

APPROXIMATION BY GENUINE LUPAŞ-BETA-STANCU OPERATORS

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ABSTRACT. In this paper, we introduce a Stancu type generalization of genuine Lupaş-Beta operators of integral type. We establish some moment estimates and the direct results in terms of classical modulus of continuity, Voronovskaja-type asymptotic theorem, weighted approximation, rate of convergence and pointwise estimates using the Lipschitz type maximal function. Lastly, we propose a king type modification of these operators to obtain better estimates.

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1. Introduction

The approximation of functions by positive linear operators is an important research area in the classical approximation theory. It provides us key tools for exploring the computer-aided geometric design, numerical analysis and the solutions of ordinary and partial differential equations that arise in the mathematical modeling of real world phenomena.

In the year 1995, Lupaş introduced an important discrete operators as follows

$$L_n(f, x) = \sum_{k=0}^{\infty} l_{n,k}(x) f(k/n), \quad x \in [0, \infty) \quad (1)$$

where

$$l_{n,k}(x) = 2^{-nx} \frac{(nx)_k}{k! \cdot 2^k}.$$

In the last four decades several operators have been modified and their approximation properties has been discussed in real and complex domain (see [11], [22], [32] etc.). In order to modify the operators (1), Govil et al. in [10] considered the

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hybrid operators by taking the weights of Szász basis functions. In [9], Gupta and Yadav considered other hybrid operators by taking weights of Beta basis function, but the operators reproduce only the constant functions. Later in [8] Gupta, Rassias and Yadav considered the following form of hybrid operators, which preserve constant as well as linear functions

$$D_n(f, x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) f(t) dt + 2^{-nx} f(0), \quad x \in [0, \infty) \quad (2)$$

where

$$b_{n,k-1}(t) = \frac{1}{B(k, n+1)} \frac{t^{k-1}}{(1+t)^{k+n+1}},$$

and $B(m, n)$ being the Beta function defined as

$$B(m, n) = \frac{\Gamma m \Gamma n}{\Gamma m + n}, \quad m, n > 0.$$

Very recently, Gupta, Rassias and Pandey [7] studied some approximation properties on the weighted modulus of continuity for these operators.

In [33], Stancu introduced the positive linear operators $P_n^{\alpha, \beta} : C[0, 1] \rightarrow C[0, 1]$ by modifying the Bernstein polynomial as

$$P_n^{\alpha, \beta}(f; x) = \sum_{k=0}^n b_{n,k}(x) f\left(\frac{k + \alpha}{n + \beta}\right),$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0, 1]$ is the Bernstein basis function and α, β are any two real numbers which satisfy the condition that $0 \leq \alpha \leq \beta$.

In the recent years, Stancu type generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [2], [17], [18], [25] [26], [30], etc.

For $f \in C[0, \infty)$, $0 \leq \alpha \leq \beta$ we introduce the following Stancu type generalization of the operators (2):

$$D_n^{\alpha, \beta}(f; x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) f\left(\frac{nt + \alpha}{n + \beta}\right) dt + 2^{-nx} f\left(\frac{\alpha}{n + \beta}\right) \quad (3)$$

For $\alpha = \beta = 0$, we denote $D_n^{\alpha, \beta}(f; x)$ by $D_n(f; x)$.

The purpose of this paper is to study the basic convergence theorem, Voronovskaja-type asymptotic result, rate of convergence, weighted approximation and point-wise estimation of the operators (3). Further, to obtain better approximation, we also propose modification of the operators (3) using King type approach.

2. Moment Estimates

In this section we collect some results about the operators $D_n^{\alpha, \beta}$ useful in the sequel.

Lemma 2.1. [7] For $n > 1$, we have

- (1) $D_n(1; x) = 1$;
- (2) $D_n(t; x) = x$;
- (3) $D_n(t^2; x) = \frac{nx^2 + 3x}{n-1}$.

Lemma 2.2. For the operators $D_n^{\alpha, \beta}(f; x)$ as defined in (3), the following equalities holds for $n > 1$

- (1) $D_n^{\alpha, \beta}(1; x) = 1$;
- (2) $D_n^{\alpha, \beta}(t; x) = \frac{nx + \alpha}{n + \beta}$;
- (3) $D_n^{\alpha, \beta}(t^2; x) = \left\{ \frac{n^3}{(n-1)(n+\beta)^2} \right\} x^2 + \left\{ \frac{3n^2 + 2\alpha n(n-1)}{(n-1)(n+\beta)^2} \right\} x + \frac{\alpha^2}{(n+\beta)^2}$.

Proof. For $x \in [0, \infty)$, in view of Lemma 2.1, we have

$$D_n^{\alpha, \beta}(1; x) = 1.$$

Next, for $f(t) = t$, again applying Lemma 2.1, we get

$$\begin{aligned} D_n^{\alpha, \beta}(f; x) &= \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) \left(\frac{nt + \alpha}{n + \beta} \right) dt + 2^{-nx} \left(\frac{\alpha}{n + \beta} \right) \\ &= \frac{n}{n + \beta} D_n(t, x) + \frac{\alpha}{n + \beta} = \frac{nx + \alpha}{n + \beta}. \end{aligned}$$

Proceeding similarly, we have

$$\begin{aligned} D_n^{\alpha, \beta}(f; x) &= \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) \left(\frac{nt + \alpha}{n + \beta} \right)^2 dt + 2^{-nx} \left(\frac{\alpha}{n + \beta} \right)^2 \\ &= \left(\frac{n}{n + \beta} \right)^2 D_n(t^2, x) + \frac{2n\alpha}{(n + \beta)^2} D_n(t, x) + \left(\frac{\alpha}{n + \beta} \right)^2 \\ &= \left\{ \frac{n^3}{(n-1)(n+\beta)^2} \right\} x^2 + \left\{ \frac{3n^2 + 2\alpha n(n-1)}{(n-1)(n+\beta)^2} \right\} x + \frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

□

Lemma 2.3. For $f \in C_B[0, \infty)$ (space of all real valued bounded and uniformly continuous functions on $[0, \infty)$ endowed with the norm $\|f\| = \sup\{|f(x)| : x \in [0, \infty)\}$), $\|D_n^{\alpha, \beta}(f)\| \leq \|f\|$.

Proof. In view of (3) and Lemma 2.2, the proof of this lemma easily follows. □

Remark 2.1. For $n > 1$, we have

$$D_n^{\alpha, \beta}((t-x); x) = \frac{\alpha - \beta x}{n + \beta}$$

and

$$D_n^{\alpha, \beta}((t-x)^2; x)$$

$$\begin{aligned}
&= \left\{ \frac{n^2 + \beta^2(n-1)}{(n-1)(n+\beta)^2} \right\} x^2 + \left\{ \frac{3n^2 + 2\alpha\beta(1-n)}{(n-1)(n+\beta)^2} \right\} x + \frac{\alpha^2}{(n+\beta)^2} \\
&= \xi_n^{\alpha, \beta}(x), (\text{say}).
\end{aligned}$$

3. Main results

In this section we establish some approximation properties in several settings. For the reader's convenience we split up this section in more subsections.

Theorem 3.1. (*Voronovskaja-type theorem*) *Let $f \in C_B[0, \infty)$. If f', f'' exists at a fixed point $x \in [0, \infty)$, then we have*

$$\lim_{n \rightarrow \infty} n (D_n^{\alpha, \beta}(f; x) - f(x)) = (\alpha - \beta x)f'(x) + \frac{x(x+3)}{2}f''(x).$$

Proof. Let $x \in [0, \infty)$ be fixed. Using Taylor's expansion of f , we obtain

$$f(t) = f(x) + (t-x)f'(x) + \frac{f''(x)}{2}(t-x)^2 + r(t, x)(t-x)^2, \quad (4)$$

where the function $r(t, x)$ is the Peano form of remainder and $\lim_{t \rightarrow x} r(t, x) = 0$.

Applying $D_n^{\alpha, \beta}(f; x)$ on both sides of (4), we have

$$\begin{aligned}
n (D_n^{\alpha, \beta}(f; x) - f(x)) &= n f'(x) D_n^{\alpha, \beta}((t-x); x) + \frac{1}{2} n f''(x) D_n^{\alpha, \beta}((t-x)^2; x) \\
&\quad + n D_n^{\alpha, \beta}((t-x)^2 r(t, x); x).
\end{aligned}$$

In view of Remark 2.1, we have

$$\lim_{n \rightarrow \infty} n D_n^{\alpha, \beta}((t-x); x) = \alpha - \beta x \quad (5)$$

and

$$\lim_{n \rightarrow \infty} n D_n^{\alpha, \beta}((t-x)^2; x) = x(x+3). \quad (6)$$

Now, we shall show that

$$\lim_{n \rightarrow \infty} n D_n^{\alpha, \beta}(r(t, x)(t-x)^2; x) = 0$$

By using Cauchy-Schwarz inequality, we have

$$D_n^{\alpha, \beta}(r(t, x)(t-x)^2; x) \leq (D_n^{\alpha, \beta}(r^2(t, x); x))^{1/2} (D_n^{\alpha, \beta}((t-x)^4; x))^{1/2} \quad (7)$$

We observe that $r^2(x, x) = 0$ and $r^2(\cdot, x) \in C_B[0, \infty)$. Then, it follows that

$$\lim_{n \rightarrow \infty} D_n^{\alpha, \beta}(r^2(t, x); x) = r^2(x, x) = 0, \quad (8)$$

in view of fact that $D_n^{\alpha, \beta}((t-x)^4; x) = O\left(\frac{1}{n^2}\right)$.

Now, from (7) and (8) we obtain

$$\lim_{n \rightarrow \infty} n D_n^{\alpha, \beta}(r(t, x)(t-x)^2; x) = 0. \quad (9)$$

From (5), (6) and (9), we get the required result. \square

3.1. Local approximation. For $C_B[0, \infty)$, let us consider the following K -functional:

$$K_2(f, \delta) = \inf_{g \in W^2} \{ \|f - g\| + \delta \|g''\| \},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$. By, p. 177, Theorem 2.4 in [1], there exists an absolute constant $M > 0$ such that

$$K_2(f, \delta) \leq M\omega_2(f, \sqrt{\delta}), \quad (10)$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of f . By

$$\omega(f, \delta) = \sup_{0 < h \leq \delta} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|,$$

we denote the first order modulus of continuity of $f \in C_B[0, \infty)$.

Theorem 3.2. *Let $f \in C_B[0, \infty)$. Then, for every $x \in [0, \infty)$, we have*

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq M\omega_2(f, \zeta_n^{\alpha, \beta}(x)) + \omega\left(f, \frac{|\alpha - \beta x|}{n + \beta}\right),$$

where M is a positive constant and

$$\zeta_n^{\alpha, \beta}(x) = \left(\xi_n^{\alpha, \beta}(x) + \left(\frac{\alpha - \beta x}{n + \beta} \right)^2 \right)^{1/2}.$$

Proof. For $x \in [0, \infty)$, we consider the auxiliary operators $\overline{D}_n^{\alpha, \beta}$ defined by

$$\overline{D}_n^{\alpha, \beta}(f; x) = D_n^{\alpha, \beta}(f; x) - f\left(\frac{nx + \alpha}{n + \beta}\right) + f(x). \quad (11)$$

From Lemma 2.2, we observe that the operators $\overline{D}_n^{\alpha, \beta}$ are linear and reproduce the linear functions.

Hence

$$\overline{D}_n^{\alpha, \beta}((t - x); x) = 0. \quad (12)$$

Let $g \in W^2$ and $x, t \in [0, \infty)$. By Taylor's expansion we have

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - v)g''(v)dv.$$

Applying $\overline{D}_n^{\alpha, \beta}$ on both sides of the above equation and using (12), we get

$$\overline{D}_n^{\alpha, \beta}(g; x) - g(x) = \overline{D}_n^{\alpha, \beta}\left(\int_x^t (t - v)g''(v)dv; x\right).$$

Thus, by (11) we get

$$\begin{aligned}
 |\overline{D}_n^{\alpha,\beta}(g; x) - g(x)| &\leq D_n^{\alpha,\beta} \left(\left| \int_x^t (t-v)g''(v)dv \right|; x \right) + \left| \int_x^{\frac{nx+\alpha}{n+\beta}} \left(\frac{nx+\alpha}{n+\beta} - v \right) g''(v)dv \right| \\
 &\leq \left(\xi_n^{\alpha,\beta}(x) + \left(\frac{\alpha - \beta x}{n + \beta} \right)^2 \right) \|g''\| \\
 &\leq \left(\zeta_n^{\alpha,\beta}(x) \right)^2 \|g''\|. \tag{13}
 \end{aligned}$$

On other hand, by (11) and Lemma 2.3, we have

$$|\overline{D}_n^{\alpha,\beta}(f; x)| \leq \|f\|. \tag{14}$$

Using (13) and (14) in (11), we obtain

$$\begin{aligned}
 |D_n^{\alpha,\beta}(f; x) - f(x)| &\leq |\overline{D}_n^{\alpha,\beta}(f - g; x)| + |(f - g)(x)| + |\overline{D}_n^{\alpha,\beta}(g; x) - g(x)| + \left| f \left(\frac{nx + \alpha}{n + \beta} \right) - f(x) \right| \\
 &\leq 2 \|f - g\| + (\zeta_n^{\alpha,\beta}(x))^2 \|g''\| + \left| f \left(\frac{nx + \alpha}{n + \beta} \right) - f(x) \right|.
 \end{aligned}$$

Taking infimum over all $g \in W^2$, we get

$$|D_n^{\alpha,\beta}(f; x) - f(x)| \leq K_2(f, (\zeta_n^{\alpha,\beta}(x))^2) + \omega \left(f, \frac{|\alpha - \beta x|}{n + \beta} \right).$$

In view of (10), we get

$$|D_n^{\alpha,\beta}(f; x) - f(x)| \leq M\omega_2(f, \zeta_n^{\alpha,\beta}(x)) + \omega \left(f, \frac{|\alpha - \beta x|}{n + \beta} \right),$$

which proves the theorem. \square

3.2. Rate of convergence. Let $\omega_a(f, \delta)$ denote the usual modulus of continuity of f on the closed interval $[0, a]$, $a > 0$, and defined as

$$\omega_a(f, \delta) = \sup_{|t-x| \leq \delta} \sup_{x, t \in [0, a]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_B[0, \infty)$, the modulus of continuity $\omega_a(f, \delta)$ tends to zero.

Now, we give a rate of convergence theorem for the operators $D_n^{\alpha,\beta}$.

Theorem 3.3. *Let $f \in C_B[0, \infty)$ and $\omega_{a+1}(f, \delta)$ be its modulus of continuity on the finite interval $[0, a + 1] \subset [0, \infty)$, where $a > 0$. Then, for every $n > 1$,*

$$|D_n^{\alpha,\beta}(f; x) - f(x)| \leq 6M_f(1 + a^2)\xi_n^{\alpha,\beta}(a) + 2\omega_{a+1} \left(f, \sqrt{\xi_n^{\alpha,\beta}(a)} \right),$$

where $\xi_n^{\alpha,\beta}(a)$ is defined in Remark 2.1 and M_f is a constant depending only on f .

Proof. For $x \in [0, a]$ and $t > a + 1$. Since $t - x > 1$, we have

$$\begin{aligned} |f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(t - x)^2(2 + 3x^2 + 2(t - x)^2) \\ &\leq 6M_f(1 + a^2)(t - x)^2. \end{aligned}$$

For $x \in [0, a]$ and $t \leq a + 1$, we have

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t - x|) \leq \left(1 + \frac{|t - x|}{\delta}\right) \omega_{a+1}(f, \delta)$$

with $\delta > 0$.

From the above, we have

$$|f(t) - f(x)| \leq 6M_f(1 + a^2)(t - x)^2 + \left(1 + \frac{|t - x|}{\delta}\right) \omega_{a+1}(f, \delta),$$

for $x \in [0, a]$ and $t \geq 0$.

Thus

$$\begin{aligned} |D_n^{\alpha, \beta}(f; x) - f(x)| &\leq 6M_f(1 + a^2)(D_n^{\alpha, \beta}(t - x)^2; x) \\ &\quad + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta}(D_n^{\alpha, \beta}(t - x)^2; x)^{\frac{1}{2}}\right) \end{aligned}$$

Applying Cauchy-Schwarz's inequality, we get

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq 6M_f(1 + a^2)\xi_n^{\alpha, \beta}(a) + 2\omega_{a+1}\left(f, \sqrt{\xi_n^{\alpha, \beta}(a)}\right),$$

on choosing $\delta = \sqrt{\xi_n^{\alpha, \beta}(a)}$. This completes the proof of theorem. \square

3.3. Weighted approximation. In this section, we obtain the Korovkin type weighted approximation by the operators defined in (3). The weighted Korovkin-type theorems were proved by Gadzhiev [3]. A real function $\nu(x) = 1 + x^2$ is called a weight function if it is continuous on R and $\lim_{|x| \rightarrow \infty} \nu(x) =$

∞ , $\nu(x) \geq 1$ for all $x \in R$.

Let $B_\nu[0, \infty)$ denote the weighted space of real-valued functions f defined on $[0, \infty)$ with the property $|f(x)| \leq M_f \nu(x)$ for all $x \in [0, \infty)$, where M_f is a constant depending on the function f . We also consider the weighted subspace $C_\nu[0, \infty)$ of $B_\nu[0, \infty)$ given by $C_\nu[0, \infty) = \{f \in B_\nu[0, \infty) : f \text{ is continuous on } [0, \infty)\}$ and $C_\nu^*[0, \infty)$ denotes the subspace of all functions $f \in C_\nu[0, \infty)$ for which $\lim_{|x| \rightarrow \infty} \frac{f(x)}{\nu(x)}$ exists finitely.

It is obvious that $C_\nu^*[0, \infty) \subset C_\nu[0, \infty) \subset B_\nu[0, \infty)$. The space $B_\nu[0, \infty)$ is a normed linear space with the following norm:

$$\|f\|_\nu = \sup_{x \in [0, \infty)} \frac{|f(x)|}{\nu(x)}.$$

Theorem 3.4. For each $f \in C_\nu^*$, we have

$$\lim_{n \rightarrow \infty} \| D_n^{\alpha, \beta}(f) - f \|_\nu = 0.$$

Proof. From [3], we know that it is sufficient to verify the following three conditions

$$\lim_{n \rightarrow \infty} \| D_n^{\alpha, \beta}(t^r) - x^r \|_\nu = 0, \quad r = 0, 1, 2. \quad (15)$$

Since $D_n^{\alpha, \beta}(1; x) = 1$, the condition in (15) holds for $r = 0$.

For $n > 1$, we have

$$\begin{aligned} \| D_n^{\alpha, \beta}(t) - x \|_\nu &= \sup_{x \in [0, \infty)} \frac{|D_n^{\alpha, \beta}(t; x) - x|}{1 + x^2} \\ &\leq \frac{\beta}{n + \beta} \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{\alpha}{n + \beta} \sup_{x \in [0, \infty)} \frac{1}{1 + x^2} \\ &\leq \frac{\alpha + \beta}{n + \beta} \end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} \| D_n^{\alpha, \beta}(t) - x \|_\nu = 0$.

Similarly, we can write for $n > 1$

$$\begin{aligned} \| D_n^{\alpha, \beta}(t^2) - x^2 \|_\nu &= \sup_{x \in [0, \infty)} \frac{|D_n^{\alpha, \beta}(t^2; x) - x^2|}{1 + x^2} \\ &\leq \left| \frac{n^3}{(n-1)(n+\beta)^2} - 1 \right| + \left| \frac{3n^2 + 2\alpha n(n-1)}{(n-1)(n+\beta)^2} \right| + \frac{\alpha^2}{(n+\beta)^2}, \end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} \| D_n^{\alpha, \beta}(t^2) - x^2 \|_\nu = 0$.

This completes the proof of theorem. \square

Now we give the following theorem to approximate all functions in C_ν^* . Such type of results are given in [4] for locally integrable functions.

Theorem 3.5. For each $f \in C_\nu^*$ and $\vartheta > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \infty)} \frac{|D_n^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+\vartheta}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned} \sup_{x \in [0, \infty)} \frac{|D_n^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+\vartheta}} &\leq \sup_{x \leq x_0} \frac{|D_n^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+\vartheta}} + \sup_{x \geq x_0} \frac{|D_n^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+\vartheta}} \\ \sup_{x \in [0, \infty)} \frac{|D_n^{\alpha, \beta}(f; x) - f(x)|}{(1 + x^2)^{1+\vartheta}} &\leq \| D_n^{\alpha, \beta}(f) - f \|_{C[0, x_0]} \\ &\quad + \| f \|_\nu \sup_{x \geq x_0} \frac{|D_n^{\alpha, \beta}(1 + t^2; x)|}{(1 + x^2)^{1+\vartheta}} + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+\vartheta}}. \end{aligned}$$

The first term of the above inequality tends to zero from Theorem 3.3. By Lemma 2.2, for any fixed $x_0 > 0$, it is easily prove that

$$\sup_{x \geq x_0} \frac{|D_n^{\alpha, \beta}(1 + t^2; x)|}{(1 + x^2)^{1+\vartheta}} \rightarrow 0$$

as $n \rightarrow \infty$. We can choose $x_0 > 0$ so large that the last part of the above inequality can be small.

Hence the proof is completed. \square

3.4. Pointwise Estimates. In this section, we establish some pointwise estimates of the rate of convergence of the operators $D_n^{\alpha, \beta}$. First, we give the relationship between the local smoothness of f and local approximation. We know that a function $f \in C[0, \infty)$ is in $Lip_M(\eta)$ on E , $\eta \in (0, 1]$, $E \subset [0, \infty)$ if it satisfies the condition

$$|f(t) - f(x)| \leq M|t - x|^\eta, \quad t \in [0, \infty) \text{ and } x \in E,$$

where M is a constant depending only on η and f .

Theorem 3.6. *Let $f \in C[0, \infty) \cap Lip_M(\eta)$, $E \subset [0, \infty)$ and $\eta \in (0, 1]$. Then, we have*

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq M \left((\xi_n^{\alpha, \beta}(x))^{\eta/2} + 2d^\eta(x, E) \right), \quad x \in [0, \infty),$$

where M is a constant depending on η and f and $d(x, E)$ is the distance between x and E defined as

$$d(x, E) = \inf\{|t - x| : t \in E\}.$$

Proof. Let \bar{E} be the closure of E in $[0, \infty)$. Then, there exists at least one point $x_0 \in \bar{E}$ such that

$$d(x, E) = |x - x_0|.$$

By our hypothesis and the monotonicity of $D_n^{\alpha, \beta}$, we get

$$\begin{aligned} |D_n^{\alpha, \beta}(f; x) - f(x)| &\leq D_n^{\alpha, \beta}(|f(t) - f(x_0)|; x) + D_n^{\alpha, \beta}(|f(x) - f(x_0)|; x) \\ &\leq M (D_n^{\alpha, \beta}(|t - x_0|^\eta; x) + |x - x_0|^\eta) \\ &\leq M (D_n^{\alpha, \beta}(|t - x|^\eta; x) + 2|x - x_0|^\eta). \end{aligned}$$

Now, applying Hölder's inequality with $p = \frac{2}{\eta}$ and $q = \frac{2}{2 - \eta}$, we obtain

$$|D_n^{\alpha, \beta}((f; x) - f(x)| \leq M \left(\{D_n^{\alpha, \beta}(|t - x|^2; x)\}^{\eta/2} + 2d^\eta(x, E) \right),$$

from which the desired result immediate. \square

Next, we obtain the local direct estimate of the operators defined in (3), using the Lipschitz-type maximal function of order η introduced by B. Lenze [21] as

$$\tilde{\omega}_\eta(f, x) = \sup_{t \neq x, t \in [0, \infty)} \frac{|f(t) - f(x)|}{|t - x|^\eta}, \quad x \in [0, \infty) \quad \text{and} \quad \eta \in (0, 1]. \quad (16)$$

Theorem 3.7. *Let $f \in C_B[0, \infty)$ and $0 < \eta \leq 1$. Then, for all $x \in [0, \infty)$ we have*

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq \tilde{\omega}_\eta(f, x) (\xi_n^{\alpha, \beta}(x))^{\eta/2}.$$

Proof. From the equation (16), we have

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq \tilde{\omega}_\eta(f, x) D_n^{\alpha, \beta}(|t - x|^\eta; x).$$

Applying the Hölder's inequality with $p = \frac{2}{\eta}$ and $q = \frac{2}{2 - \eta}$, we get

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq \tilde{\omega}_\eta(f, x) D_n^{\alpha, \beta}((t - x)^2; x)^{\frac{\eta}{2}} \leq \tilde{\omega}_\eta(f, x) (\xi_n^{\alpha, \beta}(x))^{\eta/2}.$$

Thus, the proof is completed. \square

For $a, b > 0$, Özarslan and Aktuğlu [31] consider the Lipschitz-type space with two parameters:

$$Lip_M^{(a, b)}(\eta) = \left(f \in C[0, \infty) : |f(t) - f(x)| \leq M \frac{|t - x|^\eta}{(t + ax^2 + bx)^{\eta/2}}; \quad x, t \in [0, \infty) \right),$$

where M is any positive constant and $0 < \eta \leq 1$.

Theorem 3.8. *For $f \in Lip_M^{(a, b)}(\eta)$. Then, for all $x > 0$, we have*

$$|D_n^{\alpha, \beta}(f; x) - f(x)| \leq M \left(\frac{\xi_n^{\alpha, \beta}(x)}{ax^2 + bx} \right)^{\eta/2}.$$

Proof. First we prove the theorem for $\eta = 1$. Then, for $f \in Lip_M^{(a, b)}(1)$, and $x \in [0, \infty)$, we have

$$\begin{aligned} |D_n^{\alpha, \beta}(f; x) - f(x)| &\leq D_n^{\alpha, \beta}(|f(t) - f(x)|; x) \\ &\leq M D_n^{\alpha, \beta} \left(\frac{|t - x|}{(t + ax^2 + bx)^{1/2}}; x \right) \\ &\leq \frac{M}{(ax^2 + bx)^{1/2}} D_n^{\alpha, \beta}(|t - x|; x). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} |D_n^{\alpha, \beta}(f; x) - f(x)| &\leq \frac{M}{(ax^2 + bx)^{1/2}} (D_n^{\alpha, \beta}((t - x)^2; x))^{1/2} \\ &\leq M \left(\frac{\xi_n^{\alpha, \beta}(x)}{ax^2 + bx} \right)^{1/2}. \end{aligned}$$

Thus the result holds for $\eta = 1$.

Now, we prove that the result is true for $0 < \eta < 1$. Then, for $f \in Lip_M^{(a,b)}(\eta)$, and $x \in [0, \infty)$, we get

$$|D_n^{\alpha,\beta}(f; x) - f(x)| \leq \frac{M}{(ax^2 + bx)^{\eta/2}} D_n^{\alpha,\beta}(|t - x|^\eta; x).$$

Taking $p = \frac{1}{\eta}$ and $q = \frac{2}{2-\eta}$, applying the Hölders inequality, we have

$$|D_n^{\alpha,\beta}(f; x) - f(x)| \leq \frac{M}{(ax^2 + bx)^{\eta/2}} (D_n^{\alpha,\beta}(|t - x|; x))^\eta.$$

Finally by Cauchy-Schwarz inequality, we get

$$|D_n^{\alpha,\beta}(f; x) - f(x)| \leq M \left(\frac{\xi_n^{\alpha,\beta}(x)}{ax^2 + bx} \right)^{\eta/2}.$$

Thus, the proof is completed. \square

4. King's Approach

To make the convergence faster, King [20] proposed an approach to modify the classical Bernstein polynomial, so that the sequence preserve test functions e_0 and e_2 , where $e_i(t) = t^i, i = 0, 1, 2$. After this approach many researcher contributed in this direction.

As the operator $D_n^{\alpha,\beta}(f; x)$ defined in (3) preserve only the constant functions so further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions.

For this purpose the modification of (3) is defined as

$$\begin{aligned} \hat{D}_n^{\alpha,\beta}(f; x) &= \sum_{k=1}^{\infty} l_{n,k}(r_n(x)) \int_0^{\infty} b_{n,k-1}(t) f\left(\frac{nt + \alpha}{n + \beta}\right) dt \\ &\quad + 2^{-nr_n(x)} f\left(\frac{\alpha}{n + \beta}\right) \end{aligned} \quad (17)$$

where $r_n(x) = \frac{(n+\beta)x - \alpha}{n}$ for $x \in I_n = [\frac{\alpha}{n+\beta}, \infty)$ and $n > 1$.

Lemma 4.1. *For every $x \in I_n$, we have*

- (1) $\hat{D}_n^{\alpha,\beta}(1; x) = 1;$
- (2) $\hat{D}_n^{\alpha,\beta}(t; x) = x;$
- (3) $\hat{D}_n^{\alpha,\beta}(t^2; x) = \frac{nx^2}{n-1} + \frac{(3n-2\alpha)x}{(n-1)(n+\beta)} + \frac{\alpha(\alpha-3n)}{(n-1)(n+\beta)^2}.$

Consequently, for each $x \in I_n$, we have the following equalities

$$\hat{D}_n^{\alpha,\beta}(t - x; x) = 0$$

$$\hat{D}_n^{\alpha,\beta}((t - x)^2; x) = \frac{x^2}{n-1} + \frac{(3n-2\alpha)x}{(n-1)(n+\beta)} + \frac{\alpha(\alpha-3n)}{(n-1)(n+\beta)^2}$$

$$= \lambda_n^{\alpha,\beta}(x), (\text{say}). \quad (18)$$

Theorem 4.2. For $f \in C_B(I_n)$ and $n > 1$, we have

$$|\hat{D}_n^{\alpha,\beta}(f; x) - f(x)| \leq M' \omega_2 \left(f, \sqrt{\lambda_n^{\alpha,\beta}(x)} \right),$$

where $\lambda_n^{\alpha,\beta}(x)$ is given by (18) and M' is a positive constant.

Proof. Let $g \in W^2$ and $x, t \in I_n$. Using the Taylor's expansion we have

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - v)g''(v)dv.$$

Applying $\hat{D}_n^{\alpha,\beta}$ on both sides and using Lemma 4.1, we get

$$\hat{D}_n^{\alpha,\beta}(g; x) - g(x) = \hat{D}_n^{\alpha,\beta} \left(\int_x^t (t - v)g''(v)dv; x \right).$$

Obviously, we have $\left| \int_x^t (t - v)g''(v)dv \right| \leq (t - x)^2 \|g''\|$.

Therefore

$$|\hat{D}_n^{\alpha,\beta}(g; x) - g(x)| \leq \hat{D}_n^{\alpha,\beta}((t - x)^2; x) \|g''\| = \lambda_n^{\alpha,\beta}(x) \|g''\|.$$

Since $|\hat{D}_n^{\alpha,\beta}(f; x)| \leq \|f\|$, we get

$$\begin{aligned} |\hat{D}_n^{\alpha,\beta}(f; x) - f(x)| &\leq |\hat{D}_n^{\alpha,\beta}(f - g; x)| + |(f - g)(x)| + |\hat{D}_n^{\alpha,\beta}(g; x) - g(x)| \\ &\leq 2\|f - g\| + \lambda_n^{\alpha,\beta}(x) \|g''\|. \end{aligned}$$

Finally, taking the infimum over all $g \in W^2$ and using (10) we obtain

$$|\hat{D}_n^{\alpha,\beta}(f; x) - f(x)| \leq M' \omega_2 \left(f, \sqrt{\lambda_n^{\alpha,\beta}(x)} \right),$$

which proves the theorem. \square

Theorem 4.3. Let $f \in C_B(I_n)$. If f', f'' exists at a fixed point $x \in I_n$, then we have

$$\lim_{n \rightarrow \infty} n \left(\hat{D}_n^{\alpha,\beta}(f; x) - f(x) \right) = \frac{x(x+3)}{2} f''(x).$$

The proof follows along the lines of Theorem 3.1.

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REFERENCES

1. R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer, Berlin (1993).
2. T. Acar, L.N. Mishra and V.N. Mishra, *Simultaneous Approximation for Generalized Srivastava-Gupta Operators*, Journal of Function Spaces **2015**, Article ID 936308, 11 pages.
3. A.D. Gadjiev, *Theorems of the type of P. P. Korovkin's theorems*, Matematicheskie Zametki **20** (1976), 781-786.
4. A.D. Gadjiev, R.O. Efendiyev and E. Ibikli, *On Korovkin type theorem in the space of locally integrable functions*, Czechoslovak Math. J. **1**(128) (2003), 45-53.
5. A.R. Gairola, Deepmala and L.N. Mishra, *On the q -derivatives of a certain linear positive operators*, Iranian Journal of Science and Technology, Transactions A: Science, (2017), DOI 10.1007/s40995-017-0227-8.
6. A.R. Gairola, Deepmala and L.N. Mishra, *Rate of Approximation by Finite Iterates of q -Durrmeyer Operators*, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. (April-June 2016) **86**:229-234 (2016). doi: 10.1007/s40010-016-0267-z
7. V. Gupta, Th.M. Rassias and E. Pandey, *On genuine Lupaş-Beta operators and modulus of continuity*, Int. J. Nonlinear Anal. Appl. **8** (2017), 23-32.
8. V. Gupta, Th.M. Rassias and R. Yadav, *Approximation by Lupaş-Beta integral operators*, Appl. Math. Comput. **236** (2014), 19-26.
9. V. Gupta and R. Yadav, *On approximation of certain integral operators*, Acta Math. Vietnamica **39** (2014), 193-203.
10. N.K. Govil, V. Gupta and D. Soybaş, *Certain new classes of Durrmeyer type operators*, Appl. Math. Comput. **225** (2013), 195-203.
11. V. Gupta, Th.M. Rassias and J. Sinha, *A survey on Durrmeyer operators*, Springer International Publishing Switzerland, (2016), 299-312, Contributions in Mathematics and Engineering P.M. Pardalos, T.M. Rassias (eds.).
12. G.C. Jain, *Approximation of functions by a new class of linear operators*, J. Aust. Math. Soc. **13** (1972), 271-276.
13. A. Kumar, *Voronovskaja type asymptotic approximation by general Gamma type operators*, Int. J. of Mathematics and its Applications **3** (2015), 71-78.
14. A. Kumar, *Approximation by Stancu type generalized Srivastava-Gupta operators based on certain parameter*, Khayyam J. Math. **3**, no. 2 (2017), pp. 147-159. DOI: 10.22034/kjm.2017.49477
15. A. Kumar, *General Gamma type operators in L^p spaces*, Palestine Journal of Mathematics **7** (2018), 73-79.
16. A. Kumar and D. K. Vishwakarma, *Global approximation theorems for general Gamma type operators*, Int. J. of Adv. in Appl. Math. and Mech. **3**(2) (2015), 77-83.
17. A. Kumar and L.N. Mishra, *Approximation by modified Jain-Baskakov-Stancu operators*, Tbilisi Mathematical Journal **10** (2017), pp. 185-199.
18. A. Kumar, V.N. Mishra and Dipti Tapiawala, *Stancu type generalization of modified Srivastava-Gupta operators*, Eur. J. Pure Appl. Math **10**, (2017), 890-907.
19. A. Kumar, Artee and D.K. Vishwakarma, *Approximation properties of general gamma type operators in polynomial weighted space*, Int. J. Adv. Appl. Math. and Mech. **4** (2017), pp. 7-13.
20. J.P. King, *Positive linear operators which preserve x^2* , Acta Math. Hungar. **99** (2003), 203-208.
21. B. Lenze, *On Lipschitz type maximal functions and their smoothness spaces*, Nederl. Akad. Indag. Math. **50** (1988), 53-63.
22. N.I. Mahmudov and V. Gupta, *On certain q -analogue of Szász Kantorovich operators*, J. Appl. Math. Comput. **37** (2011), 407-419.
23. V.N. Mishra, P. Sharma and M. Birou, *Approximation by Modified Jain-Baskakov Operators*, arXiv:1508.05309v2 [math.FA] 9 Sep 2015.

24. V.N. Mishra and P. Sharma, *On approximation properties of Baskakov-Schurer-Szász operators*, arXiv:1508.05292v1 [math.FA] 21 Aug 2015.
25. V.N. Mishra, H.H. Khan, K. Khatri and L.N. Mishra, *Hypergeometric Representation for Baskakov-Durrmeyer-Stancu Type Operators*, Bulletin of Mathematical Analysis and Applications **5** Issue 3 (2013), Pages 18-26.
26. V.N. Mishra, K. Khatri and L.N. Mishra, *On Simultaneous Approximation for Baskakov-Durrmeyer-Stancu type operators*, Journal of Ultra Scientist of Physical Sciences **24**, 3(A) (2012), pp. 567-577.
27. V.N. Mishra, K. Khatri, L.N. Mishra and Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, Journal of Inequalities and Applications (2013), 2013:586. doi:10.1186/1029-242X-2013-586.
28. V.N. Mishra, K. Khatri and L.N. Mishra, *Some approximation properties of q -Baskakov-Beta-Stancu type operators*, Journal of Calculus of Variations **2013**, Article ID 814824, 8 pages.
29. V.N. Mishra, Rajiv B. Gandhi and Ram N. Mohapatra, *Summation-Integral type modification of Szász-Mirakjan-Stancu operators*, J. Numer. Anal. Approx. Theory **45**(1) (2016), pp. 27-36.
30. V.N. Mishra, K. Khatri, L.N. Mishra and Deepmala, *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, J. Inequal. Appl. **586** (2013), 1-11.
31. M.A. Özarslan and H. Aktuğlu, *Local approximation for certain King type operators*, Filomat **27** (2013), 173-181.
32. E. Pandey and R.K. Mishra, *Convergence estimates in simultaneous approximation by certain Srivastava-Gupta type operators*, Adv. Studies Contemporary Math. **26** (2016), 467-480.
33. D.D. Stancu, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roum. Math. Pures Appl. **13** (1968), 1173-1194.
34. O. Szász, *Generalization of S. Bernstein polynomials to the infinite interval*, J. Res. Natl. Bur. Stand. **45** (1950), 239-245.

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