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Vaisman-Gray Manifold of Pointwise Holomorphic Sectional Conharmonic Tensor

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Abstract. The purpose of the present paper is to discuss the geometrical properties of the Vaisman-Gray manifold (VG-manifold) of a pointwise holomorphic sectional conharmonic tensor (PHT-tensor). Furthermore, the necessary and sufficient conditions required for the VG-manifold to admit such a PHT-tensor have been determined. In particular, under certain conditions, we have established that the aforementioned manifold was an Einstein manifold.

1. Introduction

The classification of the almost Hermitian structures was introduced by Gray and Hervella [4]. These structures have been categorized into sixteen different classes. Moreover, it has been found that the condition for each one of them depends on a Kozel's operator method [15].

On the other hand, there is another significant classification method for the almost Hermitian structures that were introduced by Kirichenko. This method depends on the principle fibre bundle space of all complex frames of a smooth manifold M with the unitary structure group $U(n)$. This space is called an adjoined G-structure space. For further information, refer to the following citations: [3], [8], [9], [10], and [11].

One of the interesting classes of almost Hermitian structures is a VG-manifold, which is denoted by $W_1 \oplus W_4$, where W_1 and W_4 denote the nearly Kähler manifold and the locally conformal Kähler manifold, respectively.

It is a well-known fact that the harmonic function is one whose Laplacian vanishes. In general, it is not a conformal transformation harmonic function. With

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regard to this fact, Ishi [6] introduced a tensor that remained invariant under a conharmonic transformation for an n-dimensional Riemannian manifold. In addition, Khan [7] determined the properties of the conharmonically flat Sasakian manifolds. Moreover, it has been proved that a special weakly Ricci symmetric Sasakian manifold is an Einstein manifold. Subsequently, Shihab [13] went on to determine the geometrical properties of the conharmonic curvature tensor belonging to the nearly Kähler manifold. Furthermore, it has been established that a Kähler manifold with a dimension greater than four is a conharmonic parakähler manifold if, and only if, it has a flat Ricci tensor. On the other hand, Zengin and Tasci [19] studied a pseudo conharmonically symmetric manifold. In particular, they proved that the aforementioned manifold with a non-zero scalar curvature has a closed associated 1-form. Lastly, Abood and Abdulameer [1] considered the conharmonically flat VG-manifold, exclusively and identified the necessary and sufficient conditions required for the VG-manifold to be an Einstein manifold.

In this article, we have employed the adjoined G-structure space to study the geometry of the VG-manifold that corresponds with a PHT-tensor.

2. Preliminaries

Let M be a smooth manifold of even dimension, $C^\infty(M)$ be an algebra of smooth functions on M, and $X(M)$ be the module of smooth vector fields on M. An almost Hermitian manifold (AH-manifold) is a triple $\{M, J, g = \langle .,. \rangle\}$, where M is a smooth manifold, J is an almost complex structure, and $g = \langle .,. \rangle$ is a Riemannian metric, such that the equality $\langle JX, JY \rangle = \langle X, Y \rangle$ holds for $X, Y \in X(M)$.

Suppose that $T_p^c(M)$ is the complexification of a tangent space $T_p(M)$ at the point $p \in M$ and $\{e_1, ..., e_n, Je_1, ..., Je_n\}$ is a real adapted basis of AH-manifold. Then, in the module $T_p^c(M)$, there exists a basis given by $\{\varepsilon_1, ..., \varepsilon_n, \hat{\varepsilon}_1, ..., \hat{\varepsilon}_n\}$ which is called as an adapted basis, where, $\varepsilon_a = \sigma(e_a)$, $\hat{\varepsilon}_a = \bar{\sigma}(e_a)$, and $\sigma, \bar{\sigma}$ are two endomorphisms in the module $X^c(M)$, which are given by $\sigma = \frac{1}{2}(id - \sqrt{-1}J^c)$ and $\bar{\sigma} = -\frac{1}{2}(id + \sqrt{-1}J^c)$, respectively, such that $X^c(M)$ and J^c are the complexifications of $X(M)$ and J, respectively. The corresponding frame of this basis is $\{p; \varepsilon_1, \dots \varepsilon_n, \hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n\}$. Suppose that the indexes i, j, k, and l are in the range $1, 2, \dots, 2n$ and the indexes a, b, c, d and f are in the range $1, 2, \dots, n$. Moreover, $\hat{a} = a + n$.

For a manifold M, it is a well known that the given AH-structure is equivalent to the given G-structure space in the principle fibre bundle of all complex frames of M with the unitary structure group $U(n)$. Whereas, in the adjoined G-structure space, the components matrices of the almost complex structure J and the Riemannian metric g are given as follows:

$$(2.1) \qquad (J_j^i) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \; (g_{ij}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the identity matrix of order n.

Definition 2.1.([14]) A *Riemannian curvature tensor* R of a smooth manifold M is a 4-covariant tensor $R : T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \mathbb{R}$ which is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W)$$

Moreover, the following properties:

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W)$;

(2) $R(X, Y, Z, W) = -R(X, Y, W, Z)$;

(3) $R(X, Y, Z, W) = R(Z, W, X, Y)$;

(4) $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0$

hold, where $R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z; X, Y, Z, W \in T_p(M)$.

Definition 2.2.([17]) A *Ricci tensor* is a tensor of type $(2, 0)$ which is a contracting of the Riemannian curvature tensor R, that is

$$r_{ij} = R^k_{ijk} = g^{kl} R_{kijl}.$$

Definition 2.3.([6]) A *conharmonic tensor* of an AH-manifold is a tensor T of type $(4, 0)$ which is given by the following form:

$$T_{ijkl} = R_{ijkl} - \frac{1}{2(n-1)}(r_{il}g_{jk} - r_{jl}g_{ik} + r_{jk}g_{il} - r_{ik}g_{jl})$$

where r, R and g are respectively the Ricci tensor, the Riemannian curvature tensor, and the Riemannian metric. Similar to the property of Riemannian curvature tensor, the conharmonic tensor has the following property:

$$T_{ijkl} = -T_{jikl} = -T_{ijlk} = T_{klij}.$$

Definition 2.4. An AH-manifold is called a *conharmonically flat* if the conharmonic tensor vanishes.

Definition 2.5.([3]) In the adjoined G-structure space, an AH-manifold $\{M, J, g = \langle ., . \rangle\}$ is called a *Vaisman-Gray manifold* (*VG-manifold*) if $B^{abc} = -B^{bac}, B^{ab}_c = \alpha^{[a}\delta^{b]}_c$; a *locally conformal Kähler manifold* (*LCK-manifold*) if $B^{abc} = 0$ and $B^{ab}_c = \alpha^{[a}\delta^{b]}_c$; and a *nearly Kähler manifold* (*NK-manifold*) if $B^{abc} = -B^{bac}$ and $B^{ab}_c = 0$, where $B^{abc} = \frac{\sqrt{-1}}{2} J^a_{[\hat{b},\hat{c}]}, B^{ab}_c = \frac{\sqrt{-1}}{2} J^a_{\hat{b},c]}, \alpha = \frac{1}{(n-1)}\delta F \circ J$ is a Lie form, F is a Kähler form which is given by $F(X, Y) = \langle JX, Y \rangle$, δ is a codrivative; $X, Y \in X(M)$ and the bracket $[\quad]$ denotes the antisymmetric operation.

Theorem 2.6.([3]) *In the adjoined G-structure space, the components of the Riemannian curvature tensor of the VG-manifold are given by the following forms:*

(1) $R_{abcd} = 2(B_{ab[cd]} + \alpha_{[a}B_{b]cd})$;

(2) $R_{\hat{a}bcd} = 2A^a_{bcd}$;

(3) $R_{\hat{a}\hat{b}cd} = 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]})$;

(4) $R_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}_{c}B^{d}_{hb}$,

where, $\{A^a_{bcd}\}$ are some functions on the adjoined G-structure space, $\{A^{ad}_{bc}\}$ are a system of functions in the adjoined G-structure space that are symmetric by the lower and upper indices, which are called the components of the holomorphic sectional curvature tensor.

The functions $\{\alpha^a_b, \alpha^b_a\}$ are the components of the covariant differential structure tensor of the first and second type, and $\{\alpha_{ab}, \alpha^{ab}\}$ are the components of the Lie form on the adjoined G-structure space such that:

$$d\alpha_a + \alpha_b\omega^b_a = \alpha^b_a\omega_b + \alpha_{ab}\omega^b and\ d\alpha^a - \alpha^b\omega^a_b = \alpha^a_b\omega^b + \alpha^{ab}\omega_b,$$

where, $\{\omega^a, \omega_a\}$ are the components of mixture form and $\{\omega^a_b\}$ are the components of the Riemannian connection of the metric g. Other components of the Riemannian curvature tensor R can be obtained by the property of symmetry for R.

Theorem 2.7.([5]) *In the adjoined G-structure space, the components of Ricci tensor of the VG-manifold are given by the following forms:*

(1) $r_{ab} = \frac{1-n}{2}\ (\alpha_{ab} + \alpha_{ba} + \alpha_a\alpha_b)$;

(2) $r_{\hat{a}b} = 3B^{cah}B_{cbh} - A^{ca}_{bc} + \frac{n-1}{2}\ (\alpha^a\alpha_b - \alpha^h\alpha_h) - \frac{1}{2}\alpha^h_h\delta^a_b + (n-2)\alpha^a_b$.

Whereas, the other components are conjugate to the above components.

The next theorem gives the components of the conharmonic tensor of the VG-manifold in the adjoined G-structure space.

Theorem 2.8.([1]) *In the adjoined G-structure space, the components of the conharmonic tensor of the VG-manifold are given by the following forms:*

(1) $T_{abcd} = 2(B_{ab[cd]} + \alpha_{[a}B_{b]cd})$;

(2) $T_{\hat{a}bcd} = 2A^a_{bcd} + \frac{1}{2(n-1)}(r_{bd}\delta^a_c - r_{bc}\delta^a_d)$;

(3) $T_{\hat{a}\hat{b}cd} = 2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]}) - \frac{1}{n-1}(r^{[a}_{d}\delta^{b]}_{c} + r^{[b}_{c}\delta^{a]}_{d})$;

(4) $T_{\hat{a}bc\hat{d}} = A^{ad}_{bc} + B^{adh}B_{hbc} - B^{ah}_{c}B^{d}_{hb} + \frac{1}{n-1}(r^{(a}_{(c}\delta^{d)}_{b)})$,

Whereas, the other components can be obtained by the conjugate operation regarding the above components.

Definition 2.9.([16]) A Riemannian manifold is called an *Einstein manifold* if the Ricci tensor satisfies the equation $r_{ij} = Cg_{ij}$, where C is an Einstein constant.

Definition 2.10.([12]) An AH-manifold has a *J-invariant Ricci tensor* if $J \circ r = r \circ J$.

The following Lemma shows the invariant Ricci tensor in the adjoined G-structure space.

Lemma 2.11.([18]) *An AH-manifold has a J-invariant Ricci tensor if, and only if, the equality $r^{\hat{a}}_{b} = r_{ab} = 0$ holds.*

Definition 2.12.([8]) Define two endomorphisms on $\tau^0_r(V)$ as follows:

(1) *Symmetric mapping $Sym : \tau^0_r(V) \longrightarrow \tau^0_r(V)$* by:

$$sym(t)(v_1, ..., v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} t(v_{\sigma(1)}, ..., v_{\sigma(r)}).$$

(2) *Antisymmetric mapping $Alt : \tau^0_r(V) \longrightarrow \tau^0_r(V)$* by:

$$Alt(t)(v_1, ..., v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} = \varepsilon(\sigma) t(v_{\sigma(1)}, ..., v_{\sigma(r)}).$$

The symbols ( ) and [ ] are usually used to denote the symmetric and antisymmetric respectively.

3. The main results

Definition 3.1. Let M be an AH-manifold. A holomorphic sectional conharmonic (HT-tensor) of a manifold M in the direction $X \in X(M), X \neq 0$ is a function $h(X)$, which is given by

$$\langle T(X, JX, X, JX,)\rangle = h(X)\|X\|^4; \quad \|X\|^2 = \langle X, X\rangle.$$

Definition 3.2. A manifold M has a pointwise holomorphic sectional conharmonic (PHT- tensor) if h does not depend on X, then this means

$$\langle T(X, JX, X, JX,)\rangle = h\|X\|^4; \quad X \in X(M), \quad h \in C^\infty(M).$$

Lemma 3.3.([2]) *If M is an AH-manifold of PHT- tensor, then the equation $\|X\|^4 = 2\tilde{\delta}^{bc}_{ad} X^a X^d X_b X_c$ holds, where $\tilde{\delta}^{bc}_{ad} = \delta^b_a \delta^c_d + \delta^b_d \delta^c_a$ is a Kroneker delta of the second type.*

The necessary condition for a VG-manifold to be a PHT-tensor is summarized in the following theorem.

Theorem 3.4. *Suppose that M is a VG-manifold of the conharmonic tensor and the J-invariant Ricci tenor. Then, the necessary condition for a VG-manifold to* *be a PHT- tensor is for the components of the HT-curvature tensor to satisfy the following condition:*

$$A^{bc}_{ad} = \frac{c}{2}\tilde{\delta}^{bc}_{ad} + B^{ch}_{a} B^{b}_{hd} - \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}).$$

Proof. Suppose that M is a VG-manifold of the PHT-tensor.
According to the Definition 3.2, we have

$$\langle T(X, JX, X, JX,)\rangle = c\|X\|^4.$$

By using the Lemma 3.3, it follows that

$$\langle T(X, JX, X, JX,)\rangle = 2c\tilde{\delta}^{bc}_{ad} X^a X^d X_b X_c.$$

In the adjoined G-structure space, we have

$$\begin{aligned} T_{ijkl}X^i(JX)^jX^k(JX)^l = \; & T_{abcd}X^a(JX)^bX^c(JX)^d + T_{ab\hat{c}d}X^a(JX)^bX^{\hat{c}}(JX)^d \\ & + T_{abc\hat{d}}X^a(JX)^bX^c(JX)^{\hat{d}} + T_{ab\hat{c}\hat{d}}X^a(JX)^bX^{\hat{c}}(JX)^{\hat{d}} \\ & + T_{a\hat{b}cd}X^a(JX)^{\hat{b}}X^c(JX)^d + T_{a\hat{b}\hat{c}d}X^a(JX)^{\hat{b}}X^{\hat{c}}(JX)^d \\ & + T_{a\hat{b}c\hat{d}}X^a(JX)^{\hat{b}}X^c(JX)^{\hat{d}} + T_{a\hat{b}\hat{c}\hat{d}}X^a(JX)^{\hat{b}}X^{\hat{c}}(JX)^{\hat{d}} \\ & + T_{\hat{a}bcd}X^{\hat{a}}(JX)^bX^c(JX)^d + T_{\hat{a}b\hat{c}d}X^{\hat{a}}(JX)^bX^{\hat{c}}(JX)^d \\ & + T_{\hat{a}bc\hat{d}}X^{\hat{a}}(JX)^bX^c(JX)^{\hat{d}} + T_{\hat{a}b\hat{c}\hat{d}}X^{\hat{a}}(JX)^bX^{\hat{c}}(JX)^{\hat{d}} \\ & + T_{\hat{a}\hat{b}cd}X^{\hat{a}}(JX)^{\hat{b}}X^c(JX)^d + T_{\hat{a}\hat{b}\hat{c}d}X^{\hat{a}}(JX)^{\hat{b}}X^{\hat{c}}(JX)^d \\ & + T_{\hat{a}\hat{b}c\hat{d}}X^{\hat{a}}(JX)^{\hat{b}}X^c(JX)^{\hat{d}} + T_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{a}}(JX)^{\hat{b}}X^{\hat{c}}(JX)^{\hat{d}}. \end{aligned}$$

According to the properties $(JX)^a = \sqrt{-1}X^a$ and $(JX)^{\hat{a}} = -\sqrt{-1}X^{\hat{a}}$, it follows that

$$\begin{aligned} T_{ijkl}X^i(JX)^jX^k(JX)^l = & - T_{abcd}X^aX^bX^cX^d - T_{ab\hat{c}d}X^aX^bX^{\hat{c}}X^d \\ & + T_{abc\hat{d}}X^aX^bX^cX^{\hat{d}} + T_{ab\hat{c}\hat{d}}X^aX^bX^{\hat{c}}X^{\hat{d}} \\ & + T_{a\hat{b}cd}X^a(X)^{\hat{b}}X^cX^d + T_{a\hat{b}\hat{c}d}X^aX^{\hat{b}}X^{\hat{c}}X^d \\ & - T_{a\hat{b}c\hat{d}}X^aX^{\hat{b}}X^cX^{\hat{d}} - T_{a\hat{b}\hat{c}\hat{d}}X^aX^{\hat{b}}X^{\hat{c}}X^{\hat{d}} \\ & - T_{\hat{a}bcd}X^{\hat{a}}X^bX^cX^d - T_{\hat{a}b\hat{c}d}X^{\hat{a}}X^bX^{\hat{c}}X^d \\ & + T_{\hat{a}bc\hat{d}}X^{\hat{a}}X^bX^cX^{\hat{d}} + T_{\hat{a}b\hat{c}\hat{d}}X^{\hat{a}}X^bX^{\hat{c}}X^{\hat{d}} \\ & + T_{\hat{a}\hat{b}cd}X^{\hat{a}}X^{\hat{b}}X^cX^d + T_{\hat{a}\hat{b}\hat{c}d}X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X^d \\ & + T_{\hat{a}\hat{b}c\hat{d}}X^{\hat{a}}X^{\hat{b}}X^cX^{\hat{d}} - T_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X)^{\hat{d}}. \end{aligned}$$

By using the properties of the conharmonic tensor, we get the following:

$$\begin{aligned} T_{ijkl}X^i(JX)^jX^k(JX)^l = &-T_{abcd}X^aX^bX^cX^d - 4T_{a\hat{b}\hat{c}\hat{d}}X^aX^{\hat{b}}X^{\hat{c}}X^{\hat{d}} \\ &- 4T_{\hat{a}bcd}X^{\hat{a}}X^bX^cX^d + 4T_{a\hat{b}\hat{c}d}X^aX^{\hat{b}}X^{\hat{c}}X^d \\ &- T_{\hat{a}\hat{b}\hat{c}\hat{d}}X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X)^{\hat{d}} + 2T_{\hat{a}\hat{b}cd}X^{\hat{a}}X^{\hat{b}}X^cX^d \\ = &\, 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c. \end{aligned}$$

Making use of the Theorem 2.3, we obtain

$$\begin{aligned} &-2(B_{ab[cd]}+\alpha_{[a}B_{b]cd})X^aX^bX^cX^d - 4(2A^{bcd}_a + \frac{1}{2(n-1)}(r^{[d}_{\hat{b}}\delta^{c]}_a))X^aX^{\hat{b}}X^{\hat{c}}X^{\hat{d}} \\ &-4(2A^a_{bcd} + \frac{1}{2(n-1)}(r_{bd}\delta^a_c - r_{bc}\delta^a_d))X^{\hat{a}}X^bX^cX^d \\ &+4(A^{bc}_{ad} + B^{bch}B_{had} - B^{ch}_aB^b_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}))X^aX^{\hat{b}}X^{\hat{c}}X^d \\ &-2(B^{ab[cd]}+\alpha^{[a}B^{b]cd})X^{\hat{a}}X^{\hat{b}}X^{\hat{c}}X^{\hat{d}} + 2(2(-B^{abh}B_{hcd} + \alpha^{[a}_{[c}\delta^{b]}_{d]}) \\ &-\frac{1}{n-1}(r^{[a}_d\delta^{b]}_c + r^{[b}_c\delta^{a]}_d))X^{\hat{a}}X^{\hat{b}}X^cX^d \\ &= 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c. \end{aligned}$$

Symmetrizing and antisymmetrizing the last equation by the indices (c,d), we have

$$\begin{aligned} &-4(\frac{1}{2(n-1)}(r_{bd}\delta^a_c - r_{bc}\delta^a_d))X^{\hat{a}}X^bX^cX^d \\ &+4(A^{bc}_{ad} + B^{bch}B_{had} - B^{ch}_aB^b_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}))X^aX^{\hat{b}}X^{\hat{c}}X^d \\ &= 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c. \end{aligned}$$

Since M has a J-invariant Ricci tensor, then

$$6(A^{bc}_{ad} + B^{bch}B_{had} - B^{ch}_aB^b_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}))X^aX^{\hat{b}}X^{\hat{c}}X^d = 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c.$$

Symmetrizing by the indices (b,c), we deduce

$$\begin{aligned} &4(A^{bc}_{ad} + \frac{1}{2}(B^{bch}B_{had} + B^{cbh}B_{had}) - B^{ch}_aB^b_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}))X^aX^{\hat{b}}X^{\hat{c}}X^d \\ &= 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c, \end{aligned}$$

$$4(A^{bc}_{ad} - B^{ch}_aB^b_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}))X^aX^{\hat{b}}X^{\hat{c}}X^d = 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c,$$

$$4(A^{bc}_{ad} - B^{ch}_aB^b_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}))X^aX^dX_bX_c = 2c\tilde{\delta}^{bc}_{ad}X^aX^dX_bX_c,$$

$$A^{bc}_{ad} - B^{ch}_{a} B^{b}_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) = \frac{c}{2}\tilde{\delta}^{bc}_{ad}.$$

Therefore,

$$A^{bc}_{ad} = \frac{c}{2}\tilde{\delta}^{bc}_{ad} + B^{ch}_{a} B^{b}_{hd} - \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}). \qquad \square$$

Theorem 3.5. *If M is a VG-manifold of the PHT-tensor, then M is an NK-manifold.*

Proof. Suppose that M is a manifold of the PHT-tensor. According to the Theorem 3.4, we get

$$A^{bc}_{ad} - B^{ch}_{a} B^{d}_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) = \frac{c}{2}\tilde{\delta}^{bc}_{ad}.$$

Symmetrizing and antisymmetrizing the above equation by the indices (b,c) consequently, we get

$$B^{ch}_{a} B^{b}_{hd} = 0.$$

Contracting the last equation by the indices (a, b) and (c, d), it follows that

$$B^{ch}_{a} B^{a}_{hc} = 0 \Leftrightarrow B^{ch}_{a} \bar{B}^{ch}_{a} = 0 \Leftrightarrow \sum_{a,d,h} |B^{ch}_{a}|^2 = 0 \Leftrightarrow B^{ch}_{a} = 0.$$

Therefore, according to the Definition 2.5, M is an NK-manifold. $\square$

Theorem 3.6. *Let M be a VG-manifold of the PHT-tensor with a flat holomorphic sectional curvature tensor and J-invariant Ricci tensor. Then, M is an Einstein manifold.*

Proof. Suppose that$M$ is a VG-manifold of the PHT-curvature tensor. According to the Theorem 3.4, we get

$$A^{bc}_{ad} - B^{ch}_{a} B^{d}_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) = \frac{c}{2}\tilde{\delta}^{bc}_{ad}.$$

Making use of the Theorem 3.5 consequently, we obtain

$$A^{bc}_{ad} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) = \frac{c}{2}\tilde{\delta}^{bc}_{ad}.$$

Since M has flat holomorphic sectional tensor, then

$$\frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) = \frac{c}{2}\tilde{\delta}^{bc}_{ad},$$

$$\frac{1}{2(n-1)}(r^{b}_{d}\delta^{c}_{a} + r^{c}_{a}\delta^{b}_{d}) = \frac{c}{2}(\delta^{b}_{a}\delta^{c}_{d} + \delta^{b}_{d}\delta^{c}_{a}).$$

Contracting by the indices (d, c), we have

$$\begin{aligned}
\frac{1}{2(n-1)}(r^b_d\delta^d_a + r^d_a\delta^b_d) &= \frac{c}{2}(\delta^b_a\delta^d_d + \delta^b_d\delta^d_a),\\
\frac{1}{2(n-1)}(r^b_a + r^b_a) &= \frac{c}{2}(n\delta^b_a + \delta^b_a),\\
\frac{1}{(n-1)}r^b_a &= \frac{c}{2}\delta^b_a(n+1),\\
r^b_a &= \frac{c(n^{2-1})}{2}\delta^b_a,\\
r^b_a &= e\delta^b_a.
\end{aligned}$$

Since M has a J-invariant Ricci tensor, then M is an Einstein manifold. □

Theorem 3.7. *Let M be a VG-manifold of the PHT-tensor with a J-invariant Ricci tensor, then M is an Einstein manifold if, and only if, $A^{bc}_{ac} = c_1\delta^b_a$, where c_1 is a constant.*

Proof. Suppose that M is a VG-manifold of the PHT-tensor. According to the Theorem 3.4, we have

$$A^{bc}_{ad} - B^{ch}_a B^d_{hd} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) = \frac{c}{2}\tilde{\delta}^{bc}_{ad}.$$

By using the Theorem 3.5, we obtain

$$\begin{aligned}
A^{bc}_{ad} + \frac{1}{n-1}(r^{(b}_{(d}\delta^{c)}_{a)}) &= \frac{c}{2}\tilde{\delta}^{bc}_{ad},\\
A^{bc}_{ad} + \frac{1}{2(n-1)}(r^b_d\delta^c_a + r^c_a\delta^b_d) &= \frac{c}{2}\tilde{\delta}^{bc}_{ad},\\
A^{bc}_{ad} + \frac{1}{2(n-1)}(r^b_d\delta^c_a + r^c_a\delta^b_d) &= \frac{c}{2}(\delta^b_a\delta^c_d + \delta^b_d\delta^c_a).
\end{aligned}$$

Contracting by the indices (c, d), it follows that

$$\begin{aligned}
A^{bc}_{ac} + \frac{1}{2(n-1)}(r^b_c\delta^c_a + r^c_a\delta^b_c) &= \frac{c}{2}(\delta^b_a\delta^c_c + \delta^b_c\delta^c_a),\\
A^{bc}_{ac} + \frac{1}{2(n-1)}(r^b_a + r^b_a) &= \frac{c}{2}(n\delta^b_a + \delta^b_a),\\
A^{bc}_{ac} + \frac{1}{(n-1)}r^b_a &= \frac{c\delta^b_a}{2}(n+1),
\end{aligned}$$

$$A^{bc}_{ac} = \frac{c\delta^b_a}{2}(n+1) - \frac{1}{(n-1)}r^b_a. \tag{3.1}$$

Since M is an Einstein manifold, then

$$\begin{aligned}A^{bc}_{ac} &= \frac{c\delta^b_a}{2}(n+1) - \frac{1}{(n-1)}e\delta^b_a,\\ A^{bc}_{ac} &= \{\frac{c}{2}(n+1) - \frac{1}{(n-1)}e\}\delta^b_a,\\ A^{bc}_{ac} &= c_1\delta^b_a.\end{aligned}$$

Conversely, by using the equation (3.1), we have

$$A^{bc}_{ac} = \frac{c\delta^b_a}{2}(n+1) - \frac{1}{(n-1)}r^b_a.$$

Since $A^{bc}_{ac} = c_1\delta^b_a$, then

$$\begin{aligned}c_1\delta^b_a &= \frac{c\delta^b_a}{2}(n+1) - \frac{1}{(n-1)}r^b_a,\\ \frac{1}{(n-1)}r^b_a &= \{\frac{c}{2}(n+1) - c_1\}\delta^b_a,\\ r^b_a &= \frac{(c(n+1) - 3c_1)(n-1)}{2}\delta^b_a,\\ r^b_a &= e\delta^b_a.\end{aligned}$$

Since M has a J-invariant Ricci tensor, then M is an Einstein manifold. □

4. Conclusions

This article clearly aimed to study the geometrical properties of the VG-manifold of the pointwise holomorphic sectional curvature conharmonic tensor. We have found out the necessary conditions for the VG-manifold to be a manifold of the pointwise holomorphic sectional conharmonic tensor. Furthermore, we have formulated an interesting theoretical physical application. In particular, we have concluded the necessary and sufficient conditions for a VG-manifold to be an Einstein manifold.

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