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CHARACTERIZATIONS OF ORDERED INTRA k-REGULAR SEMIRINGS BY ORDERED k-IDEALS

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Abstract. We introduce the notion of ordered intra k-regular semirings, characterize them using their ordered k-ideals and prove that an ordered semiring S is both ordered k-regular and ordered intra k-regular if and only if every ordered quasi k-ideal or every ordered k-bi-ideal of S is ordered k-idempotent.

1. Introduction

The notion of intra-regular semigroups was introduced by Lajos [7] in 1963. Then Shabir, Ali and Batool [14] introduced the notion of intra-regular semirings and gave some of their characterizations by their quasi-ideals.

In 1951, Bourne [4] called a semiring $(S, +, \cdot)$ to be regular if for every element a of S there exist $x, y \in S$ such that $a + axa = aya$. In 1996, Bourne regularity was renamed to be k-regular by Adhikari, Sen and Weinert [1]. Also in [1] k-regular semirings were characterized by their k-ideals. Later, Bhuniya and Jana introduced the notions of k-bi-ideals, quasi k-ideals of semirings and intra k-regular semirings and characterized k-regular semirings and intra k-regular semirings using their k-bi-ideals and quasi k-ideals, see in [3] and [6], respectively.

In 2011, the notion of an ordered semiring $S := (S, +, \cdot, \leq)$ was defined by Gan and Jiang [5] as a semiring $(S, +, \cdot)$ with a partially ordered set $(S, \leq)$ such that $\leq$ is compatible with the operations $+$ and $\cdot$ of S. In this paper, the notion of left (right) ordered ideals and ordered ideals were defined. Then Mandal [8] introduced and studied regular, intra-regular and k-regular ordered semirings. In our previous works [9, 10], in 2016, we introduced the notion of ordered

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quasi-ideals of ordered semirings, studied some of their properties and characterized regular ordered semirings, regular ordered duo-semirings and intra-regular ordered semirings using their ordered quasi-ideals. Later, Patchakhieo and Pibaljommee [12] introduced the notion of ordered k-ideals of ordered semirings, defined the concept of ordered k-regular semirings as the generalization of k-regular ordered semirings in sense of Mandal [8] and characterized them by their ordered k-ideals.

In our previous work [11], we introduced the notion of an ordered quasi k-ideal of an ordered semiring, and characterized ordered k-regular semirings using their ordered quasi k-ideals. As a continuation of our previous work, in this paper, we introduce the notion of ordered intra k-regular semirings and characterize them by their ordered k-ideals. In the last part of this paper, ordered quasi k-ideals are used to characterize an ordered semiring which is both ordered k-regular and ordered intra k-regular.

2. Preliminaries

A *semiring* is a tri-tuple $(S, +, \cdot)$ consisting of a nonempty set S and two binary operations $+$ and $\cdot$ on S such that $(S, +)$ and $(S, \cdot)$ are semigroups and the distributive law holds on S. A semiring S is called *additively commutative* if $x + y = y + x$ for all $x, y \in S$.

A nonempty subset A of a semiring S such that $A + A \subseteq A$ is called a *left* (*right*) *ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). We call A an *ideal* of S if A is both a left and a right ideal of S. A subsemiring A of S is called a *bi-ideal* (*interior ideal*) of S if $ASA \subseteq A$ ($SAS \subseteq A$).

An *ordered semiring* is an algebraic structure $(S, +, \cdot, \leq)$ such that $(S, +, \cdot)$ is a semiring, $(S, \leq)$ is a partially ordered set and the relation $\leq$ is compatible to the operations $+$ and $\cdot$ on S. Mandal [8] called an ordered semiring S to be *regular* if for every $a \in S$, $a \leq axa$ for some $x \in S$. In this paper, we always assume that S is an additively commutative ordered semiring.

For any nonempty subsets A, B of S, and element $a \in S$, we denote

$$\begin{aligned}
AB &= \{ab \in S \mid a \in A, b \in B\}, \\
A + B &= \{a + b \in S \mid a \in A, b \in B\}, \\
\Sigma A &= \left\{ \sum_{i=1}^{n} a_i \in S \mid a_i \in A, n \in \mathbb{N} \right\}, \\
\Sigma AB &= \left\{ \sum_{i=1}^{n} a_i b_i \in S \mid a_i \in A, b_i \in B, n \in \mathbb{N} \right\}, \\
\Sigma a &= \Sigma\{a\} \text{ and} \\
(A] &= \{x \in S \mid x \leq a \text{ for some } a \in A\}.
\end{aligned}$$

Remark 2.1. Let A, B be nonempty subsets of an ordered semiring S. Then the following statements hold:

(i) $A \subseteq \Sigma A$ and $\Sigma(\Sigma A) = \Sigma A$;
(ii) if $A \subseteq B$, then $\Sigma A \subseteq \Sigma B$;
(iii) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$;
(iv) $\Sigma(A + B) \subseteq \Sigma A + \Sigma B$;
(v) $\Sigma(A] \subseteq (\Sigma A]$;
(vi) $A \subseteq (A]$ and $((A]] = (A]$;
(vii) if $A \subseteq B$, then $(A] \subseteq (B]$;
(viii) $A(B] \subseteq (A](B] \subseteq (AB]$ and $(A]B \subseteq (A](B] \subseteq (AB]$;
(ix) $A + (B] \subseteq (A] + (B] \subseteq (A + B]$;
(x) $(A \cup B] = (A] \cup (B]$;
(xi) $(A \cap B] \subseteq (A] \cap (B]$.

Clearly, $A = \Sigma A$ if and only if A is closed under addition. It is easy to check that the equality of Remark 2.1(xi) holds if $(A] = A$ and $(B] = B$ and also true for arbitrary intersections.

Let A be a nonempty subset of S. Then the k-*closure* [12] of A in S is defined by

$$\overline{A} = \{x \in S \mid x + a \leq b \text{ for some } a, b \in A\}.$$

Remark 2.2. Let A, B be nonempty subsets of an ordered semiring S. Then the following statements hold:

(i) $\Sigma\overline{A} \subseteq \overline{\Sigma A}$;
(ii) if $A + A \subset A$, then $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{(A]} = \overline{\overline{(A]}}$;
(iii) if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$;
(iv) $A\overline{B} \subseteq \overline{AB}$ and $\overline{A}B \subseteq \overline{AB}$;
(v) $\overline{A} + \overline{B} \subseteq \overline{A + B}$;
(vi) $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$;
(vii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$;
(viii) if $A + A \subset A$, then $A \subseteq (A] \subseteq (\overline{A}] = \overline{A} \subseteq \overline{(A]}$.

We note that if a nonempty subset A of an ordered semiring S is closed under addition, then $(A]$, $\overline{A}$ and $\overline{(A]}$ are also closed.

As a consequence of Remark 2.1 and Remark 2.2, we obtain the following remark.

Remark 2.3. Let A, B be nonempty subsets of an ordered semiring S such that A and B are closed under addition. Then the following statements hold:

(i) $\Sigma\overline{(A]} = \overline{(\Sigma A]}$;
(ii) $\overline{(\overline{(A]}]} = \overline{(A]}$;
(iii) $\Sigma A\overline{(B]} \subseteq \overline{(\Sigma A\overline{(B]}]} \subseteq \overline{(\Sigma\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$ and
$\Sigma\overline{(A]}B \subseteq \overline{(\Sigma\overline{(A]}B]} \subseteq \overline{(\Sigma\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$;
(iv) $\overline{(\overline{(A]} + \overline{(B]}]} \subseteq \overline{(A + B]}$.

We recall the notions of some types of ordered k-ideals occurring in [9,11–13] as follows. A nonempty subset A of an ordered semiring S is called a *left ordered* *k-ideal* (resp. *right ordered k-ideal, ordered k-ideal, ordered k-bi-ideal, ordered k-interior ideal*) if A is a left ideal (resp. right ideal, ideal, bi-ideal, interior ideal) of S and $A = \overline{A}$. A nonempty subset Q of an ordered semiring S such that $Q+Q \subseteq Q$ is said to be an *ordered quasi k-ideal* of S if $\overline{(\Sigma SQ]} \cap \overline{(\Sigma QS]} \subseteq Q$ and $Q = \overline{Q}$.

Remark 2.4. Let S be an ordered semiring. Then the following statements hold:

(i) every left (right) ordered k-ideal of S is an ordered quasi k-ideal of S;
(ii) every ordered quasi k-ideal of S is an ordered k-bi-ideal of S;
(iii) the intersection of a left ordered k-ideal and a right ordered k-ideal of S is an ordered quasi k-ideal of S;
(iv) every ordered k-ideal of S is an ordered k-interior ideal of S.

The converses of Remark 2.4(i)-(iv) are not generally true as examples in [11] for (i)-(iii) and as the following example for (iv).

Example 2.5. Let $S = \{a, b, c, d, e\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	d
c	c	d	d	d	d
d	d	d	d	d	d
e	e	d	d	d	d

and

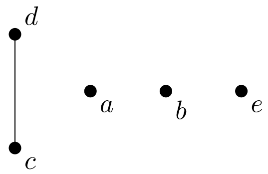
$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	a	a	a
d	a	a	a	a	a
e	a	c	a	a	a

Define a binary relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (c, d)\}.$$

We give the covering relation " $\prec$ " and the figure of S:

$$\prec := \{(c, d)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring. Let $I = \{a, b\}$. Clearly, I is a subsemiring of S. We have $SIS = \{a\} \subseteq I$ and $I = \overline{I}$. Hence, I is an ordered k-interior ideal of S, but not an ordered left k-ideal of S, since $SI = \{a, c\} \nsubseteq I$.

If an ordered semiring S has an identity (i.e., $\exists e \in S, ea = a = ae, \forall a \in S$), then the converse of Remark 2.4(iv) is true as follows.

Theorem 2.6. *If an ordered semiring S has an identity, then their ordered k-ideals and ordered k-interior ideals coincide.*

Proof. Let I be an ordered k-interior ideal and e be an identity of S. Then $IS = eIS \subseteq SIS \subseteq I$ and $SI = SIe \subseteq SIS \subseteq I$. Hence, I is an ordered k-ideal of S. □

For any nonempty subset A of an ordered semiring S, we denote $L_k(A)$, $R_k(A)$, $J_k(A)$, $Q_k(A)$ and $B_k(A)$ as the smallest left ordered k-ideal, right ordered k-ideal, ordered k-ideal, ordered quasi k-ideal and ordered k-bi-ideal of S containing A, respectively. Now we recall their constructions occurring in [11, 12] as the following lemma.

Lemma 2.7. *Let A be a nonempty subset of an ordered semiring S. Then the following statements hold*:

(i) $L_k(A) = \overline{(\Sigma A + \Sigma SA]}$;
(ii) $R_k(A) = \overline{(\Sigma A + \Sigma AS]}$;
(iii) $J_k(A) = \overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS]}$;
(iv) $Q_k(A) = \overline{(\Sigma A + (\overline{(\Sigma SA]} \cap \overline{(\Sigma AS]})]}$;
(v) $B_k(A) = \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA]}$.

3. Ordered k-regular semirings

Here, we recall the definition of ordered k-regular semirings and show that their ordered k-interior ideals and their ordered k-ideals coincide.

Definition. Let S be an ordered semiring.

(i) An element $a \in S$ is called *k-regular* [8] if $a + axa \leq aya$ for some $x, y \in S$.
(ii) An element $a \in S$ is called *ordered k-regular* [12] if $a \in \overline{(aSa]}$.

An ordered semiring S is said to be *k-regular* (resp. *ordered k-regular*) if every element $a \in S$ is k-regular (resp. ordered k-regular).

We note that every k-regular ordered semiring is an ordered k-regular semiring but the converse is not true (see Example 3.1 in [12]).

In ordered k-regular semirings, the converse of Remark 2.4(iv) is true as the following theorem shows.

Theorem 3.1. *Let S be an ordered semiring. If S is ordered k-regular, then ordered k-ideals and ordered k-interior ideals coincide in S.*

Proof. Let I be an ordered k-interior ideal of S. If $x \in IS$, then $x \in \overline{(xSx]} \subseteq \overline{(ISSIS]} \subseteq \overline{(ISI]} \subseteq \overline{(I]} = I$. Similarly, we can show that $SI \subseteq I$. Therefore, I is an ordered k-ideal of S. □

Now, we recall some properties of ordered k-regular semirings that will be used in the later sections.

Lemma 3.2 ([12])**.** *Let S be an ordered semiring. Then S is ordered k-regular if and only if $R \cap L = \overline{(RL]}$ for every right ordered k-ideal R and left ordered k-ideal L of S.*

Corollary 3.3 ([11])**.** *Let S be an ordered semiring. Then S is ordered k-regular if and only if $A \subseteq \overline{(\Sigma R_k(A)L_k(A)]}$ for each $A \subseteq S$.*

4. Ordered intra k-regular semirings

In [2], Ahsan, Mordeson and Shabir gave a general definition of intra-regular semirings, namely a semiring S is said to be *intra-regular* if every element a of S, $a = \sum_{i=1}^{n} x_i a^2 y_i$ for some $x_i, y_i \in S$ and for some $n \in \mathbb{N}$. In this section, we introduce the notion of ordered intra k-regular semirings as a generalization of the intra-regular semiring in sense of Ahsan, Mordeson and Shabir, show that their ordered k-ideals and ordered k-interior ideals coincide and characterize them using their ordered k-ideals.

Definition. Let S be an ordered semiring.

(i) An element $a \in S$ is called *intra k-regular* if $a+\sum_{i=1}^{n} x_i a^2 y_i \leq \sum_{i=1}^{m} x_i' a^2 y_i'$ for some $x_i, y_i, x_i', y_i' \in S, n, m \in \mathbb{N}$.

(ii) An element $a \in S$ is called *ordered intra k-regular* if $a \in \overline{(\Sigma S a^2 S]}$.

If every $a \in S$ is intra k-regular (resp. ordered intra k-regular), then we call S an *intra k-regular ordered semiring* (resp. *ordered intra k-regular semiring*).

Lemma 4.1. *Let S be an ordered semiring. Then S is ordered intra k-regular if and only if $A \subseteq \overline{(\Sigma S A^2 S]}$ for every $A \subseteq S$.*

It is easy to check that if an ordered semiring S is intra k-regular, then S is ordered intra k-regular, but the converse is not true that is the concept of ordered intra k-regular semirings is a generalization of the concept of intra k-regular semirings. The following example shows that there exists an ordered intra k-regular semiring which is not intra k-regular.

Example 4.2. Let $S = \{a, b, c\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

and

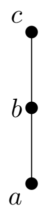
$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

Define a binary relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}.$$

We give the covering relation " $\prec$ " and the figure of S:

$$\prec := \{(a, b), (b, c)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring. We have $x \in \overline{(\Sigma S x^2 S]}$ for all $x \in S$. This means S is ordered intra k-regular. However, S is not intra k-regular, since the inequality $c + \sum_{i=1}^{n} x_i c^2 y_i \leq \sum_{i=1}^{n} x_i' c^2 y_i'$ has no a solution.

The converse of Remark 2.4(iv) is true in ordered intra k-regular semirings as the following theorem.

Theorem 4.3. *Let S be an ordered semiring. If S is ordered intra k-regular, then ordered k-ideals and ordered k-interior ideals coincide in S.*

Proof. Let I be an ordered k-interior ideal of S. If $x \in SI$, then

$$x \in \overline{(\Sigma S x^2 S]} \subseteq \overline{(\Sigma SSISIS]} \subseteq \overline{(\Sigma SIIS]} \subseteq \overline{(\Sigma SIS]} \subseteq \overline{(\Sigma I]} = I.$$

Similarly, we can show that $IS \subseteq I$. Therefore, I is an ordered k-ideal of S. □

Now, we give some characterizations of ordered intra k-regular semirings by their ordered k-ideals.

Theorem 4.4. *An ordered semiring S is ordered intra k-regular if and only if $L \cap R \subseteq \overline{(\Sigma LR]}$ for every left ordered k-ideal L and right ordered k-ideal R of S.*

Proof. Assume that S is ordered intra k-regular. Let L and R be a left and a right ordered k-ideal of S, respectively. If $x \in L \cap R$, then $x \in \overline{(\Sigma S x^2 S]} \subseteq \overline{(\Sigma SLRS]} \subseteq \overline{(\Sigma LR]}$.

Conversely, let $A \subseteq S$. By assumption and using Remark 2.3(iii) and Lemma 2.7, we obtain

$$\begin{aligned} A \subseteq L_k(A) \cap R_k(A) &\subseteq \overline{(\Sigma L_k(A) R_k(A)]} \\ &= \overline{(\Sigma \overline{(\Sigma A + \Sigma SA]}\ \overline{(\Sigma A + \Sigma AS]}]} \\ &\subseteq \overline{(\Sigma (\Sigma A + \Sigma SA)(\Sigma A + \Sigma AS)]} \\ &\subseteq \overline{(\Sigma A^2 + \Sigma A^2 S + \Sigma S A^2 + \Sigma S A^2 S]}. \qquad (1) \end{aligned}$$

Using (1) and Remark 2.3(iii), we have

$$\begin{aligned} \Sigma A^2 = \Sigma AA &\subseteq \Sigma A \overline{(\Sigma A^2 + \Sigma A^2 S + \Sigma S A^2 + \Sigma S A^2 S]} \\ &\subseteq \overline{(\Sigma A^3 + \Sigma A^3 S + \Sigma ASA^2 + \Sigma ASA^2 S]} \\ &\subseteq \overline{(\Sigma S A^2 + \Sigma S A^2 S]}. \qquad (2) \end{aligned}$$

Using (1) and Remark 2.3(iii) again, we have

$$\begin{aligned}\Sigma A^2 = \Sigma AA &\subseteq \Sigma\overline{(\Sigma A^2 + \Sigma A^2 S + \Sigma SA^2 + \Sigma SA^2 S]}A \\ &\subseteq \overline{(\Sigma A^3 + \Sigma A^2 SA + \Sigma SA^3 + \Sigma SA^2 SA]} \\ &\subseteq \overline{(\Sigma A^2 S + \Sigma SA^2 S]}. \qquad (3)\end{aligned}$$

Using (2) and Remark 2.3(iii), we have

$$\Sigma A^2 S \subseteq \Sigma\overline{(\Sigma SA^2 + \Sigma SA^2 S]}S \subseteq \overline{(\Sigma(\Sigma SA^2 S)]} = \overline{(\Sigma SA^2 S]}. \qquad (4)$$

Using (3) and Remark 2.3(iii), we have

$$\Sigma SA^2 \subseteq \Sigma S\overline{(\Sigma A^2 S + \Sigma SA^2 S]} \subseteq \overline{(\Sigma(\Sigma SA^2 S)]} = \overline{(\Sigma SA^2 S]}. \qquad (5)$$

Using (3), (5) and Remark 2.3(iv), we have

$$\Sigma A^2 \subseteq \overline{(\Sigma A^2 S + \Sigma SA^2 S]} \subseteq \overline{(\overline{(\Sigma SA^2 S]} + \Sigma SA^2 S]} \subseteq \overline{(\Sigma SA^2 S]}. \qquad (6)$$

By (1), (4), (5), (6) and using Remark 2.3(iv), we obtain

$$\begin{aligned}A &\subseteq \overline{(\Sigma A^2 + \Sigma A^2 S + \Sigma SA^2 + \Sigma SA^2 S]} \\ &\subseteq \overline{(\overline{(\Sigma SA^2 S]} + \overline{(\Sigma SA^2 S]} + \overline{(\Sigma SA^2 S]} + \Sigma SA^2 S]} \\ &\subseteq \overline{(\Sigma SA^2 S]}.\end{aligned}$$

By Lemma 4.1, S is ordered intra k-regular. □

Corollary 4.5. *An ordered semiring S is ordered intra k-regular if and only if $A \subseteq \overline{(\Sigma L_k(A)R_k(A)]}$ for each $A \subseteq S$.*

Theorem 4.6. *Let S be an ordered semiring. Then the following statements are equivalent*:

(i) *S is ordered intra k-regular*;

(ii) *$L \cap B \subseteq \overline{(\Sigma LBS]}$ for every left ordered k-ideal L and ordered k-bi-ideal B of S*;

(iii) *$L \cap Q \subseteq \overline{(\Sigma LQS]}$ for every left ordered k-ideal L and ordered quasi k-ideal Q of S*;

(iv) *$B \cap R \subseteq \overline{(\Sigma SBR]}$ for every right ordered k-ideal R and ordered k-bi-ideal B of S*;

(v) *$Q \cap R \subseteq \overline{(\Sigma SQR]}$ for every right ordered k-ideal R and ordered quasi k-ideal Q of S.*

Proof. (i)⇒(ii): Let L and B be a left ordered k-ideal and an ordered k-bi-ideal of S, respectively. If $x \in L \cap B$, then $x \in \overline{(\Sigma Sx^2 S]} \subseteq \overline{(\Sigma SLBS]} \subseteq \overline{(\Sigma LBS]}$.

(ii)⇒(iii): It follows from Remark 2.4(ii).

(iii)⇒(i): Assume that (iii) holds. Let $A \subseteq S$. By assumption, we obtain

$$A \subseteq L_k(A) \cap Q_k(A) \subseteq \overline{(\Sigma L_k(A)Q_k(A)S]} \subseteq \overline{(\Sigma L_k(A)R_k(A)S]} \subseteq \overline{(\Sigma L_k(A)R_k(A)]}.$$

By Corollary 4.5, S is ordered intra k-regular.

(i)⇒(iv)⇒(v)⇒(i) can be prove in analogous way of (i)⇒(ii)⇒(iii)⇒(i). □

Theorem 4.7. *Let S be an ordered semiring. Then the following statements are equivalent*:

(i) *S is ordered intra k-regular*;
(ii) $B \cap Q \subseteq \overline{(\Sigma SBQS]}$ *for every ordered k-bi-ideal B and ordered quasi k-ideal Q of S*;
(iii) $B \cap Q \subseteq \overline{(\Sigma SQBS]}$ *for every ordered k-bi-ideal B and ordered quasi k-ideal Q of S.*

Proof. (i)⇔(ii): Assume that S is ordered intra k-regular. Let B and Q be an ordered k-bi-ideal and an ordered quasi k-ideal of S, respectively. If $x \in B \cap Q$, then $x \in \overline{(\Sigma Sx^2S]} \subseteq \overline{(\Sigma SBQS]}$.

Conversely, assume that (ii) holds. Let $A \subseteq S$. By assumption, we obtain

$$\begin{aligned} A \subseteq B_k(A) \cap Q_k(A) &\subseteq \overline{(\Sigma SB_k(A)Q_k(A)S]} \\ &\subseteq \overline{(\Sigma SL_k(A)R_k(A)S]} \\ &\subseteq \overline{(\Sigma L_k(A)R_k(A)]}. \end{aligned}$$

By Corollary 4.5, S is ordered intra k-regular.

(i)⇔(iii) can be prove similar to (i)⇔(ii). □

5. Ordered k-regular and ordered intra k-regular semirings

In this section, we give some characterizations of an ordered semiring which is both ordered k-regular and ordered intra k-regular and show that its ordered k-bi-ideals and ordered quasi k-ideals are ordered k-idempotent.

First, we give an example of an ordered semiring which is both ordered k-regular and ordered intra k-regular.

Example 5.1. Let $S = \{a, b, c, d\}$. Define binary operations $+$ and $\cdot$ by the following tables:

$+$	a	b	c	d
a	a	b	a	d
b	b	b	b	b
c	a	b	c	d
d	d	b	d	d

and

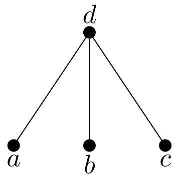
$\cdot$	a	b	c	d
a	a	d	a	d
b	a	b	a	d
c	a	d	a	d
d	a	d	a	d

Define a binary relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, d), (b, d), (c, d)\}.$$

We give the covering relation " $\prec$ " and the figure of S:

$$\prec := \{(a, d), (b, d), (c, d)\}.$$



Then $(S,+,\cdot,\leq)$ is an additively commutative ordered semiring. Clearly, a,b,d are ordered k-regular and ordered intra k-regular. We consider $c\in\overline{(cSc]}=\overline{\{a\}}=\{a,c\}$ and $c\in\overline{(\Sigma Sc^2S]}=S$. Therefore, S is ordered k-regular and ordered intra k-regular.

Lemma 5.2. *Let S be an ordered semiring. Then the following statements are equivalent*:

(i) *S is ordered k-regular and ordered intra k-regular*;
(ii) *$A\subseteq\overline{(\Sigma ASA^2SA]}$ for each $A\subseteq S$*;
(iii) *$a\in\overline{(aSa^2Sa]}$ for each $a\in S$.*

Proof. (i)⇒(ii): Assume that (i) holds. Let $A\subseteq S$. Then $A\subseteq\overline{(ASA]}$ and $A\subseteq\overline{(\Sigma SA^2S]}$. By Remark 2.3(iii), it follows that

$$\begin{aligned}A&\subseteq\overline{(\Sigma ASA]}\subseteq\overline{(\Sigma AS\overline{(\Sigma ASA]}]}\subseteq\overline{(\Sigma ASASA]}\subseteq\overline{(\Sigma AS\overline{(\Sigma SA^2S]}SA]}\\&\subseteq\overline{(\Sigma ASSA^2SSA]}\subseteq\overline{(\Sigma ASA^2SA]}.\end{aligned}$$

(ii)⇒(iii) and (iii)⇒(i) are obvious. □

Theorem 5.3. *Let S be an ordered semiring. Then S is ordered k-regular and ordered intra k-regular if and only if $R\cap L=\overline{((RL)^2]}$ for every right ordered k-ideal R and left ordered k-ideal L of S.*

Proof. Assume that S is ordered k-regular and ordered intra k-regular. Let R and L be a right and a left ordered k-ideal of S, respectively. Then

$$\overline{((RL)^2]}\subseteq\overline{((RS)^2]}\subseteq\overline{(R^2]}\subseteq\overline{(R]}=R\text{ and }\overline{((RL)^2]}\subseteq\overline{((SL)^2]}\subseteq\overline{(L^2]}\subseteq\overline{(L]}=L.$$

Thus $\overline{((RL)^2]}\subseteq R\cap L$. On the other hand, let $x\in R\cap L$. By Lemma 5.2, we get $x\in\overline{(xSx^2Sx]}\subseteq\overline{(RSLRSL]}\subseteq\overline{(RLRL]}=\overline{((RL)^2]}$. Hence, $\overline{((RL)^2]}=R\cap L$.

Conversely, assume that $\overline{((RL)^2]}=R\cap L$ for every right ordered k-ideal R and left ordered k-ideal L of S. Then $R\cap L=\overline{(RLRL]}\subseteq\overline{(RL]}\subseteq R\cap L$, i.e., $R\cap L=\overline{(RL]}$ and $R\cap L=\overline{(RLRL]}\subseteq\overline{(LR]}\subseteq\overline{(\Sigma LR]}$. By Lemma 3.2 and Theorem 4.4, we obtain that S is ordered k-regular and ordered intra k-regular. □

Theorem 5.4. *Let S be an ordered semiring. Then the following statements are equivalent*:

(i) *S is ordered k-regular and ordered intra k-regular*;

(ii) $B \cap L \cap R \subseteq \overline{(BLRB]}$ *for every ordered k-bi-ideal B, left ordered k-ideal L and right ordered k-ideal R of S*;

(iii) $Q \cap L \cap R \subseteq \overline{(QLRQ]}$ *for every ordered quasi k-ideal Q, left ordered k-ideal L and right ordered k-ideal R of S.*

Proof. (i)⇒(ii): Assume that (i) holds. Let B, L and R be ordered k-bi-ideal, left ordered k-ideal and right ordered k-ideal of S, respectively. Let $x \in B \cap L \cap R$. By Lemma 5.2, we get $x \in \overline{(xSx^2Sx]} \subseteq \overline{(BSLRSB]} \subseteq \overline{(BLRB]}$.

(ii)⇒(iii): It follows from Remark 2.4(ii).

(iii)⇒(i): Assume that (iii) holds. Let R and L be a right and a left ordered k-ideal of S, respectively. By Remark 2.4(iii), we have that $R \cap L$ is an ordered quasi k-ideal of S. By assumption, we obtain

$$\begin{aligned} R \cap L = (R \cap L) \cap L \cap R &\subseteq \overline{((R \cap L)LR(R \cap L)]} \\ &\subseteq \overline{(RLRL]} = \overline{((RL)^2]} \subseteq R \cap L. \end{aligned}$$

Thus, $R \cap L = \overline{((RL)^2]}$. By Theorem 5.3, we have that S is ordered k-regular and ordered intra k-regular. □

Let I be an ordered k-bi-ideal (ordered quasi k-ideal) of an ordered semiring S. Then we call I *ordered k-idempotent* if $I = \overline{(\Sigma I^2]}$.

Theorem 5.5. *Let S be an ordered semiring. Then the following statements are equivalent*:

(i) *S is ordered k-regular and ordered intra k-regular*;

(ii) *every ordered k-bi-ideal of S is ordered k-idempotent*;

(iii) *every ordered quasi k-ideal of S is ordered k-idempotent.*

Proof. (i)⇒(ii): Assume that (i) holds. Clearly, $\overline{(\Sigma B^2]} \subseteq \overline{(\Sigma B]} = B$. Let $x \in B$. Using Lemma 5.2, we obtain $x \in \overline{(xSx^2Sx]} \subseteq \overline{(BSBBSB]} \subseteq \overline{(BB]} = \overline{(B^2]} \subseteq \overline{(\Sigma B^2]}$. Now, $B = \overline{(\Sigma B^2]}$.

(ii)⇒(iii): It follows from Remark 2.4(ii).

(iii)⇒(i): Assume that (iii) holds. Let $A \subseteq S$. By assumption, we obtain

$$A \subseteq Q_k(A) = \overline{(\Sigma Q_k(A)Q_k(A)]} \subseteq \overline{(\Sigma R_k(A)L_k(A)]} \text{ and}$$
$$A \subseteq Q_k(A) = \overline{(\Sigma Q_k(A)Q_k(A)]} \subseteq \overline{(\Sigma L_k(A)R_k(A)]}.$$

By Corollaries 3.3 and 4.5, we have that S is ordered k-regular and ordered intra k-regular. □

References

[1] M. R. Adhikari, M. K. Sen, and H. J. Weinert, *On k-regular semirings*, Bull. Calcutta Math. Soc. **88** (1996), no. 2, 141–144.

[2] J. Ahsan, J. N. Mordeson, and M. Shabir, *Fuzzy semirings with applications to automata theory*, Studies in Fuzziness and Soft Computing, **278**, Springer, Heidelberg, 2012.

[3] A. K. Bhuniya and K. Jana, *Bi-ideals in k-regular and intra k-regular semirings*, Discuss. Math. Gen. Algebra Appl. **31** (2011), no. 1, 5–25.

[4] S. Bourne, *The Jacobson radical of a semiring*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), no. 3, 163–170.
[5] A. P. Gan and Y. L. Jiang, *On ordered ideals in ordered semirings*, J. Math. Res. Exposition **31** (2011), no. 6, 989–996.
[6] K. Jana, *Quasi k-ideals in k-regular and intra k-regular semirings*, Pure Math. Appl. (PU.M.A.) **22** (2011), no. 1, 65–74.
[7] S. Lajos, *A note on intra-regular semigroups*, Proc. Japan Acad. **39** (1963), 626–627.
[8] D. Mandal, *Fuzzy ideals and fuzzy interior ideals in ordered semirings*, Fuzzy Inf. Eng. **6** (2014), no. 1, 101–114.
[9] P. Palakawong na Ayutthaya and B. Pibaljommee, *Characterizations of regular ordered semirings by ordered quasi-ideals*, Int. J. Math. Math. Sci. **2016**, Art. ID 4272451, 8 pp.
[10] ———, *Characterizations of intra-regular ordered semirings by ordered quasi-ideals*, Proceeding of The National and International Graduate Research Conference, Graduate School, Khon Kaen University, Thailand, (2016), 102–109.
[11] ———, *Characterizations of ordered k-regular semirings by ordered quasi k-ideals*, Quasigroups Related Systems **25** (2017), no. 1, 109–120.
[12] S. Patchakhieo and B. Pibaljommee, *Characterizations of ordered k-regular semirings by ordered k-ideals*, Asian-Eur. J. Math. **10** (2017), no. 2, 1750020, 14 pp.
[13] P. Senarat and B. Pibaljommee, *Prime ordered k-bi-ideals in ordered semirings*, Quasigroups Related Systems **25** (2017), no. 1, 121–132.
[14] M. Shabir, A. Ali, and S. Batool, *A note on quasi-ideals in semirings*, Southeast Asian Bull. Math. **27** (2004), no. 5, 923–928.

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