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# ANALYTIC TREATMENT FOR GENERALIZED $(m+1)$-DIMENSIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, a recently developed semi-analytic technique, so called the residual power series method, is generalized to process higher-dimensional linear and nonlinear partial differential equations. The solutions obtained takes a form of an infinite power series which can, in turn, be expressed in a closed exact form. The results reveal that the proposed generalization is very effective, convenient and simple. This is achieved by handling the $(m+1)$-dimensional Burgers equation.

## 1. INTRODUCTION

Over the last four years, a recent developed technique, namely the residual power series method (RPSM), for solving linear and nonlinear differential equations of integer and fractional orders have been proposed [1, 2, 3, 4, 5]. In the current work, we improve this method to process linear and nonlinear partial differential equations (NPDEs) of higher dimensional and orders. Due to their broad variety of relevance, comparative studies, using the Adomian decomposition method, homotopy perturbation method, variational iteration method, differential transform method and its reduction, were discussed deeply to tackle higher dimensional NPDEs [6, 7, 8] and in a special case, the higher order Burgers type equations [9, 10, 11].

## 2. THE GRPSM

Consider the $(m+1)$-dimensional, $n^{th}$-order NPDE in general form

$$\partial_t^n u(x_i,t) = F(t,x_i;u,\partial_t u,\cdots,\partial_t^{n-1}u,\partial_{x_i}u,\partial_{x_i}^2 u,\cdots),(x_i,t)\in \mathbf{R}^m\times[0,T), \quad (2.1)$$

for $i=1,...,m$. Where $F$ is assumed to be sufficiently smooth on the indicated domain $D$ contains the starting values. $\partial_i^q u$ represents the $q^{th}$ derivative of the analytic function $u(x_i,t)$ with respect to independent variable $t$, and in the same way for other independent variables.

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The generalized RPSM(GRPSM) assumes the solution $u(x_i,t)$, of Eq. (2.1), in a form of power series

$$u(x_i,t)=\sum_{j}^{\infty}\sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}a_{i_1,\ldots,i_m,j}\left(\prod_{k=1}^{m}(x_k-x_{k_0})\right)^{i_k}(t-t_0)^j. \tag{2.2}$$

In more compact form

$$u(\mathbf{x},t)=\sum_{j=0}^{\infty}\xi_j(\mathbf{x})(t-t_0)^j, \tag{2.3}$$

where

$$\xi_j(\mathbf{x})=\sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}a_{i_1,\ldots,i_m,j}\left(\prod_{p=1}^{m}(x_k-x_{k_0})^{i_p}\right)\text{ and }\mathbf{x}=(x_1,\ldots,x_m).$$

subject to the initial conditions

$$u(\mathbf{x},t_0)=u_0(\mathbf{x}),\partial_t u(\mathbf{x},t_0)=u_1(\mathbf{x}),\ldots,\partial_t^{n-1}u(\mathbf{x},t_0)=u_{n-1}(\mathbf{x}). \tag{2.4}$$

The initial approximation of $u(\mathbf{x},t)$ will be

$$u_{n-1}(\mathbf{x},t)=\sum_{j=0}^{n-1}\frac{1}{j!}\partial_t^j\left(u(\mathbf{x},t_0)\right)(t-t_0)^j=\sum_{j=0}^{n-1}\xi_j(\mathbf{x})(t-t_0)^j. \tag{2.5}$$

The $k^{th}$-order approximate solution is defined by the truncated series

$$u_k(\mathbf{x},t)=u_{n-1}(\mathbf{x},t)+\sum_{j=n}^{k}\xi_j(\mathbf{x})(t-t_0)^j,k\geq n \tag{2.6}$$

subject to

$$\lim_{t\to t_0}\text{Res}_k(\mathbf{x},t)=0 \tag{2.7}$$

where $\text{Res}_k(\mathbf{x},t)$ is the well–defined $k^{th}$-residual function is defined by

$$\text{Res}_k(\mathbf{x},t)=\partial_t^k u_k(\mathbf{x},t)-\partial_t^{k-n}F(t,\mathbf{x};u,\partial_t u_k,\ldots,\partial_t^{n-1}u_k,\partial_{\mathbf{x}}u,\partial_{\mathbf{x}}^2u,\ldots) \tag{2.8}$$

which represents the basic idea of the GRPSM. The exact analytic solution of the initial value problem, Eq. (2.1) and Eq. (2.4), is given by

$$u(\mathbf{x},t)=\lim_{k\to\infty}u_k(\mathbf{x},t) \tag{2.9}$$

Provided that the series has exact closed form.

## 3. CONVERGENCE ANALYSIS

In this part, convergence analysis and error estimating for using GRPSM are studied. The presented scheme, under considerations mentioned in the previous section, approaches the exact analytic solution as more and more terms found.

**Theorem 3.1.** *If $F$ is an analytic operator on an open interval $I$ containing $t_0$, then the residual function $Res_k(\mathbf{x}, t)$ vanishes as $k$ approaches the infinity.*

*Proof.* Since $F$ is assumed to be analytic as well as for $u(\mathbf{x}, t)$. It is obvious by the definition of residual function Eq. (2.7). □

**Lemma 3.2.** *Suppose that $u(\mathbf{x}, t) = \sum_{j=0}^{\infty} \xi_i(\mathbf{x})(t - t_0)^j$, then*

$$[\partial_t^q u(\mathbf{x}, t)]_{t=t_0} = q!\xi_q(\mathbf{x}),\ q \in \mathbf{N}. \tag{3.1}$$

*Proof.* The $q^{th}$-derivative of $u(\mathbf{x}, t)$ with respect to $t$ is continuous at $t = t_0$. Therefore

$$\begin{aligned}
[\partial_t^q u(\mathbf{x}, t)]_{t=t_0} &= \lim_{t\to t_0} \partial_t^q u(\mathbf{x}, t) \\
&= \lim_{t\to t_0} \partial_t^q \left(\sum_{i=0}^{\infty} \xi_j(\mathbf{x})(t - t_0)^j\right) \\
&= \lim_{t\to t_0} \left(\sum_{i=0}^{\infty} \xi_j(\mathbf{x})\partial_t^q (t - t_0)^j\right) \\
&= \lim_{t\to t_0} \left(\sum_{i=0}^{\infty} \frac{(q + j)!}{j!} \xi_{q+j}(\mathbf{x})(t - t_0)^j\right) \\
&= \sum_{i=0}^{\infty} \left(\frac{(q + j)!}{j!} \xi_{q+j}(\mathbf{x}) \lim_{t\to t_0} (t - t_0)^j\right) \\
&= q!\xi_q(\mathbf{x})
\end{aligned}$$

□

**Theorem 3.3.** *The approximate truncated series solution $u_k(\mathbf{x}, t)$ defined in Eq. (2.6) and obtained by applying the GRPSM for solving the Eqs. (2.1) and (2.4) is the $k^{th}$ Taylor polynomial of $u(\mathbf{x}, t)$ about $t = t_0$. In general, as $k \to \infty$, the series solution in Eq. (2.9) concise the Taylor series expansion of $u(\mathbf{x}, t)$ centered at $t = t_0$.*

*Proof.* For $k < n$, it is clear from the initial approximation of $u(\mathbf{x}, t)$ in Eq. (2.5). For $k \geq n$ , it suffices to prove that

$$\left[\partial_t^k u(\mathbf{x}, t)\right]_{t=t_0} = \lim_{t\to t_0} \partial_t^{k-n} F(t, \mathbf{x}; u, \partial_t u_k, \ldots, \partial_t^{n-1} u_k, \partial_\mathbf{x} u, \partial_\mathbf{x}^2 u, \ldots). \tag{3.2}$$

Applying Eq. (2.7) to the $k^{th}$-order approximate solution given in Eq. (2.6), and using the result of Theorem 3.1, we get

$$\begin{aligned} 0 &= \lim_{k\to\infty} \mathrm{Res}_k(\mathbf{x},t) \\ &= \lim_{t\to t_0} \Big(\partial_t^k u_k(\mathbf{x},t) - \partial_t^{k-n} F(t,\mathbf{x}; u, \partial_t u_k, \dots, \partial_t^{n-1} u_k, \partial_{\mathbf{x}} u, \partial_{\mathbf{x}}^2 u, \dots)\Big) \\ &= \lim_{t\to t_0} \partial_t^k u_k(\mathbf{x},t) - \lim_{t\to t_0} \partial_t^{k-n} F(t,\mathbf{x}; u, \partial_t u_k, \dots, \partial_t^{n-1} u_k, \partial_{\mathbf{x}} u, \partial_{\mathbf{x}}^2 u, \dots) \end{aligned}$$

As a result of Lemma 3.2,

$$0 = q!\xi_q(\mathbf{x}) - \lim_{t\to t_0} \partial_t^{k-n} F(t,\mathbf{x}; u, \partial_t u_k, \dots, \partial_t^{n-1} u_k, \partial_{\mathbf{x}} u, \partial_{\mathbf{x}}^2 u, \dots).$$

Which completes the proof. □

**Corollary 3.4.** *Suppose that the truncated series $u_k(\mathbf{x},t)$ defined in Eq. (2.6) is used as an approximation to the solution $u(\mathbf{x},t)$ of problem Eqs. (2.1)–(2.4) on*

$$S = \big\{(\mathbf{x},t) : \mathbf{x} \in \mathbf{R}^{\mathbf{m}}, |t - t_0| < \rho\big\}$$

*then numbers $\eta(t)$, satisfies $|\eta(t) - t_0| \le \rho$, and $\mu_k > 0$ exist with*

$$|u(\mathbf{x},t) - u_k(\mathbf{x},t)| \le \frac{\mu_k}{(k+1)!}\rho^{k+1}. \tag{3.3}$$

*Proof.* Theorem 3.1 implies that

$$u(\mathbf{x},t) - u_k(\mathbf{x},t) = \sum_{j=k+1}^{\infty} \frac{1}{j!}\partial_t^j \left(u(\mathbf{x},t)\right)(t-t_0)^j.$$

Following the proof of Taylor's Theorem [12], a number $\eta(t) \in (t_0 - \rho, t_0 + \rho)$ exists with

$$u(\mathbf{x},t) - u_k(\mathbf{x},t) = \frac{\partial_t^{k+1} u\left(\mathbf{x}, \eta(t)\right)}{(k+1)!}(t-t_0)^{k+1}.$$

Since the $(k+1)^{st}$-derivative of the analytic function $u(\mathbf{x},t)$ with respect to $t$ is bounded on $S$, a number $\mu_k$ also exists with $|\partial_t^{k+1} u(\mathbf{x},t)| \le \mu_k$ for all $t \in [t_0 - \rho, t_0 + \rho]$ . Hence error bound in Eq. (3.3) is obtained. □

**Corollary 3.5.** *The GRPSM results the exact analytic solution $u(\mathbf{x},t)$ if it is a polynomial of t.*

## 4. NUMERICAL ILLUSTRATION

To illustrate the technique discussed in Section 2, we consider the $(m+1)$-dimensional nonlinear Burgers equation [13, 14]

$$\partial_t u(\mathbf{x},t) = u(\mathbf{x},t)\partial_{x_1} u(\mathbf{x},t) + \sum_{j=1}^{m} \partial_{x_j}^2 u(\mathbf{x},t), |t| < 1 \tag{4.1}$$

subject to the initial condition

$$u(\mathbf{x},0)=\sum_{j=1}^{m}x_j.$$

Eq. (4.1) is also known as Richards equation, which is used in the study of cellular automata, and interacting particle systems. It describes the flow pattern of the particle in a lattice fluid past an impenetrable obstacle; it can be also used as a model to describe the water flow in soils. Applying the generalized residual power series mechanism to suggested problem, the initial approximation is $u_0(\mathbf{x},t)=\sum_{j=1}^{m}x_j$, and the $k^{th}$-order approximate solution has the form

$$u_k(\mathbf{x},t)=u_0(\mathbf{x},t)+\sum_{j=n}^{k}\xi_j(\mathbf{x})t^j,\quad k\geq 1$$

which satisfies

$$\lim_{t\to 0}\left(\partial_t^k u_k(\mathbf{x},t)-\partial_t^{k-1}\left(u_k(\mathbf{x},t)\partial_{x_1}u_k(\mathbf{x},t)+\sum_{j=1}^{m}\partial_{x_j}^2 u_k(\mathbf{x},t)\right)\right)=0.$$

For $k=1$, we have $u_1(\mathbf{x},t)=u_0(\mathbf{x},t)+\xi_1(\mathbf{x})t$ and

$$\lim_{t\to 0}\left(\partial_t u_1(\mathbf{x},t)-\left(u_1(\mathbf{x},t)\partial_{x_1}u_1(\mathbf{x},t)+\sum_{j=1}^{m}\partial_{x_j}^2 u_1(\mathbf{x},t)\right)\right)=0.$$

Hence we get, $\xi_1(\mathbf{x})-\sum_{j=1}^{m}x_j=0$ and therefore $\xi_1(\mathbf{x})=\sum_{j=1}^{m}x_j$. Repeating this procedure for $k=2,3,\ldots$, we obtain that $\xi_2(\mathbf{x})=\xi_3(\mathbf{x})=\cdots=\sum_{j=1}^{m}x_j$. As $k\to\infty$, the solution takes the form

$$u(\mathbf{x},t)=\sum_{j=1}^{\infty}x_j t^j=\frac{\sum_{j=1}^{\infty}x_j}{1-t}.$$

The series solution leads to the exact solution obtained by Taylor's expansion.

## 5. Conclusions

In this work, we have improved an analytic solution procedure, called the generalized residual power series method, for solving higher dimensional partial differential equations. The results validate the efficiency and reliability of the aforesaid technique that are achieved by handling the $(m+1)$-dimensional Burgers equation. The method is a powerful mathematical tool for solving a wide range of problems arising in engineering and sciences.

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