

FINITE ELEMENT SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATION WITH MULTIPLE CONCAVE CORNERS

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Abstract. In [8] they introduced a new finite element method for accurate numerical solutions of Poisson equations with corner singularities. They consider the Poisson equations with homogeneous Dirichlet boundary condition with one corner singularity at the origin, and compute the finite element solution using standard FEM and use the extraction formula to compute the stress intensity factor, then pose a PDE with a regular solution by imposing the non-homogeneous boundary condition using the computed stress intensity factor, which converges with optimal speed. From the solution they could get an accurate solution just by adding the singular part. This approach uses the polar coordinate and the cut-off function to control the singularity and the boundary condition.

In this paper we consider Poisson equations with multiple singular points, which involves different cut-off functions which might overlaps together and shows the way of coding in FreeFEM++ to control the singular functions and cut-off functions with numerical experiments.

1. Introduction

Let Ω be an open, bounded polygonal domain in \mathbb{R}^2 and let Γ be the boundary of Ω . For a given function $f \in L^2(\Omega)$, as a model problem, we consider the following Poisson equation with Dirichlet boundary condition:

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

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where Δ stands for the Laplacian operator.

If the domain is convex or smooth, the solution belongs to $H^2(\Omega)$ and we expect to have an optimal convergence rate with the standard finite element method. But this is not true for Poisson problems defined on non-convex domains. In these cases, the solutions of Poisson problems have singular behavior at those concave corners and such singular behavior affects the accuracy of numerical solution throughout the whole domain.

For overcoming this difficulty, roughly speaking, there were two groups of people who use two different approaches: mesh refinements and augmenting the space of trial/test functions. (See [1, 5, 3, 8] and references therein.) Obviously, the method in [8] belongs to the second approach.

In this paper we consider the case the model problem (1.1) with multiple concave corners, which make the solution involves the same number of singular functions.

It is well-known that the singular function s and its dual singular function s_- can be expressed by

$$s = s(r, \theta) = r^{\frac{\pi}{\omega}} \sin \frac{\pi\theta}{\omega}, \quad s_- = s_-(r, \theta) = r^{-\frac{\pi}{\omega}} \sin \frac{\pi\theta}{\omega} \quad (1.2)$$

for the case with only one concave corner at the origin, and the unique solution $u \in H_0^1(\Omega)$ has the representation

$$u = w + \lambda \eta s, \quad (1.3)$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$, and η is a smooth cut-off function which equals one identically in a neighborhood of the origin and the support of η is small enough so that the function ηs vanishes identically on Γ . (Here, (r, θ) is polar coordinate at the origin.)

We need to use the polar coordinate systems (r_i, θ_i) and the corresponding cut-off functions η_i which equals one identically in a neighborhood of the concave corner P_i with inner angle ω_i and the support of η_i is small enough so that the function $\eta_i s_i$ vanishes identically on Γ . Here, we have that the singular function s_i and its dual singular function s_{-i} can be expressed by

$$s_i = s_i(r_i, \theta_i) = r_i^{\frac{\pi}{\omega_i}} \sin \frac{\pi\theta_i}{\omega_i}, \quad s_{-i} = s_{-i}(r_i, \theta_i) = r_i^{-\frac{\pi}{\omega_i}} \sin \frac{\pi\theta_i}{\omega_i}, \quad (1.4)$$

and the unique solution $u \in H_0^1(\Omega)$ has the representation (see [6, 4])

$$u = w + \sum_i \lambda_i \eta_i s_i, \quad (1.5)$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$, and η_i is a smooth cut-off function which equals one identically in a neighborhood of the concave point P_i and the support of η_i is small enough so that the function $\eta_i s_i$ vanishes identically on Γ . (Here, (r_i, θ_i) is polar coordinate around P_i .) Note that each s_i and s_{-i} are harmonic functions in Ω .

The coefficient, λ_i , is called ‘stress intensity factor’ and can be computed by the following extraction formula (see [4]):

$$\lambda_i = \frac{1}{\pi} \int_{\Omega} f \eta_i s_{-i} dx + \frac{1}{\pi} \int_{\Omega} u \Delta(\eta_i s_{-i}) dx. \quad (1.6)$$

In [8] they introduced new partial differential equation, whose solution is in $H^2(\Omega)$ with the same input function by simple changing of the boundary condition. Using this partial differential equation, they suggested an efficient algorithm to compute the numerical solution for Poisson equation with singular domain.

In this paper we consider a partial differential equation with multiple singular points. We use the algorithm suggested in [8] with minor changes suitable for the case.

Step 1) Solve the partial differential equation (1.1) using the standard finite element method.

Step 2) Compute the stress intensity factor λ_i using the extraction formula (1.6).

Step 3) Pose new partial differential equation which has zero stress intensity factor and find the solution w

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = -\sum_i \lambda_i s_i|_{\Gamma} & \text{on } \Gamma, \end{cases} \quad (1.7)$$

Step 4) Set $u = w + \sum_i \lambda_i s_i$.

Remark 1.1. *The stress intensity factor computed from the extraction formula depends on the regularity of the solution u . So, the convergence of the solution depend on the accuracy of the stress intensity factors, by which we use to find u in the algorithm.*

In Section 2, we suggest a modified algorithm which is introduced in [8] originally and some basic theorems and finite element approximation theories in Section 2 and 3, an example will be given in Section 4 with computational results using FreeFEM++ code.([7])

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot, \cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which

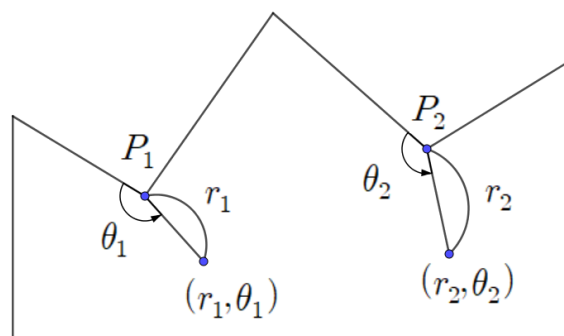


FIGURE 1. A domain with multiple concave corners and corresponding polar coordinates

case the inner product and norm will be denoted by $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively, although we will omit Ω if there is no chance of misunderstanding. $H_D^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$.

2. Stress Intensity Factors and Corresponding Algorithm

We need cut-off functions to derive the singular behavior of the problem. We set

$$B_i(p_1; p_2) = \{(r_i, \theta_i) : p_1 < r < p_2 \text{ and } 0 < \theta_i < \omega_i\} \cap \Omega$$

and

$$B_i(p_1) = B_i(0; p_1),$$

and define a smooth enough cut-off function of r as follows:

$$\eta_{i,\rho}(r) = \begin{cases} 1 & \text{in } B_i(\frac{1}{2}\rho), \\ \frac{1}{16}\{8 - 15p(r) + 10p(r)^3 - 3p(r)^5\} & \text{in } \overline{B_i}(\frac{1}{2}\rho; \rho), \\ 0 & \text{in } \Omega \setminus \overline{B_i}(\rho), \end{cases} \quad (2.1)$$

with $p(r) = 4r/\rho - 3$. Here, ρ is a parameter which will be determined so that the singular part $\eta_{i,\rho}s$ has the same boundary condition as the solution u of the Model problem, where s_i is the singular function which is given in (1.4). Note each $\eta_{i,\rho}(r)$ is C^2 .

2.1. Singularity and Extraction Formula

The solution of the Poisson equation on the polygonal domain is well known as in [2, 6]. Given $f \in L^2(\Omega)$, if we assume there are I reentrant

corners P_i with inner angles $\pi < \omega_i < 2\pi$, then there exists a unique solution u and in addition there exist fixed numbers λ_i such that

$$u - \sum_{i \in I} \lambda_i s_i \in H^2(\Omega). \quad (2.2)$$

By using the cut-off function $\eta_i = \eta_{i,\rho}$ we may write

$$u = w + \sum_{i \in I} \lambda_i \eta_i s_i, \quad (2.3)$$

with $w \in H^2(\Omega) \cap H_0^1(\Omega)$.

The constant λ_i is referred as stress intensity factors and computed by the following formula ([4]);

Lemma 2.1. *The stress intensity factor λ_i can be expressed in terms of u and f by the following extraction formula*

$$\lambda_i = \frac{1}{\pi} \int_{\Omega} f \eta s_{-i} dx + \frac{1}{\pi} \int_{\Omega} u \Delta(\eta_i s_{-i}) dx. \quad (2.4)$$

Assume that (1.1) has a solution u as in (2.3) and the stress intensity factor λ_i are known, then we introduce the following boundary value problem:

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = -\sum_{i \in I} \lambda_i s_i & \text{on } \Gamma, \end{cases} \quad (2.5)$$

Note the input function f is the same as in (1.1) and $s_i = s_i|_{\Gamma}$ is the restriction of the singular function s_i to the boundary Γ .

2.2. Regularity of New Partial Differential Equation

The following theorems show (2.5) has a regular solution.

Theorem 2.2. *If (1.1) has a solution u as in (2.3) with the stress intensity factors λ_i , then (2.5) has a unique solution w in $H^2(\Omega)$.*

Proof. The theorem comes from the uniqueness of the solution of the following Poisson problem;

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega, \\ p = -\sum_{i \in I} \lambda_i s_i & \text{on } \Gamma. \end{cases} \quad (2.6)$$

(Note $p = -\sum_{i \in I} \lambda_i s_i$ is the unique solution and the coefficient of the singular function s_i is the stress intensity factors.) By adding two equations, (1.1) and (2.6), we have the following equation

$$\begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = -\sum_{i \in I} \lambda_i s_i & \text{on } \Gamma, \end{cases} \quad (2.7)$$

whose solution $w = u + p$ belongs to $H^2(\Omega)$. \square

Theorem 2.3. *If λ_i is the stress intensity factors given by (2.4) with the solution u in (1.1) and w is the solution of (2.5), then $u = w + \sum_{i \in I} \lambda_i s_i$ is the unique solution of (1.1).*

Proof. We only need to show $u = w + \sum_{i \in I} \lambda_i s_i$ is the solution to (1.1) when w is the solution of (2.5). The proof comes from the fact that $\Delta s_i = 0$ and the linearity of the operator. \square

Note we have $u = w - p = w + \sum_{i \in I} \lambda_i s_i$ from the above theorems.

2.3. Corresponding Algorithm

Now we suggest an algorithm in a variational form for the solution u of the model problem (1.1). First we use the standard Finite Element Method and get an approximated solution, then compute the stress intensity factors λ_i from the formula in (2.4). Then we solve a non-homogeneous problem with more regular solution with the changed boundary data using the approximated stress intensity factors.

The following is the proposed algorithm:

The algorithm (V)

V-1: To find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\Omega). \quad (2.8)$$

V-2: Then compute λ_i by (2.4) with u .

V-3: To find w such that $w + \sum_i \lambda_i s_i \in H^1(\Omega)$ and

$$(\nabla w, \nabla v) = (f, v), \quad \forall v \in H^1(\Omega). \quad (2.9)$$

V-4: Finally set $u = w + \sum_i \lambda_i s_i$.

The existence and uniqueness of the solution u and w is clear. By Theorem 2.2 and Theorem 2.3 we have the solution $w \in H^2(\Omega)$ and u is the solution of (1.1).

3. Finite Element Approximation

In this section we present standard finite element approximation for u obtained in the algorithm in the L^2 and H^1 norms. Let T_h be a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \cup_{K \in T_h} K$ with $h = \max\{\text{diam} K : K \in T_h\}$. Let V_h be continuous piecewise linear finite element space; i.e.,

$$V_h = \{\phi_h \in C^0(\Omega) : \phi_h|_K \in P_1(K) \quad \forall K \in T_h, \phi_h = 0 \text{ on } \Gamma_0\} \subset H_0^1(\Omega),$$

where $P_1(K)$ is the space of linear functions on K .

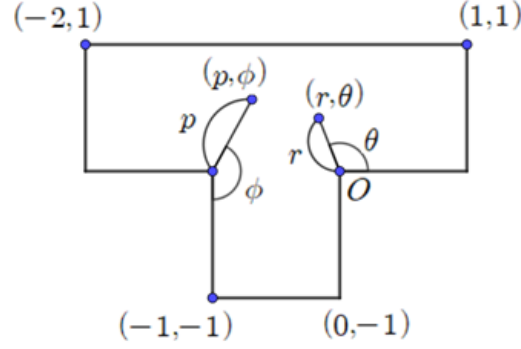


FIGURE 2. Two polar coordinates on a T-shaped domain

Now the error analysis of the method in the standard norms, $\|\cdot\|$ and $|\cdot|_1$, is carried out with a regular triangulation and continuous piecewise linear finite element space V_h . (See [8])

Note we can find approximated solution u_h using the following Algorithm: Algorithm (A) :

A-1: To find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall v \in V_h. \quad (3.1)$$

A-2: Then compute $\lambda_{i,h}$ by

$$\lambda_{i,h} = \frac{1}{\pi} \int_{\Omega} f \eta_i s_{-i} dx + \frac{1}{\pi} \int_{\Omega} u_h \Delta(\eta_i s_{-i}) dx. \quad (3.2)$$

A-3: To find w_h such that $w_h + \sum_i \lambda_{i,h} s_i \in V_h$ and

$$(\nabla w_h, \nabla v) = (f, v) \quad \forall v \in V_h. \quad (3.3)$$

A-4: Then $u_h = w_h + \sum_i \lambda_{i,h} s_i$.

4. Numerical Results and Conclusions

In this section we consider two examples with two concave boundary, which reduce two singular points, with inner angles $\omega_1 = \omega_2 = \frac{3\pi}{2}$.

Example 4.1. Consider the Poisson equation in (1.1) with Dirichlet boundary conditions on a T-type domain $\Omega = ((-2, 1) \times (-1, 1)) \setminus ((-2, -1] \times (-1, 0] \cup [0, 1) \times (-1, 0])$ as in **Figure 2**. On this type of domain it is obvious that there are two singular points where we may use two polar coordinate systems; one the usual polar system (r, θ) and the

other one (p, ϕ) centered at $(-1, 0)$ as in **Figure 2**. We also need two cut-off functions $\eta_{1,\rho}(r)$ and $\eta_{2,\rho}(p)$, where r and p are the distances from the origin $(0, 0)$ and $(-1, 0)$, respectively. Let $f = -\Delta(\eta_{1,3/4}s_1) + \Delta(\eta_{2,3/4}s_2)$ be the input function so that the exact solution of the underlying problem is

$$u = \eta_{1,3/4}s_1 - \eta_{2,3/4}s_2.$$

The exact stress intensity factors are $\lambda_1 = 1$ and $\lambda_2 = -1$. The errors of the stress intensity factors $\lambda_{1,h}$ and $\lambda_{2,h}$, computed by using the standard finite element solution u_h , are given in Table 1. The errors and rates of approximated solutions by the standard finite element method and by the algorithm (A), are presented in Table 2 and 3, respectively.

Mesh Size	$ \lambda_1 - \lambda_{1,h} $	RATE	$ \lambda_2 - \lambda_{2,h} $	RATE
$h = \frac{1}{4}$	4.28379E-01	-	6.37161E-01	-
$h = \frac{1}{8}$	1.13626E-01	1.91459	1.11555E-01	2.5139
$h = \frac{1}{16}$	3.14410E-02	1.85357	2.71210E-02	2.04027
$h = \frac{1}{32}$	6.63000E-03	2.24557	6.25000E-03	2.11748
$h = \frac{1}{64}$	1.55700E-03	2.09024	1.50000E-03	2.05889
$h = \frac{1}{128}$	3.01000E-04	2.37093	3.02000E-04	2.31234
$h = \frac{1}{256}$	4.80000E-05	2.64866	4.70000E-05	2.68382

TABLE 1. Errors and convergence rates of the $\lambda_{1,h}$ and $\lambda_{2,h}$

Mesh Size	L^2 -NORM	RATE	H^1 -NORM	RATE
$h = \frac{1}{4}$	1.126630E-01	-	1.399020	-
$h = \frac{1}{8}$	2.962570E-02	1.92709	7.282650E-01	0.94188
$h = \frac{1}{16}$	8.670090E-03	1.77273	3.938880E-01	0.88668
$h = \frac{1}{32}$	2.437850E-03	1.83044	2.013420E-01	0.96814
$h = \frac{1}{64}$	7.776640E-04	1.64839	1.065820E-01	0.91768
$h = \frac{1}{128}$	2.549740E-04	1.60880	5.542100E-02	0.94346
$h = \frac{1}{256}$	9.121630E-05	1.48299	2.959570E-02	0.90505

TABLE 2. Errors and convergence rates for u_h with the Standard FEM

Mesh Size	L^2 -NORM	RATE	H^1 -NORM	RATE
$h = \frac{1}{4}$	1.069880E-01		1.388820	
$h = \frac{1}{8}$	3.029070E-02	1.82050	7.575880E-01	0.87437
$h = \frac{1}{16}$	7.487060E-03	2.01640	3.774330E-01	1.00519
$h = \frac{1}{32}$	1.906160E-03	1.97373	1.899190E-01	0.99084
$h = \frac{1}{64}$	4.856170E-04	1.97278	9.624000E-02	0.98068
$h = \frac{1}{128}$	1.206410E-04	2.00910	4.795980E-02	1.00481
$h = \frac{1}{256}$	3.021700E-05	1.99729	2.402360E-02	0.99737

TABLE 3. Errors and convergence rates for u_h with our algorithm **A**

We also emphasize that the solution process **A-3** in the algorithm **A** does not include any cut-off function and this is unique and strong point compared to other methods using singular functions. ([3])

Now we have the following conclusions from the theorems together with the example and corresponding numerical results:

Conclusion 1 : We may use the method given in [8] for the Poisson problem with multiple singular corners.

Conclusion 2 : As we see in TABLE 2 AND 3, the algorithm **A** may give better results than the standard finite element method.

Conclusion 3 : In the case that we have multiple singular points, the support of cut-off functions may not be disjoint. The algorithm **A** allows us to use the cut-off functions, whose supports make non-empty intersections.

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