

STABILITY OF A CLASS OF DISCRETE-TIME PATHOGEN INFECTION MODELS WITH LATENTLY INFECTED CELLS

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ABSTRACT. This paper studies the global stability of a class of discrete-time pathogen infection models with latently infected cells. The rate of pathogens infect the susceptible cells is taken as bilinear, saturation and general. The continuous-time models are discretized by using nonstandard finite difference scheme. The basic and global properties of the models are established. The global stability analysis of the equilibria is performed using Lyapunov method. The theoretical results are illustrated by numerical simulations.

1. INTRODUCTION

Pathogen infections (such as HIV, HBV, HCV, CHIKV and HTLV, etc.) models which describe within-host dynamics have been described by system of nonlinear ordinary or delay differential equations (see e.g. [1]-[32]). The use of digital computers in performing simulations necessitated the investigation of discrete-time systems. Further, it is important to note that scientists often collect the data and analyze the results at discrete times. For the models presented in the above mentioned papers, the exact analytical solutions are unknown. Therefore, a discretization can be used to obtain discrete-time model which is an approximation of the exact one. One of the very important task is to choose a discretization scheme which preserves the properties of the corresponding continuous time model. In 1994 Mickens [33] has introduced nonstandard finite difference (NSFD) scheme for solving differential equations. It has been proven that NSFD can preserve the main properties of several types of continuous time models. NSFD has been used to investigate the global stability of equilibria of the corresponding continuous time models in epidemiology [35]-[39] and virology [40]-[52].

The basic and pioneering model describing the pathogen dynamics is due Nowak and Bangham [1]. The model contains three compartments: susceptible (or uninfected) cells, infected

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cells and free pathogens. A lot of considerations have been added that aim to get the best representation of the pathogen infection. Most notable are latent pathogen reservoirs which serve as a major barrier in curing pathogen infection. Despite the fact that the antiretroviral therapy significantly limits the level of pathogen in the blood, there is still a low viral load due to ongoing reactivation of latent infected cells reservoirs. Variant models have been developed to study the dynamics of pathogen in the presence of latent reservoirs (see, e.g., [17], [26]-[32]). However, all the models presented in these papers are given by continuous-time. In this paper, our target is to study a class of discrete time pathogen infection models with latently infected cells. We study the qualitative behavior of the models with different forms of infection rates. We investigate global stability of the equilibria of the models using Lyapunov method. The theoretical results are supported by numerical simulations.

2. MODEL WITH BILINEAR INCIDENCE

We propose the following continuous pathogen infection model with latency:

$$\dot{s} = \beta - \delta s - \kappa sp, \quad (2.1)$$

$$\dot{w} = (1 - \epsilon)\kappa sp - (\alpha + m)w, \quad (2.2)$$

$$\dot{u} = \epsilon\kappa sp + mw - \gamma u, \quad (2.3)$$

$$\dot{p} = \theta u - \eta p. \quad (2.4)$$

where s , w , u , and p are the concentrations of susceptible cells, latently infected cells, actively infected cells and free pathogens, respectively. Parameters β and δ represent the birth rate and death rate constants of the susceptible cells, respectively. Susceptible cells become infected at rate κsp , where κ is the incidence rate constant. Parts ϵ and $(1 - \epsilon)$ with $0 < \epsilon < 1$ of infected cells are assumed to be latent and active, respectively. The term mw represents the activation rate of the latently infected cells. Free pathogens are produced at rate θu . The death rate of latently infected cells, actively infected cells and pathogen particles are given by αw , γu and ηp , respectively.

Discretizing system (2.1)-(2.4) using NSFD method [33]-[34] we obtain

$$s_{n+1} - s_n = \beta - \delta s_{n+1} - \kappa s_{n+1} p_n, \quad (2.5)$$

$$w_{n+1} - w_n = (1 - \epsilon)\kappa s_{n+1} p_n - (\alpha + m)w_{n+1}, \quad (2.6)$$

$$u_{n+1} - u_n = \epsilon\kappa s_{n+1} p_n + mw_{n+1} - \gamma u_{n+1}, \quad (2.7)$$

$$p_{n+1} - p_n = \theta u_{n+1} - \eta p_{n+1}, \quad (2.8)$$

where $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

We consider the initial conditions:

$$(s_0, w_0, u_0, p_0) \in \mathbb{R}_+^4 = \{(s, w, u, p) \mid s > 0, w > 0, u > 0, p > 0\}. \quad (2.9)$$

2.1. Preliminaries. Let us consider the region

$$\Gamma_1 = \{(s, w, u, p) : 0 < s, w, u < N_1, 0 < p < N_2\}$$

where $N_1 = \frac{\beta}{\delta}$, $N_2 = \frac{2\theta}{\gamma} N_1$ and $\xi = \min \{\delta, \alpha, \frac{\gamma}{2}, \eta\}$.

Lemma 2.1. Any solution (s_n, w_n, u_n, p_n) of model (2.5)-(2.8) with initial conditions (2.9) is positive and ultimately bounded.

Proof. From Eqs. (2.5)-(2.8) we obtain

$$s_{n+1} = \frac{\beta + s_n}{1 + \delta + \kappa p_n}, \tag{2.10}$$

$$w_{n+1} = \frac{w_n}{1 + \alpha + m} + \frac{(1 - \epsilon)\kappa(\beta + s_n)p_n}{(1 + \alpha + m)(1 + \delta + \kappa p_n)}, \tag{2.11}$$

$$u_{n+1} = \frac{u_n}{1 + \gamma} + \frac{\epsilon\kappa(\beta + s_n)p_n}{(1 + \gamma)(1 + \delta + \kappa p_n)} + \frac{mw_n}{(1 + \gamma)(1 + \alpha + m)} + \frac{m(1 - \epsilon)\kappa(\beta + s_n)p_n}{(1 + \gamma)(1 + \alpha + m)(1 + \delta + \kappa p_n)}, \tag{2.12}$$

$$p_{n+1} = \frac{p_n}{1 + \eta} + \frac{\theta u_n}{(1 + \eta)(1 + \gamma)} + \frac{\theta\epsilon\kappa(\beta + s_n)p_n}{(1 + \eta)(1 + \gamma)(1 + \delta + \kappa p_n)} + \frac{\theta mw_n}{(1 + \eta)(1 + \gamma)(1 + \alpha + m)} + \frac{\theta m(1 - \epsilon)\kappa(\beta + s_n)p_n}{(1 + \eta)(1 + \gamma)(1 + \alpha + m)(1 + \delta + \kappa p_n)}. \tag{2.13}$$

Since all parameters of model (2.5)-(2.8) are positive, then by induction we get $s_n > 0, w_n > 0, u_n > 0$ and $p_n > 0$ for all $n \in \mathbb{N}$.

To investigate the boundedness of solutions we define the following sequence M_n :

$$M_n = s_n + w_n + u_n + \frac{\gamma}{2\theta}p_n.$$

Then

$$M_{n+1} = M_n + \beta - \delta s_{n+1} - \alpha w_{n+1} - \frac{\gamma}{2}u_{n+1} - \frac{\gamma\eta}{2\theta}p_{n+1} \leq M_n + \beta - \xi M_{n+1}.$$

Hence

$$M_{n+1} \leq \frac{M_n}{1 + \xi} + \frac{\beta}{1 + \xi}.$$

By Lemma 2.2 in [41] we have

$$M_n \leq \left(\frac{1}{1 + \xi}\right)^n M_0 + \frac{\beta}{\xi} \left[1 - \left(\frac{1}{1 + \xi}\right)^n\right].$$

Consequently, $\limsup_{n \rightarrow \infty} M_n \leq N_1, \limsup_{n \rightarrow \infty} s_n \leq N_1, \limsup_{n \rightarrow \infty} w_n \leq N_1, \limsup_{n \rightarrow \infty} u_n \leq N_1$ and $\limsup_{n \rightarrow \infty} p_n \leq N_2$. Therefore, the solution (s_n, w_n, u_n, p_n) converges to Γ_1 as $n \rightarrow \infty$. □

The basic reproduction number of model (2.5)-(2.8) is given by:

$$\mathcal{R}_0 = \frac{\kappa\theta\beta(\alpha\epsilon + m)}{\delta\gamma\eta(\alpha + m)}. \tag{2.14}$$

System (2.5)-(2.8) has two equilibria,

- (i) pathogen-free equilibrium $Q^0(s^0, 0, 0, 0)$ where $s^0 = \beta/\delta$.
(ii) persistent pathogen equilibrium $Q^*(s^*, w^*, u^*, p^*)$, where

$$s^* = \frac{s^0}{\mathcal{R}_0}, \quad w^* = \frac{\delta\gamma\eta(1-\epsilon)}{\theta\kappa(\alpha\epsilon+m)}(\mathcal{R}_0-1), \quad u^* = \frac{\delta\eta}{\theta\kappa}(\mathcal{R}_0-1), \quad p^* = \frac{\delta}{\kappa}(\mathcal{R}_0-1).$$

Clearly, Q^* exists only when $\mathcal{R}_0 > 1$.

2.2. Global Stability. We define the function $G(x) \geq 0$ as $G(x) = x - \ln x - 1$. Hence,

$$\ln x \leq x - 1. \quad (2.15)$$

Theorem 2.2. *If $\mathcal{R}_0 \leq 1$, then Q^0 is globally asymptotically stable.*

Proof. Construct a discrete Lyapunov function $L_n(s_n, w_n, u_n, p_n)$ as:

$$L_n = s^0 G\left(\frac{s_n}{s^0}\right) + \frac{m}{\alpha\epsilon+m}w_n + \frac{\alpha+m}{\alpha\epsilon+m}u_n + \frac{\gamma(\alpha+m)}{\theta(\alpha\epsilon+m)}(1+\eta)p_n.$$

Hence, $L_n > 0$ for all $s_n > 0$, $w_n > 0$, $u_n > 0$ and $p_n > 0$. In addition, $L_n = 0$ if and only if $s_n = s^0$, $w_n = 0$, $u_n = 0$ and $p_n = 0$. Computing the difference $\Delta L_n = L_{n+1} - L_n$ as:

$$\begin{aligned} \Delta L_n &= s^0 G\left(\frac{s_{n+1}}{s^0}\right) + \frac{m}{\alpha\epsilon+m}w_{n+1} + \frac{\alpha+m}{\alpha\epsilon+m}u_{n+1} + \frac{\gamma(\alpha+m)}{\theta(\alpha\epsilon+m)}(1+\eta)p_{n+1} \\ &\quad - \left[s^0 G\left(\frac{s_n}{s^0}\right) + \frac{m}{\alpha\epsilon+m}w_n + \frac{\alpha+m}{\alpha\epsilon+m}u_n + \frac{\gamma(\alpha+m)}{\theta(\alpha\epsilon+m)}(1+\eta)p_n \right] \\ &= s^0 \left(\frac{s_{n+1}}{s^0} - \frac{s_n}{s^0} + \ln \frac{s_n}{s_{n+1}} \right) + \frac{m}{\alpha\epsilon+m}(w_{n+1} - w_n) + \frac{\alpha+m}{\alpha\epsilon+m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha+m)}{\theta(\alpha\epsilon+m)}(1+\eta)(p_{n+1} - p_n). \end{aligned}$$

Using inequality (2.15), we have

$$\begin{aligned} \Delta L_n &\leq s_{n+1} - s_n + s^0 \left(\frac{s_n}{s_{n+1}} - 1 \right) + \frac{m}{\alpha\epsilon+m}(w_{n+1} - w_n) + \frac{\alpha+m}{\alpha\epsilon+m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha+m)}{\theta(\alpha\epsilon+m)}(1+\eta)(p_{n+1} - p_n) \\ &= \left(1 - \frac{s^0}{s_{n+1}} \right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon+m}(w_{n+1} - w_n) + \frac{\alpha+m}{\alpha\epsilon+m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha+m)}{\theta(\alpha\epsilon+m)}(1+\eta)(p_{n+1} - p_n). \end{aligned}$$

From Eqs. (2.5)-(2.8), we have

$$\begin{aligned} \Delta L_n &\leq \left(1 - \frac{s^0}{s_{n+1}}\right) (\beta - \delta s_{n+1} - \kappa s_{n+1} p_n) + \frac{m}{\alpha\epsilon + m} ((1 - \epsilon)\kappa s_{n+1} p_n - (\alpha + m)w_{n+1}) \\ &\quad + \frac{(\alpha + m)}{\alpha\epsilon + m} (\epsilon\kappa s_{n+1} p_n + mw_{n+1} - \gamma u_{n+1}) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (\theta u_{n+1} - \eta p_{n+1}) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} (p_{n+1} - p_n) \\ &= \left(1 - \frac{s^0}{s_{n+1}}\right) (\beta - \delta s_{n+1}) + \kappa s^0 p_n - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p_n \\ &= \frac{-\delta}{s_{n+1}} (s_{n+1} - s^0)^2 + \left(\kappa s^0 - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)}\right) p_n \\ &= \frac{-\delta}{s_{n+1}} (s_{n+1} - s^0)^2 + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(\frac{\kappa\theta\beta(\alpha\epsilon + m)}{\delta\gamma\eta(\alpha + m)} - 1\right) p_n \\ &= \frac{-\delta}{s_{n+1}} (s_{n+1} - s^0)^2 + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} (\mathcal{R}_0 - 1) p_n. \end{aligned}$$

Hence, for $\mathcal{R}_0 \leq 1$, we have $\Delta L_n \leq 0$ for all $n \geq 0$, hence L_n is a monotone decreasing sequence. We have $L_n \geq 0$, then there is a limit $\lim_{n \rightarrow \infty} L_n \geq 0$. Therefore, $\lim_{n \rightarrow \infty} \Delta L_n = 0$, which implies that $\lim_{n \rightarrow \infty} s_{n+1} = s^0$ and $\lim_{n \rightarrow \infty} (\mathcal{R}_0 - 1)p_n = 0$. For the case $\mathcal{R}_0 < 1$, we have $\lim_{n \rightarrow \infty} s_{n+1} = s^0$ and $\lim_{n \rightarrow \infty} p_n = 0$. From Eqs. (2.5)-(2.7), we obtain $\lim_{n \rightarrow \infty} w_n = 0$ and $\lim_{n \rightarrow \infty} u_n = 0$. For the case $\mathcal{R}_0 = 1$, we have $\lim_{n \rightarrow \infty} s_{n+1} = s^0$. From Eqs. (2.5)-(2.7), we obtain $\lim_{n \rightarrow \infty} p_n = 0$, $\lim_{n \rightarrow \infty} u_n = 0$ and $\lim_{n \rightarrow \infty} w_n = 0$. Hence, in the case $\mathcal{R}_0 \leq 1$, the pathogen-free equilibrium Q^0 is globally asymptotically stable. \square

Theorem 2.3. *If $\mathcal{R}_0 > 1$, then Q^* is globally asymptotically stable.*

Proof. Define

$$\begin{aligned} U_n(s_n, w_n, u_n, p_n) &= s^* G\left(\frac{s_n}{s^*}\right) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_n}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_n}{u^*}\right) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_n}{p^*}\right). \end{aligned}$$

Clearly, $U_n(s_n, w_n, u_n, p_n) > 0$ for all $s_n, w_n, u_n, p_n > 0$ and $U_n(s^*, w^*, u^*, p^*) = 0$. Computing $\Delta U_n = U_{n+1} - U_n$ as:

$$\begin{aligned} \Delta U_n &= s^* G\left(\frac{s_{n+1}}{s^*}\right) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_{n+1}}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_{n+1}}{u^*}\right) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_{n+1}}{p^*}\right) \\ &\quad - \left[s^* G\left(\frac{s_n}{s^*}\right) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_n}{w^*}\right) + \frac{(\alpha + m)}{\alpha\epsilon + m} u^* G\left(\frac{u_n}{u^*}\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_n}{p^*}\right) \right] \\ &= s^* \left(\frac{s_{n+1}}{s^*} - \frac{s_n}{s^*} + \ln \frac{s_n}{s_{n+1}} \right) + \frac{mw^*}{\alpha\epsilon + m} \left(\frac{w_{n+1}}{w^*} - \frac{w_n}{w^*} + \ln \frac{w_n}{w_{n+1}} \right) \\ &\quad + \frac{(\alpha + m)u^*}{\alpha\epsilon + m} \left(\frac{u_{n+1}}{u^*} - \frac{u_n}{u^*} + \ln \frac{u_n}{u_{n+1}} \right) + \frac{\gamma(\alpha + m)p^*}{\theta(\alpha\epsilon + m)} \left(\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} \right) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right]. \end{aligned}$$

Using inequality (2.15), we get

$$\begin{aligned} \Delta U_n &\leq s^* \left(\frac{s_{n+1} - s_n}{s^*} + \frac{s_n}{s_{n+1}} - 1 \right) + \frac{mw^*}{\alpha\epsilon + m} \left(\frac{w_{n+1} - w_n}{w^*} + \frac{w_n}{w_{n+1}} - 1 \right) \\ &\quad + \frac{(\alpha + m)u^*}{\alpha\epsilon + m} \left(\frac{u_{n+1} - u_n}{u^*} + \frac{u_n}{u_{n+1}} - 1 \right) + \frac{\gamma(\alpha + m)p^*}{\theta(\alpha\epsilon + m)} \left(\frac{p_{n+1} - p_n}{p^*} + \frac{p_n}{p_{n+1}} - 1 \right) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right] \\ &= \left(1 - \frac{s^*}{s_{n+1}} \right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}} \right) (w_{n+1} - w_n) \\ &\quad + \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}} \right) (u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}} \right) (p_{n+1} - p_n) + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right]. \end{aligned}$$

From Eqs. (2.5)-(2.8), we have

$$\begin{aligned} \Delta U_n \leq & \left(1 - \frac{s^*}{s_{n+1}}\right) (\beta - \delta s_{n+1} - \kappa s_{n+1} p_n) \\ & + \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}}\right) ((1 - \epsilon)\kappa s_{n+1} p_n - (\alpha + m)w_{n+1}) \\ & + \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}}\right) (\epsilon\kappa s_{n+1} p_n + mw_{n+1} - \gamma u_{n+1}) \\ & + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}}\right) (\theta u_{n+1} - \eta p_{n+1}) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right]. \end{aligned}$$

Since $\beta = \delta s^* + \kappa s^* p^*$, then

$$\begin{aligned} \Delta U_n \leq & \left(1 - \frac{s^*}{s_{n+1}}\right) (\delta s^* + \kappa s^* p^* - \delta s_{n+1} - \kappa s_{n+1} p_n) \\ & + \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}}\right) ((1 - \epsilon)\kappa s_{n+1} p_n - (\alpha + m)w_{n+1}) \\ & + \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}}\right) (\epsilon\kappa s_{n+1} p_n + mw_{n+1} - \gamma u_{n+1}) \\ & + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}}\right) (\theta u_{n+1} - \eta p_{n+1}) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} \right]. \\ = & -\frac{\delta}{s_{n+1}} (s_{n+1} - s^*)^2 + \kappa s^* p^* \left(1 - \frac{s^*}{s_{n+1}}\right) + \kappa s^* p_n - \frac{m\kappa(1 - \epsilon)}{\alpha\epsilon + m} \frac{s_{n+1} p_n w^*}{w_{n+1}} \\ & + \frac{m(\alpha + m)}{\alpha\epsilon + m} w^* - \frac{\epsilon\kappa(\alpha + m)}{\alpha\epsilon + m} \frac{s_{n+1} p_n u^*}{u_{n+1}} - \frac{m(\alpha + m)}{\alpha\epsilon + m} \frac{u^* w_{n+1}}{u_{n+1}} + \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} u^* \\ & - \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} \frac{p^* u_{n+1}}{p_{n+1}} + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p_n + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \ln \frac{p_n}{p_{n+1}}. \end{aligned}$$

We have

$$\kappa s^* p_n - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p_n = 0.$$

Using the conditions of Q^*

$$\begin{aligned} (1 - \epsilon)\kappa s^* p^* &= (\alpha + m)w^* \\ \epsilon\kappa s^* p^* + mw^* &= \gamma u^* \\ \theta u^* &= \eta p^*, \end{aligned}$$

we get

$$\begin{aligned}\frac{m(\alpha+m)}{\alpha\epsilon+m}w^* &= \frac{m(1-\epsilon)\kappa s^*p^*}{\alpha\epsilon+m}, \\ \frac{\gamma(\alpha+m)}{\alpha\epsilon+m}u^* &= \frac{\gamma\eta(\alpha+m)}{\theta(\alpha\epsilon+m)}p^* = \kappa s^*p^*, \\ \kappa s^*p^* &= \frac{m(1-\epsilon)\kappa s^*p^*}{\alpha\epsilon+m} + \frac{\epsilon(\alpha+m)\kappa s^*p^*}{\alpha\epsilon+m},\end{aligned}$$

and

$$\begin{aligned}\Delta U_n &\leq \frac{-\delta}{s_{n+1}}(s_{n+1}-s^*)^2 + \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* \left(1 - \frac{s^*}{s_{n+1}}\right) + \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* \left(1 - \frac{s^*}{s_{n+1}}\right) \\ &\quad - \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* \frac{s_{n+1}p_n w^*}{s^*p^*w_{n+1}} + \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* - \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* \frac{s_{n+1}p_n u^*}{s^*p^*u_{n+1}} \\ &\quad - \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* \frac{u^*w_{n+1}}{u_{n+1}w^*} + \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* + \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* - \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* \frac{p^*u_{n+1}}{p_{n+1}u^*} \\ &\quad - \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* \frac{p^*u_{n+1}}{p_{n+1}u^*} + \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* + \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* + \kappa s^*p^* \ln \frac{p_n}{p_{n+1}} \\ &= \frac{-\delta}{s_{n+1}}(s_{n+1}-s^*)^2 \\ &\quad + \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* \left[4 - \frac{s^*}{s_{n+1}} - \frac{s_{n+1}p_n w^*}{s^*p^*w_{n+1}} - \frac{u^*w_{n+1}}{u_{n+1}w^*} - \frac{p^*u_{n+1}}{p_{n+1}u^*} + \ln \frac{p_n}{p_{n+1}}\right] \\ &\quad + \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* \left[3 - \frac{s^*}{s_{n+1}} - \frac{s_{n+1}p_n u^*}{s^*p^*u_{n+1}} - \frac{p^*u_{n+1}}{p_{n+1}u^*} + \ln \frac{p_n}{p_{n+1}}\right] \\ &= \frac{-\delta}{s_{n+1}}(s_{n+1}-s^*)^2 - \frac{m(1-\epsilon)}{\alpha\epsilon+m}\kappa s^*p^* \left[G\left(\frac{s^*}{s_{n+1}}\right) + G\left(\frac{s_{n+1}p_n w^*}{s^*p^*w_{n+1}}\right)\right. \\ &\quad \left.+ G\left(\frac{u^*w_{n+1}}{u_{n+1}w^*}\right) + G\left(\frac{p^*u_{n+1}}{p_{n+1}u^*}\right)\right] \\ &\quad - \frac{\epsilon(\alpha+m)}{\alpha\epsilon+m}\kappa s^*p^* \left[G\left(\frac{s^*}{s_{n+1}}\right) + G\left(\frac{s_{n+1}p_n u^*}{s^*p^*u_{n+1}}\right) + G\left(\frac{p^*u_{n+1}}{p_{n+1}u^*}\right)\right].\end{aligned}$$

Thus, U_n is monotone decreasing sequence. Because $U_n \geq 0$, there is a limit $\lim_{n \rightarrow \infty} U_n \geq 0$. Therefore, $\lim_{n \rightarrow \infty} \Delta U_n = 0$, which implies $\lim_{n \rightarrow \infty} s_n = s^*$, $\lim_{n \rightarrow \infty} w_n = w^*$, $\lim_{n \rightarrow \infty} u_n = u^*$ and $\lim_{n \rightarrow \infty} p_n = p^*$. \square

3. MODEL WITH SATURATED INCIDENCE

It has been reported in [15]-[16] that pathogen dynamics model with saturated incidence is more accurate in case of high concentration of the pathogens. Thus we consider the following

model:

$$\dot{s} = \beta - \delta s - \frac{\kappa s p}{1 + \mu p}, \tag{3.1}$$

$$\dot{w} = \frac{(1 - \epsilon)\kappa s p}{1 + \mu p} - (\alpha + m)w, \tag{3.2}$$

$$\dot{u} = \frac{\epsilon\kappa s p}{1 + \mu p} + mw - \gamma u, \tag{3.3}$$

$$\dot{p} = \theta u - \eta p. \tag{3.4}$$

where μ is the saturation constant. Using the NSFD method we obtain

$$s_{n+1} - s_n = \beta - \delta s_{n+1} - \frac{\kappa s_{n+1} p_n}{1 + \mu p_n}, \tag{3.5}$$

$$w_{n+1} - w_n = \frac{(1 - \epsilon)\kappa s_{n+1} p_n}{1 + \mu p_n} - (\alpha + m)w_{n+1}, \tag{3.6}$$

$$u_{n+1} - u_n = \frac{\epsilon\kappa s_{n+1} p_n}{1 + \mu p_n} + mw_{n+1} - \gamma u_{n+1}, \tag{3.7}$$

$$p_{n+1} - p_n = \theta u_{n+1} - \eta p_{n+1}. \tag{3.8}$$

3.1. Preliminaries.

Lemma 3.1. *Any solution (s_n, w_n, u_n, p_n) of model (3.5)-(3.8) with initial conditions (2.9) is positive and ultimately bounded.*

Proof. From Eqs. (3.5)-(3.8) we obtain

$$s_{n+1} = \frac{(\beta + s_n)(1 + \mu p_n)}{1 + \delta + (\mu(1 + \delta) + \kappa)p_n}, \tag{3.9}$$

$$w_{n+1} = \frac{w_n}{1 + \alpha + m} + \frac{(1 - \epsilon)\kappa p_n(\beta + s_n)}{(1 + \alpha + m)(1 + \delta + (\mu(1 + \delta) + \kappa)p_n)}, \tag{3.10}$$

$$u_{n+1} = \frac{u_n}{1 + \gamma} + \frac{\epsilon\kappa p_n(\beta + s_n)}{(1 + \gamma)(1 + \delta + (\mu(1 + \delta) + \kappa)p_n)} + \frac{mw_n}{(1 + \gamma)(1 + \alpha + m)} + \frac{m(1 - \epsilon)\kappa p_n(\beta + s_n)}{(1 + \gamma)(1 + \alpha + m)(1 + \delta + (\mu(1 + \delta) + \kappa)p_n)}, \tag{3.11}$$

$$p_{n+1} = \frac{p_n}{1 + \eta} + \frac{\theta u_n}{(1 + \eta)(1 + \gamma)} + \frac{\theta\epsilon\kappa p_n(\beta + s_n)}{(1 + \eta)(1 + \gamma)(1 + \delta + (\mu(1 + \delta) + \kappa)p_n)} + \frac{\theta mw_n}{(1 + \eta)(1 + \gamma)(1 + \alpha + m)} + \frac{\theta m(1 - \epsilon)\kappa p_n(\beta + s_n)}{(1 + \eta)(1 + \gamma)(1 + \alpha + m)(1 + \delta + (\mu(1 + \delta) + \kappa)p_n)}. \tag{3.12}$$

The solution of (3.5)-(3.8) with initial (2.9) satisfies $s_n > 0, w_n > 0, u_n > 0$ and $p_n > 0$. The boundedness of solutions of model (3.5)-(3.8) is similar to the proof of Lemma 2.1. \square

System (3.5)-(3.8) has two equilibria

- (i) pathogen-free equilibrium $Q^0(s^0, 0, 0, 0)$ where $s^0 = \beta/\delta$.
(ii) persistent pathogen equilibrium $Q^*(s^*, w^*, u^*, p^*)$, where

$$s^* = \frac{\gamma\eta(\alpha + m) + \beta\theta\mu(\alpha\epsilon + m)}{\theta(\alpha\epsilon + m)(\mu\delta + \kappa)}, \quad w^* = \frac{\delta\gamma\eta(1 - \epsilon)}{\theta(\mu\delta + \kappa)(\alpha\epsilon + m)}(\mathcal{R}_0 - 1),$$

$$u^* = \frac{\delta\eta}{\theta(\mu\delta + \kappa)}(\mathcal{R}_0 - 1), \quad p^* = \frac{\delta}{(\mu\delta + \kappa)}(\mathcal{R}_0 - 1),$$

where \mathcal{R}_0 is given by Eq. (2.14)

3.2. Global Stability.

Theorem 3.2. *Suppose that $\mathcal{R}_0 \leq 1$, then Q^0 is globally asymptotically stable.*

Proof. Construct a Lyapunov function L_n as:

$$L_n(s_n, w_n, u_n, p_n) = s^0 G\left(\frac{s_n}{s^0}\right) + \frac{m}{\alpha\epsilon + m}w_n + \frac{\alpha + m}{\alpha\epsilon + m}u_n + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)p_n.$$

Hence, $L_n \geq 0$ for all $s_n > 0$, $w_n > 0$, $u_n > 0$ and $p_n > 0$. In addition, $L_n = 0$ if and only if $s_n = s^0$, $w_n = 0$, $u_n = 0$ and $p_n = 0$. Computing the difference $\Delta L_n = L_{n+1} - L_n$ as:

$$\begin{aligned} \Delta L_n &= s^0 G\left(\frac{s_{n+1}}{s^0}\right) + \frac{m}{\alpha\epsilon + m}w_{n+1} + \frac{\alpha + m}{\alpha\epsilon + m}u_{n+1} + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)p_{n+1} \\ &\quad - \left[s^0 G\left(\frac{s_n}{s^0}\right) + \frac{m}{\alpha\epsilon + m}w_n + \frac{\alpha + m}{\alpha\epsilon + m}u_n + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)p_n \right] \\ &= s^0 \left(\frac{s_{n+1}}{s^0} - \frac{s_n}{s^0} + \ln \frac{s_n}{s_{n+1}} \right) + \frac{m}{\alpha\epsilon + m}(w_{n+1} - w_n) + \frac{\alpha + m}{\alpha\epsilon + m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)(p_{n+1} - p_n). \end{aligned}$$

Using inequality (2.15), we have

$$\begin{aligned} \Delta L_n &\leq s_{n+1} - s_n + s^0 \left(\frac{s_n}{s_{n+1}} - 1 \right) + \frac{m}{\alpha\epsilon + m}(w_{n+1} - w_n) + \frac{\alpha + m}{\alpha\epsilon + m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)(p_{n+1} - p_n) \\ &= \left(1 - \frac{s^0}{s_{n+1}} \right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m}(w_{n+1} - w_n) + \frac{\alpha + m}{\alpha\epsilon + m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)(p_{n+1} - p_n). \end{aligned}$$

From Eqs. (3.5)-(3.8), we have

$$\begin{aligned} \Delta L_n &\leq \left(1 - \frac{s^0}{s_{n+1}}\right) \left(\beta - \delta s_{n+1} - \frac{\kappa s_{n+1} p_n}{1 + \mu p_n}\right) \\ &\quad + \frac{m}{\alpha\epsilon + m} \left(\frac{(1 - \epsilon)\kappa s_{n+1} p_n}{1 + \mu p_n} - (\alpha + m)w_{n+1}\right) \\ &\quad + \frac{\alpha + m}{\alpha\epsilon + m} \left(\frac{\epsilon\kappa s_{n+1} p_n}{1 + \mu p_n} + mw_{n+1} - \gamma u_{n+1}\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(\theta u_{n+1} - \eta p_{n+1}) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)}(p_{n+1} - p_n) \\ &= \left(1 - \frac{s^0}{s_{n+1}}\right) (\beta - \delta s_{n+1}) + \frac{\kappa s^0 p_n}{1 + \mu p_n} - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p_n \\ &= \frac{-\delta}{s_{n+1}}(s_{n+1} - s^0)^2 + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(\frac{\kappa\theta\beta(\alpha\epsilon + m)}{\delta\gamma\eta(\alpha + m)(1 + \mu p_n)} - 1\right) p_n \\ &= \frac{-\delta}{s_{n+1}}(s_{n+1} - s^0)^2 + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(\frac{\mathcal{R}_0}{1 + \mu p_n} - 1\right) p_n \\ &= \frac{-\delta}{s_{n+1}}(s_{n+1} - s^0)^2 - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} \frac{\mu\mathcal{R}_0}{(1 + \mu p_n)} p_n^2 + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} (\mathcal{R}_0 - 1) p_n. \end{aligned}$$

Hence, for $\mathcal{R}_0 \leq 1$, we have $\Delta L_n \leq 0$ for all $n \geq 0$. Hence, L_n is a monotone decreasing sequence. The proof can be completed similar to that of Theorem 2.2.

□

Theorem 3.3. *If $\mathcal{R}_0 > 1$, then Q^* is globally asymptotically stable.*

Proof. Let us consider

$$\begin{aligned} U_n(s_n, w_n, u_n, p_n) &= s^* G\left(\frac{s_n}{s^*}\right) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_n}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_n}{u^*}\right) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_n}{p^*}\right). \end{aligned}$$

Clearly, $U_n(s_n, w_n, u_n, p_n) > 0$ for all $s_n, w_n, u_n, p_n > 0$ and $U_n(s^*, w^*, u^*, p^*) = 0$. Computing $\Delta U_n = U_{n+1} - U_n$ as:

$$\begin{aligned} \Delta U_n &= s^* G\left(\frac{s_{n+1}}{s^*}\right) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_{n+1}}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_{n+1}}{u^*}\right) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_{n+1}}{p^*}\right) \\ &\quad - \left[s^* G\left(\frac{s_n}{s^*}\right) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_n}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_n}{u^*}\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_n}{p^*}\right) \right] \\ &= s^* \left(\frac{s_{n+1}}{s^*} - \frac{s_n}{s^*} + \ln \frac{s_n}{s_{n+1}} \right) + \frac{m}{\alpha\epsilon + m} w^* \left(\frac{w_{n+1}}{w^*} - \frac{w_n}{w^*} + \ln \frac{w_n}{w_{n+1}} \right) \\ &\quad + \frac{\alpha + m}{\alpha\epsilon + m} u^* \left(\frac{u_{n+1}}{u^*} - \frac{u_n}{u^*} + \ln \frac{u_n}{u_{n+1}} \right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left(\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} \right) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right]. \end{aligned}$$

Using inequality (2.15), we get

$$\begin{aligned} \Delta U_n &\leq s^* \left(\frac{s_{n+1} - s_n}{s^*} + \frac{s_n}{s_{n+1}} - 1 \right) + \frac{m}{\alpha\epsilon + m} w^* \left(\frac{w_{n+1} - w_n}{w^*} + \frac{w_n}{w_{n+1}} - 1 \right) \\ &\quad + \frac{\alpha + m}{\alpha\epsilon + m} u^* \left(\frac{u_{n+1} - u_n}{u^*} + \frac{u_n}{u_{n+1}} - 1 \right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left(\frac{p_{n+1} - p_n}{p^*} + \frac{p_n}{p_{n+1}} - 1 \right) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right] \\ &= \left(1 - \frac{s^*}{s_{n+1}} \right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}} \right) (w_{n+1} - w_n) \\ &\quad + \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}} \right) (u_{n+1} - u_n) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}} \right) (p_{n+1} - p_n) \\ &\quad + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right]. \end{aligned}$$

From Eqs. (3.5)-(3.8), we have

$$\begin{aligned} \Delta U_n &\leq \left(1 - \frac{s^*}{s_{n+1}}\right) \left(\beta - \delta s_{n+1} - \frac{\kappa s_{n+1} p_n}{1 + \mu p_n}\right) \\ &+ \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}}\right) \left(\frac{(1 - \epsilon)\kappa s_{n+1} p_n}{1 + \mu p_n} - (\alpha + m)w_{n+1}\right) \\ &+ \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}}\right) \left(\frac{\epsilon\kappa s_{n+1} p_n}{1 + \mu p_n} + mw_{n+1} - \gamma u_{n+1}\right) \\ &+ \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}}\right) (\theta u_{n+1} - \eta p_{n+1}) + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left(\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}}\right) \\ &= \left(1 - \frac{s^*}{s_{n+1}}\right) (\beta - \delta s_{n+1}) + \frac{\kappa s^* p_n}{1 + \mu p_n} - \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s_{n+1} p_n w^*}{(1 + \mu p_n) w_{n+1}} + \frac{m(\alpha + m)}{\alpha\epsilon + m} w^* \\ &- \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s_{n+1} p_n u^*}{(1 + \mu p_n) u_{n+1}} - \frac{m(\alpha + m)}{\alpha\epsilon + m} \frac{w_{n+1} u^*}{u_{n+1}} + \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} u^* - \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} \frac{p^* u_{n+1}}{p_{n+1}} \\ &+ \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left(-\frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}}\right). \end{aligned}$$

Using the conditions of Q^*

$$\begin{aligned} \beta &= \delta s^* + \frac{\kappa s^* p^*}{1 + \mu p^*}, \\ \frac{(1 - \epsilon)\kappa s^* p^*}{1 + \mu p^*} &= (\alpha + m)w^*, \\ \frac{\epsilon\kappa s^* p^*}{1 + \mu p^*} + mw^* &= \gamma u^*, \\ \theta u^* &= \eta p^*, \end{aligned}$$

we get

$$\begin{aligned} \frac{m(\alpha + m)}{\alpha\epsilon + m} w^* &= \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \\ \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} u^* &= \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* = \frac{\kappa s^* p^*}{1 + \mu p^*} \\ \frac{\kappa s^* p^*}{1 + \mu p^*} &= \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*}, \end{aligned}$$

and

$$\begin{aligned}
\Delta U_n &\leq \frac{-\delta}{s_{n+1}}(s_{n+1} - s^*)^2 + \frac{\kappa s^* p^*}{1 + \mu p_n} \left(1 - \frac{s^*}{s_{n+1}}\right) + \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{p_n(1 + \mu p^*)}{p^*(1 + \mu p_n)} \\
&\quad - \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{s_{n+1} p_n w^* (1 + \mu p^*)}{s^* p^* w_{n+1} (1 + \mu p_n)} + \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \\
&\quad - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{s_{n+1} p_n u^* (1 + \mu p^*)}{s^* p^* u_{n+1} (1 + \mu p_n)} - \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{u^* w_{n+1}}{u_{n+1} w^*} \\
&\quad + \frac{\kappa s^* p^*}{1 + \mu p^*} - \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{u_{n+1} p^*}{u^* p_{n+1}} + \frac{\kappa s^* p^*}{1 + \mu p^*} + \frac{\kappa s^* p^*}{1 + \mu p^*} \left(-\frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}}\right) \\
&= \frac{-\delta}{s_{n+1}}(s_{n+1} - s^*)^2 + \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \left(1 - \frac{s^*}{s_{n+1}}\right) + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \left(1 - \frac{s^*}{s_{n+1}}\right) \\
&\quad - \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{s_{n+1} p_n w^* (1 + \mu p^*)}{s^* p^* w_{n+1} (1 + \mu p_n)} + \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \\
&\quad - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{s_{n+1} p_n u^* (1 + \mu p^*)}{s^* p^* u_{n+1} (1 + \mu p_n)} - \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{u^* w_{n+1}}{u_{n+1} w^*} \\
&\quad + \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} - \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{u_{n+1} p^*}{u^* p_{n+1}} \\
&\quad - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{u_{n+1} p^*}{u^* p_{n+1}} + \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \\
&\quad + \frac{\kappa s^* p^*}{1 + \mu p^*} \left(-\frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} + \frac{p_n(1 + \mu p^*)}{p^*(1 + \mu p_n)}\right) \\
&= \frac{-\delta}{s_{n+1}}(s_{n+1} - s^*)^2 + \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \left[5 - \frac{s^*}{s_{n+1}} - \frac{s_{n+1} p_n w^* (1 + \mu p^*)}{s^* p^* w_{n+1} (1 + \mu p_n)} - \frac{u^* w_{n+1}}{u_{n+1} w^*}\right. \\
&\quad \left. - \frac{u_{n+1} p^*}{u^* p_{n+1}} - \frac{1 + \mu p_n}{1 + \mu p^*} + \ln \frac{p_n}{p_{n+1}}\right] + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \left[4 - \frac{s^*}{s_{n+1}} - \frac{s_{n+1} p_n u^* (1 + \mu p^*)}{s^* p^* u_{n+1} (1 + \mu p_n)}\right. \\
&\quad \left. - \frac{u_{n+1} p^*}{u^* p_{n+1}} - \frac{1 + \mu p_n}{1 + \mu p^*} + \ln \frac{p_n}{p_{n+1}}\right] + \frac{\kappa s^* p^*}{1 + \mu p^*} \left(-1 - \frac{p_n}{p^*} + \frac{p_n(1 + \mu p^*)}{p^*(1 + \mu p_n)} + \frac{1 + \mu p_n}{1 + \mu p^*}\right).
\end{aligned}$$

We have

$$-1 - \frac{p_n}{p^*} + \frac{p_n(1 + \mu p^*)}{p^*(1 + \mu p_n)} + \frac{1 + \mu p_n}{1 + \mu p^*} = -\frac{\mu(p_n - p^*)^2}{p^*(1 + \mu p_n)(1 + \mu p^*)}.$$

Then

$$\begin{aligned} \Delta U_n \leq & \frac{-\delta}{s_{n+1}}(s_{n+1} - s^*)^2 \\ & - \frac{m(1-\epsilon)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \left[G\left(\frac{s^*}{s_{n+1}}\right) + G\left(\frac{s_{n+1} p_n w^*}{s^* p^* w_{n+1}} \frac{1 + \mu p^*}{1 + \mu p_n}\right) + G\left(\frac{u^* w_{n+1}}{u_{n+1} w^*}\right) \right. \\ & \left. + G\left(\frac{u_{n+1} p^*}{u^* p_{n+1}}\right) + G\left(\frac{1 + \mu p_n}{1 + \mu p^*}\right) \right] \\ & - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{\kappa s^* p^*}{1 + \mu p^*} \left[G\left(\frac{s^*}{s_{n+1}}\right) + G\left(\frac{s_{n+1} p_n u^*}{s^* p^* u_{n+1}} \frac{1 + \mu p^*}{1 + \mu p_n}\right) \right. \\ & \left. + G\left(\frac{u_{n+1} p^*}{u^* p_{n+1}}\right) + G\left(\frac{1 + \mu p_n}{1 + \mu p^*}\right) \right] - \frac{\kappa s^* p^*}{1 + \mu p^*} \frac{\mu(p_n - p^*)^2}{p^*(1 + \mu p_n)(1 + \mu p^*)}. \end{aligned}$$

Thus, U_n is monotone decreasing sequence. Since $U_n \geq 0$, then there is a limit $\lim_{n \rightarrow \infty} U_n \geq 0$ and hence, $\lim_{n \rightarrow \infty} \Delta U_n = 0$, which implies that $\lim_{n \rightarrow \infty} s_n = s^*$, $\lim_{n \rightarrow \infty} w_n = w^*$, $\lim_{n \rightarrow \infty} u_n = u^*$ and $\lim_{n \rightarrow \infty} p_n = p^*$. □

4. MODEL WITH GENERAL INCIDENCE

In the literature, several forms of the incidence rate have been considered see e.g. [17]-[25]. In this section, we assume that the incidence rate is given by $K(s, p)p$, where K is a general function.

$$\dot{s} = \beta - \delta s - K(s, p)p, \tag{4.1}$$

$$\dot{w} = (1 - \epsilon)K(s, p)p - (\alpha + m)w, \tag{4.2}$$

$$\dot{u} = \epsilon K(s, p)p + mw - \gamma u, \tag{4.3}$$

$$\dot{p} = \theta u - \eta p. \tag{4.4}$$

Using NSFD method we get

$$s_{n+1} - s_n = \beta - \delta s_{n+1} - K(s_{n+1}, p_n)p_n, \tag{4.5}$$

$$w_{n+1} - w_n = (1 - \epsilon)K(s_{n+1}, p_n)p_n - (\alpha + m)w_{n+1}, \tag{4.6}$$

$$u_{n+1} - u_n = \epsilon K(s_{n+1}, p_n)p_n + mw_{n+1} - \gamma u_{n+1}, \tag{4.7}$$

$$p_{n+1} - p_n = \theta u_{n+1} - \eta p_{n+1}. \tag{4.8}$$

4.1. Preliminaries. The function $K(s, p)$ is assumed to satisfy the following conditions:

(A1) $K(s, p) > 0$, for all $s > 0, p > 0$, and $K(0, p) = 0$ for all $p \geq 0$,

(A2) $\frac{\partial K(s, p)}{\partial s} > 0$ for all $s > 0$ and $p \geq 0$,

(A3) $\frac{\partial K(s, p)}{\partial p} \leq 0$ for all $s \geq 0$ and $p \geq 0$.

Lemma 4.1. Any solution (s_n, w_n, u_n, p_n) of model (4.5)-(4.8) with initial conditions (2.9) is positive and converges on Γ_1 as $n \rightarrow \infty$, and Γ_1 is positive invariable for model (4.5)-(4.8).

Proof. From Eqs. (4.6)-(4.8) we obtain

$$w_{n+1} = \frac{w_n + (1 - \epsilon)K(s_{n+1}, p_n)p_n}{(1 + \alpha + m)}, \tag{4.9}$$

$$u_{n+1} = \frac{u_n + \epsilon K(s_{n+1}, p_n)p_n + mw_{n+1}}{1 + \gamma}, \tag{4.10}$$

$$p_{n+1} = \frac{p_n + \theta u_{n+1}}{1 + \eta}. \tag{4.11}$$

When $n = 0$ we prove that (s_1, w_1, u_1, p_1) exists and is positive. From Eq. (4.5) we have

$$(1 + \delta) s_1 - s_0 - \beta + K(s_1, p_0)p_0 = 0.$$

Let

$$\begin{aligned} \varphi(s_1) &= (1 + \delta) s_1 - s_0 - \beta + K(s_1, p_0)p_0 = 0 \\ \varphi(0) &= -s_0 - \beta < 0 \\ \lim_{s_1 \rightarrow \infty} \varphi(s_1) &= \infty. \end{aligned}$$

From Assumption (A2), φ is a strictly increasing function in s_1 . Hence, there exists a unique $s_1 > 0$ such that $\varphi(s_1) = 0$. From Eqs. (4.9)-(4.11) we have $w_1 > 0$, $u_1 > 0$ and $p_1 > 0$. Therefore, by using the induction, we obtain $s_n > 0$, $w_n > 0$, $u_n > 0$ and $p_n > 0$ for all $n \geq 0$. The boundedness of solutions can be shown similar to Lemma 2.1. □

Lemma 4.2. For model (4.5)-(4.8) let (A1)-(A2) hold true, then there exists a threshold parameter $\mathcal{R}_0 > 0$ such that

- (i) if $\mathcal{R}_0 \leq 1$, then there exists only pathogen-free equilibrium Q^0 ,
- (ii) if $\mathcal{R}_0 > 1$, then there exist two equilibria, Q^0 and a persistent pathogen equilibrium Q^* .

Proof. Let $Q(s, w, u, p)$ be any equilibrium of model (4.5)-(4.8) satisfying

$$\beta - \delta s - K(s, p)p = 0, \tag{4.12}$$

$$(1 - \epsilon)K(s, p)p - (\alpha + m)w = 0, \tag{4.13}$$

$$\epsilon K(s, p)p + mw - \gamma u = 0, \tag{4.14}$$

$$\theta u - \eta p = 0. \tag{4.15}$$

From Eqs. (4.12)-(4.14) we have

$$w = \frac{(1 - \epsilon)K(s, p)p}{(\alpha + m)}, \quad u = \frac{(\alpha\epsilon + m)K(s, p)p}{\gamma(\alpha + m)}, \quad K(s, p)p = \beta - \delta s. \tag{4.16}$$

Substituting from Eq. (4.16) into Eq. (4.15) we get

$$\frac{\theta(\alpha\epsilon + m)}{\gamma(\alpha + m)}K(s, p)p - \eta p = 0. \tag{4.17}$$

From Eq. (4.16) we get

$$s = s^0 - \frac{\gamma\eta(\alpha + m)}{\delta\theta(\alpha\epsilon + m)}p.$$

Eq. (4.17) has two possible solutions $p = 0$ or $p \neq 0$. If $p = 0$, then from Eqs. (4.12)-(4.14), we get $s = s^0$, $w = 0$ and $u = 0$ which gives the pathogen-free equilibrium $Q^0(s^0, 0, 0, 0)$ where $s^0 = \frac{\beta}{\delta}$. If $p \neq 0$, then we have

$$\frac{\theta(\alpha\epsilon + m)}{\gamma(\alpha + m)}K\left(s^0 - \frac{\gamma\eta(\alpha + m)}{\delta\theta(\alpha\epsilon + m)}p, p\right)p - \eta p = 0.$$

Let

$$\psi(p) = \frac{\theta(\alpha\epsilon + m)}{\gamma(\alpha + m)}K\left(s^0 - \frac{\gamma\eta(\alpha + m)}{\delta\theta(\alpha\epsilon + m)}p, p\right)p - \eta p = 0.$$

We have $\psi(0) = 0$, and $\psi(\bar{p}) = -\eta\bar{p} < 0$ where $\bar{p} = \frac{s^0\delta\theta(\alpha\epsilon + m)}{\gamma\eta(\alpha + m)}$. Moreover

$$\begin{aligned} \psi'(0) &= \frac{\theta(\alpha\epsilon + m)}{\gamma(\alpha + m)}K(s^0, 0) - \eta \\ &= \eta \left[\frac{\theta(\alpha\epsilon + m)}{\gamma\eta(\alpha + m)}K(s^0, 0) - 1 \right]. \end{aligned}$$

Therefore, $\psi'(0) > 0$ if

$$\frac{\theta(\alpha\epsilon + m)}{\gamma\eta(\alpha + m)}K(s^0, 0) > 1. \tag{4.18}$$

It follows that, if condition (4.18) is satisfied, then there exist $p^* \in (0, \bar{p})$ such that $\psi(p^*) = 0$. Hence, the basic reproduction number of system (4.5)-(4.8) can be defined as:

$$\mathcal{R}_0 = \frac{\theta(\alpha\epsilon + m)}{\gamma\eta(\alpha + m)}K(s^0, 0).$$

Moreover, let $p = p^*$ in Eq. (4.12) we get

$$\beta - \delta s - K(s, p^*)p^* = 0.$$

Let us define

$$\psi_1(s) = \beta - \delta s - K(s, p^*)p^*.$$

We have $\psi_1(0) = \beta > 0$ and $\psi_1(s^0) = -K(s^0, p^*)p^* < 0$. Since $K(s, p)$ is strictly increasing with respect to s , then $\psi_1(s)$ is strictly decreasing with respect to s . Hence, there exists a unique

$s^* \in (0, s^0)$ such that $\psi_1(s^*) = 0$. From Eq. (4.16) and Assumption (A1) we have

$$w^* = \frac{(1 - \epsilon)K(s^*, p^*)p^*}{(\alpha + m)} > 0,$$

$$u^* = \frac{\epsilon K(s^*, p^*)p^* + mw^*}{\gamma} > 0.$$

This shows that if $\mathcal{R}_0 > 1$, then there exists a persistent-pathogen equilibrium $Q^*(s^*, w^*, u^*, p^*)$. \square

4.2. Global stability.

Lemma 4.3. [40] *Let \bar{s}, \bar{p} and σ be three positive real numbers and $\bar{Q}(\bar{s}, \bar{w}, \bar{u}, \bar{p})$ be any equilibrium point. The function $\Psi_{(\bar{Q}, \sigma)}$ defined on interval $[0, \infty)$ by*

$$\Psi_{(\bar{Q}, \sigma)}(s) = s - \sigma - \int_{\sigma}^s \frac{K(\bar{s}, \bar{p})}{K(\tau, \bar{p})} d\tau$$

has the global minimum at $s = \bar{s}$ and satisfies

$$\left(1 - \frac{K(\bar{s}, \bar{p})}{K(\sigma, \bar{p})}\right) (s - \sigma) \leq \Psi_{(\bar{Q}, \sigma)}(s) \leq \left(1 - \frac{K(\bar{s}, \bar{p})}{K(s, \bar{p})}\right) (s - \sigma), \quad \text{for all } s > 0. \quad (4.19)$$

Theorem 4.4. *Suppose that $\mathcal{R}_0 \leq 1$, then Q^0 of system (4.5)-(4.8) is globally asymptotically stable.*

Proof. Construct a Lyapunov function L_n as:

$$L_n = \Psi_{(Q^0, s^0)}(s_n) + \frac{m}{\alpha\epsilon + m}w_n + \frac{\alpha + m}{\alpha\epsilon + m}u_n + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)p_n.$$

From Lemma 4.3, we obtain $\Psi_{(Q^0, s^0)}(s_n) \geq 0$. Hence, $L_n > 0$ for all $s_n, w_n, u_n, p_n > 0$ and $L_n = 0$ if and only if $s_n = s^0, w_n = 0, u_n = 0$ and $p_n = 0$. Computing the difference $\Delta L_n = L_{n+1} - L_n$ as:

$$\begin{aligned} \Delta L_n &= \Psi_{(Q^0, s^0)}(s_{n+1}) + \frac{m}{\alpha\epsilon + m}w_{n+1} + \frac{\alpha + m}{\alpha\epsilon + m}u_{n+1} + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)p_{n+1} \\ &\quad - \left[\Psi_{(Q^0, s^0)}(s_n) + \frac{m}{\alpha\epsilon + m}w_n + \frac{\alpha + m}{\alpha\epsilon + m}u_n + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)p_n \right] \\ &= s_{n+1} - s_n - \int_{s_n}^{s_{n+1}} \frac{K(s^0, 0)}{K(\tau, 0)} d\tau + \frac{m}{\alpha\epsilon + m}(w_{n+1} - w_n) + \frac{\alpha + m}{\alpha\epsilon + m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)(p_{n+1} - p_n) \\ &= \Psi_{(Q^0, s_n)}(s_{n+1}) + \frac{m}{\alpha\epsilon + m}(w_{n+1} - w_n) + \frac{\alpha + m}{\alpha\epsilon + m}(u_{n+1} - u_n) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)}(1 + \eta)(p_{n+1} - p_n). \end{aligned}$$

Using Lemma 4.3, we can get

$$\begin{aligned} \Delta L_n \leq & \left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m} (w_{n+1} - w_n) + \frac{\alpha + m}{\alpha\epsilon + m} (u_{n+1} - u_n) \\ & + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta)(p_{n+1} - p_n). \end{aligned}$$

From Eqs. (4.5)-(4.8), we have

$$\Delta L_n \leq \left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) (\beta - \delta s_{n+1} - K(s_{n+1}, p_n)p_n) \tag{4.20}$$

$$\begin{aligned} & + \frac{m}{\alpha\epsilon + m} ((1 - \epsilon)K(s_{n+1}, p_n)p_n - (\alpha + m)w_{n+1}) \\ & + \frac{\alpha + m}{\alpha\epsilon + m} (\epsilon K(s_{n+1}, p_n)p_n + mw_{n+1} - \gamma u_{n+1}) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (\theta u_{n+1} - \eta p_{n+1}) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} (p_{n+1} - p_n). \end{aligned} \tag{4.21}$$

Collecting terms of Eq. (4.21) and using $s^0 = \frac{\beta}{\delta}$, we obtain

$$\begin{aligned} \Delta L_n \leq & \left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) (\beta - \delta s_{n+1}) + \frac{K(s^0, 0)}{K(s_{n+1}, 0)} K(s_{n+1}, p_n)p_n - \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p_n \\ = & \delta \left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) (s^0 - s_{n+1}) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(\frac{\theta(\alpha\epsilon + m)}{\gamma\eta(\alpha + m)} K(s^0, 0) \frac{K(s_{n+1}, p_n)}{K(s_{n+1}, 0)} - 1\right) p_n \\ = & \delta s^0 \left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) \left(1 - \frac{s_{n+1}}{s^0}\right) + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(\frac{K(s_{n+1}, p_n)}{K(s_{n+1}, 0)} \mathcal{R}_0 - 1\right) p_n. \end{aligned}$$

Since $K(s, p)$ is decreasing with respect to p , then $K(s_{n+1}, p_n) \leq K(s_{n+1}, 0)$. Thus

$$\Delta L_n \leq \delta s^0 \left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) \left(1 - \frac{s_{n+1}}{s^0}\right) + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} (\mathcal{R}_0 - 1) p_n.$$

Because $K(s, p)$ is strictly increasing with respect to s , we have

$$\left(1 - \frac{K(s^0, 0)}{K(s_{n+1}, 0)}\right) \left(1 - \frac{s_{n+1}}{s^0}\right) \leq 0.$$

Hence, if $\mathcal{R}_0 \leq 1$, we have $\Delta L_n \leq 0$ for all $n \geq 0$. Obviously, $\Delta L_n = 0$ if and only if $s_n = s^0$ and $(\mathcal{R}_0 - 1)p_n = 0$. We discuss two cases:

- If $\mathcal{R}_0 < 1$, then $\lim_{n \rightarrow \infty} p_n = 0$. then we get $\lim_{n \rightarrow \infty} w_n = 0$ and $\lim_{n \rightarrow \infty} u_n = 0$.
 - If $\mathcal{R}_0 = 1$. By using $\lim_{n \rightarrow \infty} s_n = s^0$ and from Eq. (4.5), we obtain $\lim_{n \rightarrow \infty} K(s^0, p_n)p_n = 0$.
- Because $s^0 > 0$, we have $K(s^0, p_n) > K(0, p_n) = 0$ (use Assumptions (A1) and (A2)). Thus, $p_n = 0$.

By the aforementioned discussion, we deduce that the largest compact invariant set in $\{(s_n, w_n, u_n, p_n) | \Delta L_n = 0\}$ is the just the singleton Q^0 . Therefore, Q^0 is globally asymptotically stable by the LaSalle's invariance principle [53]. \square

To establish the global stability of the persistent pathogen equilibrium Q^* we require the following condition

$$(A4) \quad \left(1 - \frac{K(s, p)}{K(s, p^*)}\right) \left(\frac{K(s, p^*)}{K(s, p)} - \frac{p}{p^*}\right) \leq 0, \text{ for all } s, p > 0.$$

Theorem 4.5. *Suppose that $R_0 > 1$, then Q^* of system (4.5)-(4.8) is globally asymptotically stable.*

Proof. Consider

$$U_n(s_n, w_n, u_n, p_n) = \Psi_{(Q^*, s^*)}(s_n) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_n}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_n}{u^*}\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_n}{p^*}\right).$$

By Lemma 4.3, we get $\Psi_{(Q^*, s^*)}(s_n) \geq 0$. Clearly, $U_n(s_n, w_n, u_n, p_n) > 0$ for all $s_n, w_n, u_n, p_n > 0$ and $U_n(s^*, w^*, u^*, p^*) = 0$. Computing $\Delta U_n = U_{n+1} - U_n$ as:

$$\begin{aligned} \Delta U_n &= \Psi_{(Q^*, s^*)}(s_{n+1}) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_{n+1}}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_{n+1}}{u^*}\right) \\ &\quad + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_{n+1}}{p^*}\right) \\ &\quad - \left[\Psi_{(Q^*, s^*)}(s_n) + \frac{m}{\alpha\epsilon + m} w^* G\left(\frac{w_n}{w^*}\right) + \frac{\alpha + m}{\alpha\epsilon + m} u^* G\left(\frac{u_n}{u^*}\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* G\left(\frac{p_n}{p^*}\right) \right] \\ &= s_{n+1} - s_n - \int_{s_n}^{s_{n+1}} \frac{K(s^*, p^*)}{K(\tau, p^*)} d\tau + \frac{m}{\alpha\epsilon + m} w^* \left[G\left(\frac{w_{n+1}}{w^*}\right) - G\left(\frac{w_n}{w^*}\right) \right] \\ &\quad + \frac{\alpha + m}{\alpha\epsilon + m} u^* \left[G\left(\frac{u_{n+1}}{u^*}\right) - G\left(\frac{u_n}{u^*}\right) \right] + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} (1 + \eta) p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right] \\ &= \Psi_{(Q^*, s_n)}(s_{n+1}) + \frac{mw^*}{\alpha\epsilon + m} \left(\frac{w_{n+1}}{w^*} - \frac{w_n}{w^*} + \ln \frac{w_n}{w_{n+1}} \right) \\ &\quad + \frac{(\alpha + m)u^*}{\alpha\epsilon + m} \left(\frac{u_{n+1}}{u^*} - \frac{u_n}{u^*} + \ln \frac{u_n}{u_{n+1}} \right) \\ &\quad + \frac{\gamma(\alpha + m)p^*}{\theta(\alpha\epsilon + m)} \left(\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} \right) + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right) \right] \end{aligned}$$

From inequality (4.19), we have

$$\begin{aligned} \Delta U_n \leq & \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m} \left(w_{n+1} - w_n + w^* \ln \frac{w_n}{w_{n+1}}\right) \\ & + \frac{(\alpha + m)}{\alpha\epsilon + m} \left(u_{n+1} - u_n + u^* \ln \frac{u_n}{u_{n+1}}\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(p_{n+1} - p_n + p^* \ln \frac{p_n}{p_{n+1}}\right) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right)\right]. \end{aligned}$$

Using inequality (2.15), we obtain

$$\begin{aligned} \Delta U_n \leq & \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m} \left(w_{n+1} - w_n + w^* \left(\frac{w_n}{w_{n+1}} - 1\right)\right) \\ & + \frac{\alpha + m}{\alpha\epsilon + m} \left(u_{n+1} - u_n + u^* \left(\frac{u_n}{u_{n+1}} - 1\right)\right) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(p_{n+1} - p_n + p^* \left(\frac{p_n}{p_{n+1}} - 1\right)\right) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[G\left(\frac{p_{n+1}}{p^*}\right) - G\left(\frac{p_n}{p^*}\right)\right] \\ = & \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) (s_{n+1} - s_n) + \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}}\right) (w_{n+1} - w_n) \\ & + \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}}\right) (u_{n+1} - u_n) + \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}}\right) (p_{n+1} - p_n) \\ & + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}}\right]. \end{aligned}$$

From Eqs. (4.5)-(4.8), we have

$$\begin{aligned}
\Delta U_n &\leq \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) (\beta - \delta s_{n+1} - K(s_{n+1}, p_n) p_n) \\
&+ \frac{m}{\alpha\epsilon + m} \left(1 - \frac{w^*}{w_{n+1}}\right) ((1 - \epsilon)K(s_{n+1}, p_n) p_n - (\alpha + m)w_{n+1}) \\
&+ \frac{\alpha + m}{\alpha\epsilon + m} \left(1 - \frac{u^*}{u_{n+1}}\right) (\epsilon K(s_{n+1}, p_n) p_n + mw_{n+1} - \gamma u_{n+1}) \\
&+ \frac{\gamma(\alpha + m)}{\theta(\alpha\epsilon + m)} \left(1 - \frac{p^*}{p_{n+1}}\right) (\theta u_{n+1} - \eta p_{n+1}) + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left(\frac{p_{n+1}}{p^*} - \frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}}\right) \\
&= \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) (\beta - \delta s_{n+1}) + \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} K(s_{n+1}, p_n) p_n \\
&- \frac{m(1 - \epsilon)}{\alpha\epsilon + m} \frac{w^*}{w_{n+1}} K(s_{n+1}, p_n) p_n + \frac{m(\alpha + m)}{\alpha\epsilon + m} w^* - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} \frac{u^*}{u_{n+1}} K(s_{n+1}, p_n) p_n \\
&- \frac{m(\alpha + m)}{\alpha\epsilon + m} \frac{u^*}{u_{n+1}} w_{n+1} + \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} u^* - \frac{\gamma(\alpha + m)}{\alpha\epsilon + m} \frac{p^*}{p_{n+1}} u_{n+1} + \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \\
&+ \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* \left[-\frac{p_n}{p^*} + \ln \frac{p_n}{p_{n+1}}\right].
\end{aligned}$$

Using the conditions of Q^*

$$\begin{aligned}
\beta &= \delta s^* + K(s^*, p^*) p^*, \\
(1 - \epsilon)K(s^*, p^*) p^* &= (\alpha + m)w^*, \\
\epsilon K(s^*, p^*) p^* + mw^* &= \gamma u^*, \\
\theta u^* &= \eta p^*,
\end{aligned}$$

we get

$$\begin{aligned}
\frac{m(\alpha + m)}{\alpha\epsilon + m} w^* &= \frac{m(1 - \epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^*, \\
\frac{\gamma(\alpha + m)}{\alpha\epsilon + m} u^* &= \frac{\gamma\eta(\alpha + m)}{\theta(\alpha\epsilon + m)} p^* = K(s^*, p^*) p^*, \\
K(s^*, p^*) p^* &= \frac{m(1 - \epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^*,
\end{aligned}$$

and

$$\begin{aligned}
 \Delta U_n &\leq \delta s^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} \right) \left(1 - \frac{s_{n+1}}{s^*} \right) + K(s^*, p^*) p^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} \right) \\
 &+ K(s^*, p^*) p^* \frac{K(s_{n+1}, p_n) p_n}{K(s_{n+1}, p^*) p^*} - \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{w^*}{w_{n+1}} \\
 &+ \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{u^*}{u_{n+1}} \\
 &- \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{u^* w_{n+1}}{u_{n+1} w^*} + K(s^*, p^*) p^* - K(s^*, p^*) p^* \frac{p^* u_{n+1}}{p_{n+1} u^*} \\
 &+ K(s^*, p^*) p^* + K(s^*, p^*) p^* \left[\frac{-p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} \right] \\
 &= \delta s^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} \right) \left(1 - \frac{s_{n+1}}{s^*} \right) + \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} \right) \\
 &+ \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} \right) - \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{w^*}{w_{n+1}} \\
 &+ \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{u^*}{u_{n+1}} \\
 &- \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{u^* w_{n+1}}{u_{n+1} w^*} + \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \\
 &- \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{p^* u_{n+1}}{p_{n+1} u^*} - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \frac{p^* u_{n+1}}{p_{n+1} u^*} \\
 &+ \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* + \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \\
 &+ K(s^*, p^*) p^* \left[\frac{-p_n}{p^*} + \ln \frac{p_n}{p_{n+1}} + \frac{K(s_{n+1}, p_n) p_n}{K(s_{n+1}, p^*) p^*} \right],
 \end{aligned}$$

$$\begin{aligned}
\Delta U_n &\leq \delta s^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) \left(1 - \frac{s_{n+1}}{s^*}\right) \\
&+ \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \left[5 - \frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{w^*}{w_{n+1}} - \frac{u^* w_{n+1}}{u_{n+1} w^*} \right. \\
&\left. - \frac{p^* u_{n+1}}{p_{n+1} u^*} - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} - \frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)} + \ln \frac{p_n}{p_{n+1}}\right] \\
&+ \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) \\
&\times p^* \left[4 - \frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{u^*}{u_{n+1}} - \frac{p^* u_{n+1}}{p_{n+1} u^*} - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)} - \frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)} + \ln \frac{p_n}{p_{n+1}}\right] \\
&+ K(s^*, p^*) p^* \left(-1 - \frac{p_n}{p^*} + \frac{K(s_{n+1}, p_n) p_n}{K(s_{n+1}, p^*) p^*} + \frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)}\right) \\
&= \delta s^* \left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) \left(1 - \frac{s_{n+1}}{s^*}\right) - \frac{m(1-\epsilon)}{\alpha\epsilon + m} K(s^*, p^*) p^* \left[G\left(\frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{w^*}{w_{n+1}}\right)\right. \\
&+ G\left(\frac{u^* w_{n+1}}{u_{n+1} w^*}\right) + G\left(\frac{p^* u_{n+1}}{p_{n+1} u^*}\right) + G\left(\frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) + G\left(\frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)}\right) \\
&\left. - \frac{\epsilon(\alpha + m)}{\alpha\epsilon + m} K(s^*, p^*) p^* \left[G\left(\frac{K(s_{n+1}, p_n) p_n}{K(s^*, p^*) p^*} \frac{u^*}{u_{n+1}}\right) + G\left(\frac{p^* u_{n+1}}{p_{n+1} u^*}\right) + G\left(\frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right)\right.\right. \\
&\left. + G\left(\frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)}\right)\right] + K(s^*, p^*) p^* \left(-1 - \frac{p_n}{p^*} + \frac{K(s_{n+1}, p_n) p_n}{K(s_{n+1}, p^*) p^*} + \frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)}\right)
\end{aligned}$$

Because $K(s, p)$ is strictly increasing function with respect to s , we obtain that

$$\left(1 - \frac{K(s^*, p^*)}{K(s_{n+1}, p^*)}\right) \left(1 - \frac{s_{n+1}}{s^*}\right) \leq 0.$$

Based on the assumption (A4), we have

$$-1 - \frac{p_n}{p^*} + \frac{K(s_{n+1}, p_n) p_n}{K(s_{n+1}, p^*) p^*} + \frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)} = \left(1 - \frac{K(s_{n+1}, p_n)}{K(s_{n+1}, p^*)}\right) \left(\frac{K(s_{n+1}, p^*)}{K(s_{n+1}, p_n)} - \frac{p_n}{p^*}\right) \leq 0.$$

Thus, U_n is monotone decreasing sequence. Because $U_n \geq 0$, there is a limit $\lim_{n \rightarrow \infty} U_n \geq 0$.

Therefore, $\lim_{n \rightarrow \infty} \Delta U_n = 0$, which implies that $\lim_{n \rightarrow \infty} s_n = s^*$, $\lim_{n \rightarrow \infty} w_n = w^*$, $\lim_{n \rightarrow \infty} u_n = u^*$ and $\lim_{n \rightarrow \infty} p_n = p^*$. □

5. NUMERICAL SIMULATIONS

We perform our simulation by choosing

$$K(s, p) = \frac{\kappa s}{1 + \lambda s}$$

where $\lambda > 0$ is the holling-type II infection rate constant. Therefore, system (4.5)-(4.8) becomes

$$s_{n+1} - s_n = \beta - \delta s_{n+1} - \frac{\kappa s_{n+1} p_n}{1 + \lambda s_{n+1}}, \tag{5.1}$$

$$w_{n+1} - w_n = \frac{(1 - \epsilon)\kappa s_{n+1} p_n}{1 + \lambda s_{n+1}} - (\alpha + m)w_{n+1}, \tag{5.2}$$

$$u_{n+1} - u_n = \frac{\epsilon\kappa s_{n+1} p_n}{1 + \lambda s_{n+1}} + mw_{n+1} - \gamma u_{n+1}, \tag{5.3}$$

$$p_{n+1} - p_n = \theta u_{n+1} - \eta p_{n+1}. \tag{5.4}$$

For this system, the basic reproduction number is given by

$$\mathcal{R}_0 = \frac{\theta\kappa\beta(\alpha\epsilon + m)}{\gamma\eta(\alpha + m)(\delta + \lambda\beta)}.$$

We verify the assumptions (A1)-(A4)

$$K(s, p) = \frac{\kappa s}{1 + \lambda s} > 0, \text{ for all } s > 0, p > 0, \text{ and } K(0, p) = 0 \text{ for all } p \geq 0,$$

$$\frac{\partial K(s, p)}{\partial s} = \frac{\kappa}{(1 + \lambda s)^2} > 0 \text{ for all } s > 0, \text{ and } p \geq 0,$$

$$\frac{\partial K(s, p)}{\partial p} = 0 \text{ for all } s \geq 0, \text{ and } p \geq 0,$$

$$\left(1 - \frac{K(s, p)}{K(s, p^*)}\right) \left(\frac{K(s, p^*)}{K(s, p)} - \frac{p}{p^*}\right) = 0.$$

Then, function $K(s, p)$ satisfies Assumptions (A1)-(A4) and hence Theorems 4.4 and 4.5 are applicable for such function.

The numerical simulations for system (5.1)-(5.4) will be conducted using the following data: $\beta = 10, \delta = 0.1, \alpha = 0.4, m = 0.1, \gamma = 0.2, \theta = 1$ and $\eta = 1$. The other parameters will be chosen bellow.

Let us consider the initial values

IV1: $s(0) = 800, w(0) = 20, u(0) = 60, p(0) = 80,$

IV2: $s(0) = 600, w(0) = 15, u(0) = 40, p(0) = 40,$

IV3: $s(0) = 400, w(0) = 10, u(0) = 20, p(0) = 20$ and

IV4: $s(0) = 600, w(0) = 2, u(0) = 15, p(0) = 15.$

Case(I) Effect of κ of stability of steady states:

We choose $\epsilon = 0.5, \lambda = 0.0005$ and κ is varied as:

(i) $\kappa = 0.0001$. This yields $\mathcal{R}_0 = 0.2000 < 1$. Figures 1-4 show that, the concentration of susceptible cells increases and tends to the value $s^0 = 1000$. In addition, the concentrations of latently infected cells, actively infected cells and free pathogen decrease and tend to zero for the initial values IV1-IV3. This shows that Q^0 is globally asymptotically stable and Theorem 4.4 is valid.

(ii) $\kappa = 0.001$. With this value we obtain $\mathcal{R}_0 = 2.0000 > 1$. Figures 1-4 show that for the initial values IV1-IV3, the solutions of the system tend to the equilibrium $Q^* = (400.0135, 6.0036, 18.0104, 18.0103)$. Therefore, Q^* exists and it is globally asymptotically stable. This validate the result of Theorem 4.5.

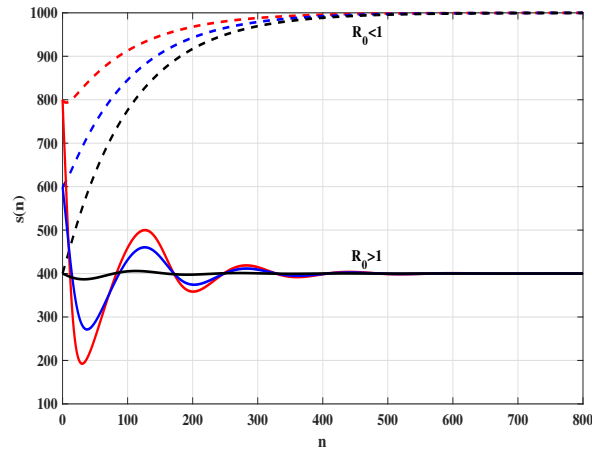


FIGURE 1. The simulation of susceptible cells of system (5.1)-(5.4) for Case(I).

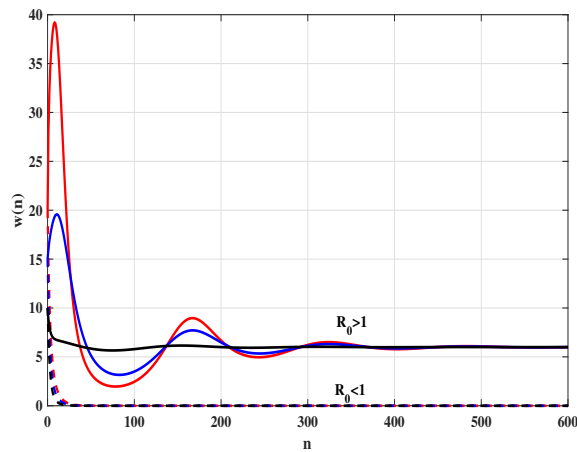


FIGURE 2. The simulation of latently infected cells of system (5.1)-(5.4) for Case(I).

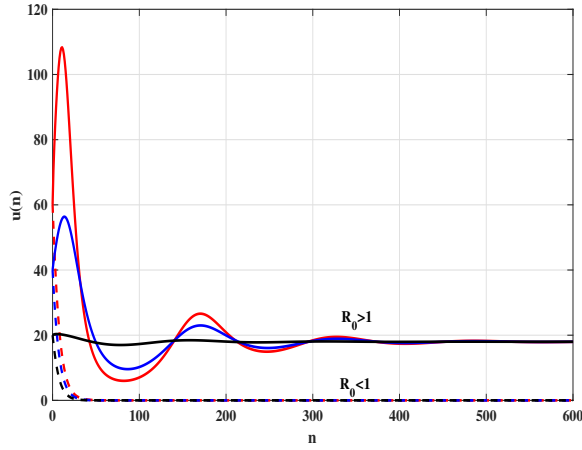


FIGURE 3. The simulation of actively infected cells of system (5.1)-(5.4) for Case(I).

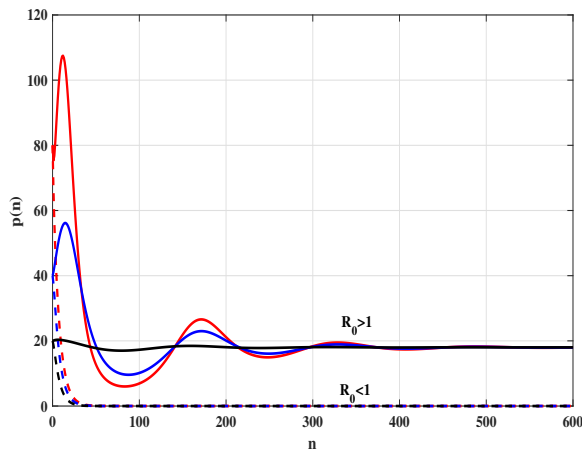


FIGURE 4. The simulation of pathogens of system (5.1)-(5.4) for Case(I).

Case(II) Effect of the Holling type II on the pathogen dynamics:

For this case, we take IV4 and choose the values $\epsilon = 0.5$, $\kappa = 0.001$. Figures 5-8 and Table 1 show the effect of Holling-type II parameter λ on the stability of the system. We observe that, as λ is increased, the pathogen-to-susceptible and infected-to-susceptible transmission rates are decreased. Then, the concentration of the susceptible cells are increased, while the concentrations of the latently infected cells, actively infected cells and free pathogens are decreased. In addition

- (i) if $\lambda < 0.002$, then Q^* exists and it is globally asymptotically stable,
- (ii) if $\lambda \geq 0.002$, then Q^0 exists and it is globally asymptotically stable.

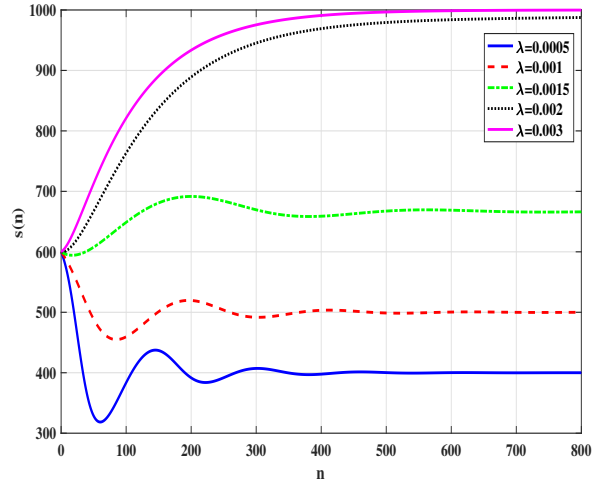


FIGURE 5. The simulation of susceptible cells of system (5.1)-(5.4) for Case(II).

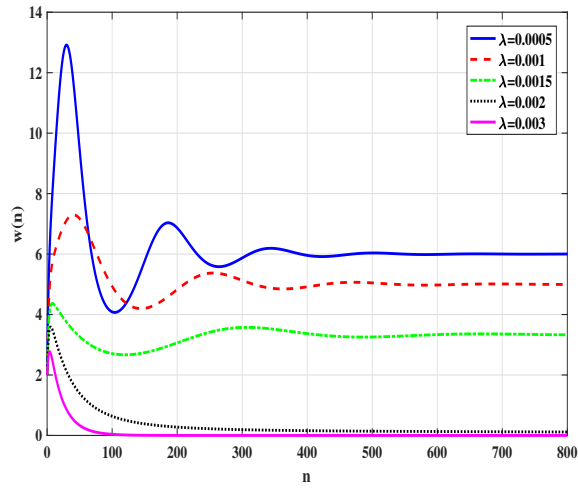


FIGURE 6. The simulation of latently infected cells of system (5.1)-(5.4) for Case(II).

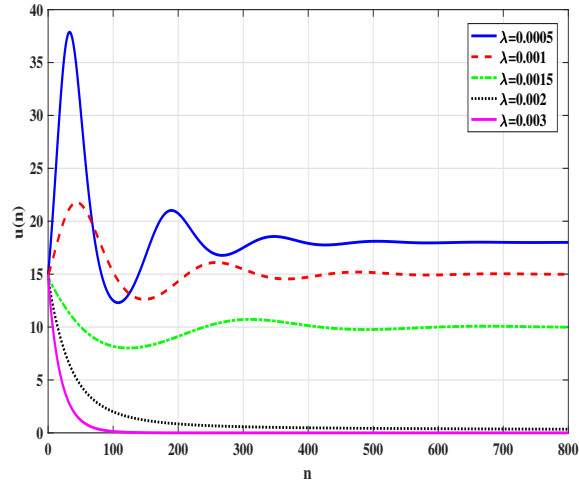


FIGURE 7. The simulation of actively infected cells of system (5.1)-(5.4) for Case(II).

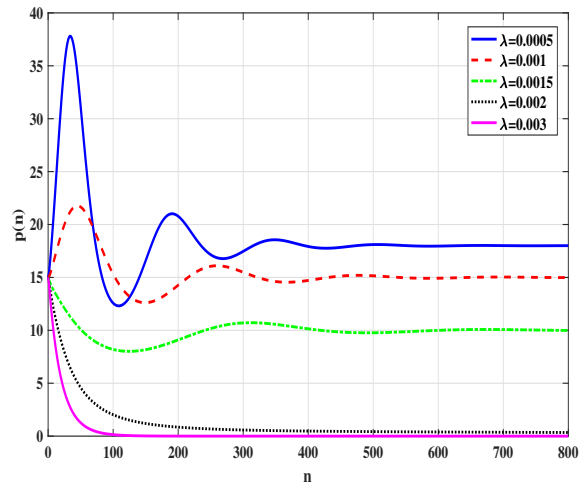


FIGURE 8. The simulation of pathogens of system (5.1)-(5.4) for Case(II).

Case(III) Effect of latency on the dynamical behavior of the system:

In this case, we show the HIV dynamics for different values of ϵ , the fraction of uninfected cells that become latently infected cells. We take $\kappa = 0.0006$, $\lambda = 0.0005$ and the initial conditions IV4. Figures 9-12 show the effect of ϵ on the evolution of system states. When ϵ increases, it is observed an increase in the concentration of the latently infected cells. This means that the reservoirs of these cells are enlarged, which promotes an increase in the amount of virus that escapes treatment [55]. Subsequently, after activation of the latently infected cells, new HIV will be produced and released into the blood stream [56]. Using the values of the parameters given in Table 1 we have the following:

- (i) if $0.0 < \epsilon < 0.3750$, then $\mathcal{R}_0 \leq 1$ and Q^0 exists and is globally asymptotically stable,
- (ii) if $\epsilon \geq 0.3750$, then $\mathcal{R}_0 > 1$ and Q^* exists and is globally asymptotically stable.

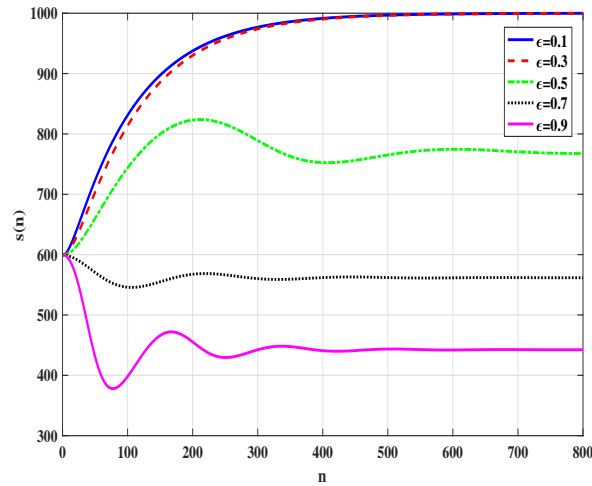


FIGURE 9. The simulation of susceptible cells of system (5.1)-(5.4) for Case(III).

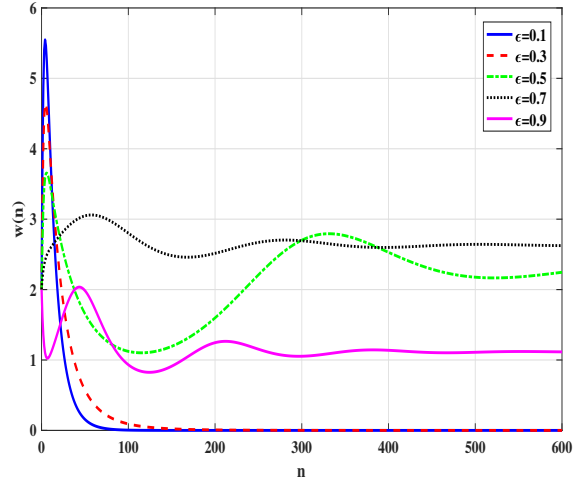


FIGURE 10. The simulation of latently infected cells of system (5.1)-(5.4) for Case(III).

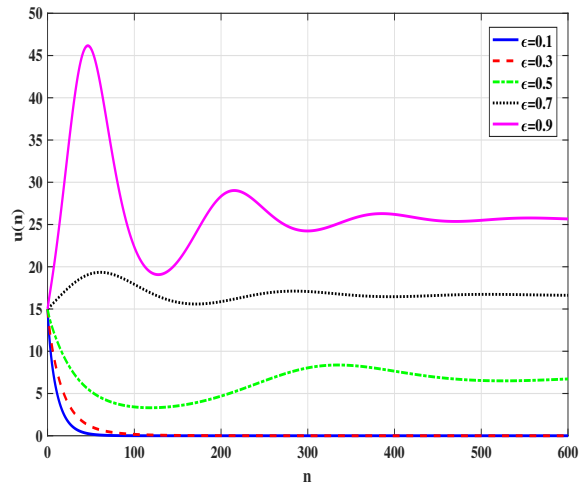


FIGURE 11. The simulation of actively infected cells of system (5.1)-(5.4) for Case(III).

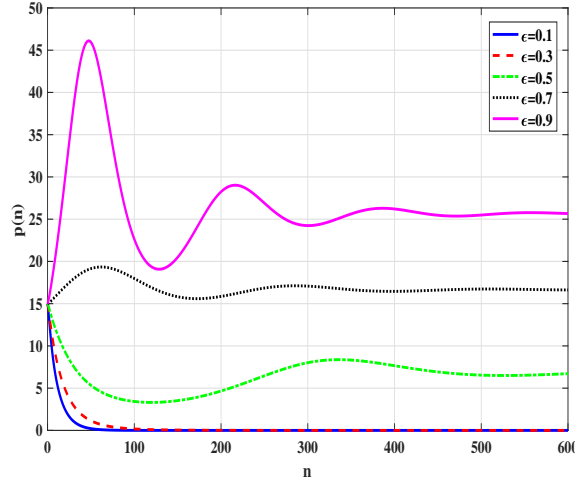


FIGURE 12. The simulation of pathogens of system (5.1)-(5.4) for Case(III).

Table 1 shows that as λ is increased the values of \mathcal{R}_0 are decreased.

TABLE 1. The values of \mathcal{R}_0 for system (5.1)-(5.4) with different values of λ .

λ	Equilibria	\mathcal{R}_0
0.0005	Q^*	2
0.001	Q^*	1.5000
0.0015	Q^*	1.2000
0.002	Q^0	1
0.003	Q^0	0.7500
0.004	Q^0	0.6000

6. CONCLUSION

In this paper, we have proposed and analyzed three discrete-time pathogen infection models with different incidence rate. We have considered two types of infected cells, latently infected cells and actively infected cells. We have discretized the continuous-time models by nonstandard finite difference scheme. We have determined the basic reproduction number \mathcal{R}_0 . We have proven the positivity and boundedness of the models’s solutions. Using Lyapunov method, we have established the global stability of the two equilibria of the models. We have proven that if $\mathcal{R}_0 \leq 1$, then the pathogen-free equilibrium Q^0 is globally asymptotically stable and if $\mathcal{R}_0 > 1$, then the persistent pathogen equilibrium Q^* exists and is globally asymptotically stable. We have performed some numerical simulations to support our theoretical results.

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