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COMMUTATIVITY OF JORDAN IDEALS IN 3-PRIME NEAR-RINGS WITH DERIVATIONS

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Abstract. We prove some theorems showing that a right Jordan ideal or a left Jordan ideal of a 3-prime near-ring must be commutative if it admits a nonzero derivation acting as a homomorphism or an antihomomorphism. Moreover, we give examples proving necessity of the conditions given.

1. Introduction

Throughout this paper $\mathcal{N}$ will be a zero-symmetric right near-ring with multiplicative center $Z(\mathcal{N})$; and usually $\mathcal{N}$ will be 3-prime, that is, for all $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $0x = 0$). According to the reference [7], an abelian near-ring $\mathcal{N}$ is a near-ring such that $(\mathcal{N}, +)$ is abelian. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [8], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. For any pair of elements $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product respectively. Recall that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. An additive subgroup $\mathcal{J}$ of $\mathcal{N}$ is said to be Jordan left (resp. right) ideal of $\mathcal{N}$ if $n \circ j \in \mathcal{J}$ (resp. $j \circ n \in \mathcal{J}$) for all $j \in \mathcal{J}$, $n \in \mathcal{N}$ and $\mathcal{J}$ is said to be a Jordan ideal of $\mathcal{N}$ if $j \circ n \in \mathcal{J}$ and $n \circ j \in \mathcal{J}$ for all $j \in \mathcal{J}$, $n \in \mathcal{N}$. A derivation d acts as a homomorphism (resp. as an anti-homomorphism) on a subset $\mathcal{S}$ of $\mathcal{N}$ if $d(xy) = d(x)d(y)$ (resp. $d(xy) = d(y)d(x)$) for all $x, y \in \mathcal{S}$. Recently many author have studied commutativity of prime and semiprime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the ring. In [3], Bell and Kappe proved that if d is a derivation of a semiprime ring $\mathcal{R}$ which is either an endomorphism or an anti-endomorphism on $\mathcal{R}$, then $d = 0$; whereas, the behavior of d is somewhat restricted in case of prime rings in the way that if d is a derivation of a prime ring $\mathcal{R}$ acting

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as a homomorphism or an anti-homomorphism on a nonzero right ideal of $\mathcal{R}$, then $d = 0$ on $\mathcal{R}$. Afterwards Yenigul and Argac [9] generalized these results with α-derivations and M. Ashraf, Rehman and Quadri [1] obtained the similar results with (σ, τ)-derivations. Recently A. Ali, N. Rehman and A. Shakir in [2] considered a (θ, ϕ)-derivation d acting as a homomorphism or an anti-homomorphism on a nonzero Lie ideal of a prime ring and concluded that $d = 0$. By the same motivation, we continue the line of investigation regarding the study of commutativity for Jordan right ideals or Jordan left ideals satisfying certain differential identities involving derivations acting as a homomorphism or as an anti-homomorphism. More precisely, we will extend the above-mentioned results for near-rings and obtain similar conclusion in case of the underlying subsets as a Jordan right ideal or a Jordan left ideal of a 3-prime near-ring $\mathcal{N}$.

2. Some preliminaries

We begin with the following known results which will be used extensively to prove our theorems.

Lemma 1. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{J}$ a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If $x \in \mathcal{N}$ and $\mathcal{J}x = \{0\}$, then $x = 0$.*

Proof. Suppose that $x \in \mathcal{N}$ and $jx = 0$ for all $j \in \mathcal{J}$. Replacing j by $j \circ n$ where $n \in \mathcal{N}$, we have $jnx + njx = 0$ for all $j \in \mathcal{J}$, $n \in \mathcal{N}$. Using the initial hypothesis, we get $jnx = 0$ for all $j \in \mathcal{J}, n \in \mathcal{N}$, this reduces to $j\mathcal{N}x = \{0\}$ for all $j \in \mathcal{J}$. Since $\mathcal{J} \neq \{0\}$, by 3-primeness of $\mathcal{N}$ we conclude that $x = 0$. □

Lemma 2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d such that $d(\mathcal{J}) \subseteq Z(\mathcal{N})$, then $d = 0$ or $\mathcal{J}$ is commutative.*

Proof. A proof can be given by using a similar approach as in the proof of [5, Theorem 3.1]. □

Lemma 3 ([4, Lemma 3])**.** *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) *If $\mathcal{N}$ is 2-torsion free and d a nonzero derivation on $\mathcal{N}$, then $d^2 \neq 0$.*
(ii) *If d is a derivation, then $x \in Z(\mathcal{N})$ implies $d(x) \in Z(\mathcal{N})$.*

Lemma 4 ([6, Lemma 2.1])**.** *A near-ring $\mathcal{N}$ admits a multiplicative derivation if and only if it is zero-symmetric.*

Using Lemma 4, we deduce that in all our results in the present paper that $\mathcal{N}$ is a zero-symmetric near-ring.

Lemma 5. *Let $\mathcal{N}$ be a near-ring and d a derivation of $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive law*

$$z\Big(xd(y) + d(x)y\Big) = zxd(y) + zd(x)y \quad \text{for all } x, y, z \in \mathcal{N}.$$

3. Derivations as homomorphisms and anti-homomorphisms

Motivated by the results in [1], [2], [3], and [9], our objective in the present paper is to establish similar results in the setting of Jordan right ideals or Jordan left ideals on 3-prime near-rings admitting a nonzero derivation d satisfies the conditions: $d(ij) = d(i)d(j)$, $d(nj) = d(j)d(n)$ and $d(jn) = d(n)d(j)$ for all $i, j \in \mathcal{J}$, $n \in \mathcal{N}$.

Theorem 1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring such that $(\mathcal{N}, +)$ is a torsion group and $\mathcal{J}$ a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If d acts a homomorphism on $\mathcal{J}$, then $\mathcal{J}$ is commutative.*

Proof. (i) Suppose that $\mathcal{J}$ is a nonzero right Jordan ideal of $\mathcal{N}$ and

$$d(ij) = d(i)d(j) \quad \text{for all } i, j \in \mathcal{J}. \tag{1}$$

By definition of d, (1) implies that

$$d(i)j = \big(d(i) - i\big)d(j) \quad \text{for all } i, j \in \mathcal{J} \tag{2}$$

and

$$id(j) = d(i)d(j) - d(i)j \quad \text{for all } i, j \in \mathcal{J}. \tag{3}$$

Replacing i by $2i^2$ in (1), we see that

$$\begin{aligned} d(2i^2j) &= d(2i^2)d(j) \\ &= 2d(i^2)d(j) \quad \text{for all } i, j \in \mathcal{J}. \end{aligned}$$

Since $d(2i^2j) = 2d(i^2j)$ for all $i, j \in \mathcal{J}$, according to the last expression after using the fact that $(\mathcal{N}, +)$ is a torsion group, we can conclude that

$$d(i^2j) = d(i^2)d(j) \quad \text{for all } i, j \in \mathcal{J}. \tag{4}$$

By application of equations (1) and (4), we have

$$\begin{aligned} d(i)ij + id(i)d(j) &= d(i)ij + id(ij) \\ &= d(i^2j) \\ &= d(i^2)d(j) \\ &= \big(d(i)i + id(i)\big)d(j) \\ &= d(i)id(j) + id(i)d(j) \quad \text{for all } i, j \in \mathcal{J}. \end{aligned}$$

This reduces to

$$d(i)ij = d(i)id(j) \quad \text{for all } i, j \in \mathcal{J}. \tag{5}$$

Putting j^2 instead of j in (5) and using it together with Lemma 5, we obtain

$$\begin{aligned} d(i)ij^2 &= d(i)id(j^2) \\ &= d(i)i\big(d(j)j + jd(j)\big) \\ &= d(i)id(j)j + d(i)ijd(j) \\ &= d(i)ij^2 + d(i)ijd(j) \quad \text{for all } i, j \in \mathcal{J}. \end{aligned}$$

Which implies that

$$d(i)ijd(j) = 0 \quad \text{for all } i, j \in \mathcal{J}. \tag{6}$$

Replacing j by $j \circ nj$ where $n \in \mathcal{N}$ in (5) and applying it with Lemma 5, we find that

$$\begin{aligned}
d(i)i(j \circ nj) &= d(i)id(j \circ nj) \\
&= d(i)id((j \circ n)j) \\
&= d(i)i\big(d(j \circ n)j + (j \circ n)d(j)\big) \\
&= d(i)id(j \circ n)j + d(i)i(j \circ n)d(j) \\
&= d(i)i(j \circ n)j + d(i)i(j \circ n)d(j) \\
&= d(i)i(j \circ nj) + d(i)i(j \circ n)d(j) \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}.
\end{aligned}$$

This expression gives

$$d(i)i(j \circ n)d(j) = 0 \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}.$$

Which can be rewritten as

$$d(i)i(jnd(j) + njd(j)) = 0 \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}. \tag{7}$$

Replacing n by $nd(i)i$ in (7) and invoking (6), we arrive at

$$d(i)ijnd(i)id(j) = 0 \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}.$$

Equivalently,

$$d(i)ij\mathcal{N}d(i)id(j) = \{0\} \quad \text{for all } i, j \in \mathcal{J}. \tag{8}$$

By 3-primeness of $\mathcal{N}$, (8) becomes

$$d(i)ij = d(i)id(j) = 0 \quad \text{for all } i, j \in \mathcal{J}. \tag{9}$$

Returning to (2) and replacing j by $j \circ nj$, we have

$$\begin{aligned}
d(i)(j \circ nj) &= (d(i) - i)d(j \circ nj) \\
&= (d(i) - i)d((j \circ n)j) \\
&= (d(i) - i)d(j \circ n)j + (d(i) - i)(j \circ n)d(j) \\
&= d(i)(j \circ n)j + (d(i) - i)(j \circ n)d(j) \\
&= d(i)(j \circ jn) + (d(i) - i)(j \circ n)d(j) \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}
\end{aligned}$$

which implies that

$$(d(i) - i)(j \circ n)d(j) = 0 \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}.$$

Equivalently,

$$(d(i) - i)(jnd(j) + njd(j)) = 0 \quad \text{for all } i, j \in \mathcal{J}, n \in \mathcal{N}. \tag{10}$$

Replacing n by $nd(j)$ in (10) and invoking (8), we arrive at

$$(d(i) - i)j\mathcal{N}d(j)d(j) = \{0\} \quad \text{for all } i, j \in \mathcal{J}. \tag{11}$$

Hence by 3-primeness of $\mathcal{N}$, we see that for each pair $i, j \in \mathcal{J}$ we get, either $(d(i) - i)j = 0$ or $d(j)d(j) = 0$. Using (1), we have

$$(d(i) - i)j = 0 \text{ or } d(j^2) = 0 \quad \text{for all } i, j \in \mathcal{J}. \tag{12}$$

If there exists $j_0 \in \mathcal{J}$ such that $(d(i) - i)j_0 = 0$, then $ij_0 = d(i)j_0$ for all $i \in \mathcal{J}$, replacing i by $i \circ ni$, then

$$\begin{aligned}(i \circ ni)j_0 &= d(i \circ ni)j_0 \\ &= d((i \circ n)i)j_0 \\ &= d(i \circ n)ij_0 + (i \circ n)d(i)j_0 \\ &= d(i \circ n)ij_0 + (i \circ n)ij_0 \\ &= d(i \circ n)ij_0 + (i \circ ni)j_0 \quad \text{for all } i \in \mathcal{J},\ n \in \mathcal{N}\end{aligned}$$

this implies that $d(i \circ n)ij_0 = 0$ for all $i \in \mathcal{J}$, $n \in \mathcal{N}$ and using (2), we get

$$(d(i \circ n) - (i \circ n))d(i)j_0 = 0 \quad \text{for all } i \in \mathcal{J},\ n \in \mathcal{N}.$$

By assumption, we have $(d(i \circ n) - (i \circ n))ij_0 = 0$ for all $i \in \mathcal{J}$, $n \in \mathcal{N}$ so that $(i \circ n)ij_0 = 0$ for all $i \in \mathcal{J}$, $n \in \mathcal{N}$ which gives

$$inij_0 = -ni^2j_0 \quad \text{for all } i \in \mathcal{J},\ n \in \mathcal{N}. \tag{13}$$

Replacing n by mn in (13) and using it again, we have

$$\begin{aligned}imnij_0 &= -mni^2j_0 \\ &= (-m)(-inij_0) \\ &= (-m)(-i)nij_0 \quad \text{for all } i \in \mathcal{J},\ m, n \in \mathcal{N}\end{aligned}$$

this implies that

$$(im - (-m)(-i))\mathcal{N}ij_0 = \{0\} \quad \text{for all } i \in \mathcal{J},\ m \in \mathcal{N}.$$

Putting $-i$ instead of i in the latter expression, we obtain

$$(-im + mi)\mathcal{N}ij_0 = \{0\} \quad \text{for all } i \in \mathcal{J},\ m \in \mathcal{N}. \tag{14}$$

Using 3-primeness of $\mathcal{N}$, (14) can be written in the form

$$i \in Z(\mathcal{N}) \text{ or } ij_0 = 0 \quad \text{for all } i \in \mathcal{J}. \tag{15}$$

If there exists $i_0 \in J \cap Z(\mathcal{N})$, then (13) implies that $2ni_0^2j_0 = 0$ for all $n \in \mathcal{N}$. By the 2-torsion freeness and the 3-primeness of $\mathcal{N}$, we arrive at $i_0^2j_0 = 0$. Recalling (15), we get $i^2j_0 = 0$ for all $i \in \mathcal{J}$ and from it and (13) it follows that $i\mathcal{N}ij_0 = \{0\}$ for all $i \in \mathcal{J}$. Since $\mathcal{N}$ is 3-prime and $\mathcal{J} \neq \{0\}$, then $ij_0 = 0$ for all $i \in \mathcal{J}$. Replacing i by $i \circ n$ where $n \in \mathcal{N}$, we get $i\mathcal{N}j_0 = \{0\}$ for all $i \in \mathcal{J}$. Since $\mathcal{N}$ is 3-prime and $\mathcal{J} \neq \{0\}$, then $j_0 = 0$. In which case equation (12) forces that $d(j^2) = 0$ for all $j \in \mathcal{J}$. Using the latter expression after replacing i by $2i^2$ in (3), we obtain $2i^2d(j) = 0$ for all $i, j \in \mathcal{J}$ which implies that $i^2d(j) = 0$ for all $i, j \in \mathcal{J}$. Putting $j \circ nj^2$ where $n \in \mathcal{N}$, in place of j, we have

$$0 = i^2d(j \circ nj^2)$$

$$= i^2(jnj + nj^2)d(j)$$
$$= i^2\big(jnjd(j) + nj^2d(j)\big)$$
$$= i^2jnjd(j) \quad \text{for all } \ i, j \in \mathcal{J}, \ n \in \mathcal{N}$$

which reduces to

$$i^2j\mathcal{N}jd(j) = \{0\} \quad \text{for all } \ i, j \in \mathcal{J}.$$

By 3-primeness of $\mathcal{N}$, we arrive at

$$i^2j = 0 \text{ or } jd(j) = 0 \quad \text{for all } \ i, j \in \mathcal{J}. \tag{16}$$

If there is $j_0 \in \mathcal{J}$ such that $j_0d(j_0) = 0$, then according to (10) we get

$$(d(i) - i)j_0\mathcal{N}d(j_0) = \{0\} \quad \text{for all } \ i \in \mathcal{J}.$$

3-primeness of $\mathcal{N}$ gives $(d(i) - i)j_0 = 0$ or $d(j_0) = 0$ for all $i \in \mathcal{J}$ from the above, in all cases we can easily find that $d(j_0) = 0$. Then (16) becomes

$$i^2j = 0 \text{ or } d(j) = 0 \quad \text{for all } \ i, j \in \mathcal{J}.$$

Suppose that there is $j_0 \in \mathcal{J}$ such that $i^2j_0 = 0$ for all $i \in \mathcal{J}$, then using the same techniques as we have used previously, we arrive at $j_0 = 0$. Therefore, in all cases, we obtain $d(\mathcal{J}) = \{0\}$ and Lemma 2 forces that $\mathcal{J}$ is commutative.

(ii) For a nonzero Jordan left ideal $\mathcal{J}$ of $\mathcal{N}$, using the same previous demonstration with minor changes, we can easily find the required result. □

Theorem 2. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $\mathcal{J}$ a nonzero Jordan right ideal or a nonzero Jordan left ideal of $\mathcal{N}$. If N admits a nonzero derivation d satisfying any one of the following conditions*:

(i) $d(nj) = d(j)d(n)$ *for all* $j \in \mathcal{J}$, $n \in \mathcal{N}$,

(ii) $d(jn) = d(n)d(j)$ *for all* $j \in \mathcal{J}$, $n \in \mathcal{N}$,

then $\mathcal{J}$ is commutative.

Proof. (i) By our hypothesis, we have

$$d(nj) = d(j)d(n) \quad \text{for all } \ j \in \mathcal{J}, \ n \in \mathcal{N}. \tag{17}$$

Replacing n by nj in (17) and using the definition of d, (17) becomes

$$d(nj)j + njd(j) = d(j)d(nj) \quad \text{for all } \ j \in \mathcal{J}, \ n \in \mathcal{N}. \tag{18}$$

Using (18) and definition of d, it follows by straightforward computation that

$$njd(j) = d(j)nd(j) \quad \text{for all } \ j \in \mathcal{J}, \ n \in \mathcal{N}. \tag{19}$$

Taking mn instead of n in (19) and using it again, we can easily arrive at $[d(j), m]\mathcal{N}d(j) = \{0\}$ for all $j \in \mathcal{J}$, $m \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we conclude that $d(\mathcal{J}) \subseteq Z(\mathcal{N})$ and application of Lemma 2 yields the required result.

(ii) Assume that

$$d(jn) = d(n)d(j) \quad \text{for all } \ j \in \mathcal{J}, \ n \in \mathcal{N}. \tag{20}$$

Replacing n by nm in (20) and using the definition of d, we get

$$d(j)nm + jd(n)m + jnd(m) = d(nm)d(j) \quad \text{for all } \ j \in \mathcal{J}, \ m, n \in \mathcal{N}. \tag{21}$$

By Lemma 5, (21) implies that

$$(d(j)n + jd(n))m + jnd(m) = d(nm)d(j) \quad \text{for all } j \in \mathcal{J},\ m, n \in \mathcal{N}. \tag{22}$$

In view of (20), (22) becomes

$$d(n)d(j)m + jnd(m) = d(nm)d(j) \quad \text{for all } j \in \mathcal{J},\ m, n \in \mathcal{N}$$

which reduces to

$$jnd(m) = -d(n)d(j)m + d(n)md(j) + nd(m)d(j) \quad \text{for all } j \in \mathcal{J},\ m, n \in \mathcal{N}.$$

Putting $d(j)$ in place of m, we obtain

$$jnd^2(j) = nd^2(j)d(j) \quad \text{for all } j \in \mathcal{J},\ n \in \mathcal{N}. \tag{23}$$

Substituting nm instead of n in (23), from the above, it is easy to see that $[j, n]\mathcal{N}d^2(j) = \{0\}$ for all $j \in \mathcal{J}, n \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we arrive at

$$j \in Z(\mathcal{N}) \text{ or } d^2(j) = 0 \quad \text{for all } j \in \mathcal{J}. \tag{24}$$

If there is $j_0 \in Z(\mathcal{N}) \cap \mathcal{J}$, then (20) yields $d(nj_0) = d(n)d(j_0)$ for all $n \in \mathcal{N}$ which gives

$$nd(j_0) = (d(j_0) - j_0)d(n) \quad \text{for all } n \in \mathcal{N}. \tag{25}$$

Taking mn instead of n in (25) and using it again, we can easily write

$$(d(j_0) - j_0)\mathcal{N}d(n) = \{0\} \quad \text{for all } n \in \mathcal{N}.$$

Since $d \neq 0$, then from the 3-primeness of $\mathcal{N}$ it follows that $d(j_0) = j_0$. Returning to (25), we can conclude that $nd(j_0) = 0$ for all $n \in \mathcal{N}$ so that $n\mathcal{N}d(j_0) = \{0\}$ for all $n \in \mathcal{N}$. In view of 3-primeness of $\mathcal{N}$, we get $d(j_0) = 0$, in which case equation (23) implies $d^2(j) = 0$ for all $j \in \mathcal{J}$. Replacing n by $d(i)$ in (20) where $i \in \mathcal{J}$, we arrive at $d(j)d(i) = 0$ for all $i, j \in \mathcal{J}$ which implies that

$$d(ij) = d(i)j + id(j) = 0 \quad \text{for all } i, j \in \mathcal{J}. \tag{26}$$

Replacing i by $2i^2$ in (26), we find that $2i^2d(j) = 0$ for all $i, j \in \mathcal{J}$ so that $i^2d(j) = 0$ for all $i, j \in \mathcal{J}$. Now, putting $i \circ ni$ instead of i where $n \in \mathcal{N}$ in (26), we obtain $(i \circ ni)d(j) = 0$ for all $i, j \in \mathcal{J}$, $n \in \mathcal{N}$ which becomes $inid(j) = 0$ for all $i, j \in \mathcal{J}$, $n \in \mathcal{N}$ and 3-primeness of $\mathcal{N}$ forces that $id(j) = 0$ for all $i, j \in \mathcal{J}$. In this case, taking $i \circ ni$ instead of i in (26) where $n \in \mathcal{N}$, we arrive at $ind(j) = 0$ for all $i, j \in \mathcal{J}$, $n \in \mathcal{N}$. By 3-primeness of $\mathcal{N}$, we arrive at $d(\mathcal{J}) = \{0\}$, another appeal to Lemma 2 we conclude that $\mathcal{J}$ is commutative.

For a nonzero Jordan left ideal $\mathcal{J}$ of $\mathcal{N}$, using the same previous demonstrations with necessary changes, we can easily find the required result. □

The following example demonstrates that the 3-primeness of $\mathcal{N}$ in Theorem 2 is crucial.

Example 1. Let $\mathcal{S}$ be a 2-torsion free near-ring which is not abelian. Define $\mathcal{N}, \mathcal{J}, d$ by $\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} \mid x, y, z \in \mathcal{S} \right\}$, $\mathcal{J} = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid m \in \mathcal{S} \right\}$ and $d\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & z & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then it can be easily seen that $\mathcal{N}$ is a left near-ring which is not 3-prime, $\mathcal{J}$ is a nonzero Jordan ideal of $\mathcal{N}$ and d is a derivation on $\mathcal{N}$ such that

(i) $d(AC) = d(C)d(A)$,
(ii) $d(CB) = d(B)d(C)$

for all $A, B \in \mathcal{J}$, $C \in \mathcal{N}$. However, $\mathcal{J}$ is not commutative.

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