codim}(R/I)$. The induction hypothesis implies that the module $(R/I)/C$ is Noetherian. On the other hand, the module C is Noetherian by assumption. Thus we conclude that R is right Noetherian, as desired. The converse is obvious. □

Recall that a module M is called a CS *module* (or *an extending module*) if every submodule of M is essential in a direct summand of M.

We finish this section with the following theorem:

Theorem 2.10. *Let R be a simple right Camillo-Krause ring. If every non-simple local right R-module is projective or injective, then R is right Noetherian.*

Proof. We may assume that R is a domain and every non-simple local right R-module is projective or injective. First we claim, for every nonzero right ideal I of R, the module R/I is a direct sum of a semisimple module and an injective module. To show this we use induction on $\mathrm{codim}(R/I)$. This is a finite number since the module R/I is Artinian. If $\mathrm{codim}(R/I) = 1$, then the assertion is trivial. So assume that $\mathrm{codim}(R/I) = n$ with $n > 1$. Since R/I is Artinian, we can write R/I as a finite sum of n local submodules, say $R/I = H_1 + \cdots + H_n$. If all H_i are simple, then R/I is semisimple, and so the assertion holds. Otherwise, we may assume that H_n is not simple. Since $n > 1$, we have $\mathrm{codim}((R/I)/H_n) < \mathrm{codim}(R/I)$ and so the induction hypothesis implies that the module $(R/I)/H_n$ is a direct sum of a semisimple module and an injective module. On the other hand, H_n is cyclic singular module and so it can not be projective. Hence it is injective and so we have $R/I \cong H_n \oplus (R/I)/H_n$. This proves claim. Now [5, Corollary 13.4] implies that every cyclic singular right R-module is CS. Hence, by [8, Theorem 9.5], R is right Noetherian. □

Acknowledgments. The author would like to express his deep gratitude to his supervisor Saeid Bagheri for his elaborated guidance. The author thanks Zahra Nazemian for his valuable comments.

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