KYUNGPOOK Math. J. 58(2018), 761-779
https://doi.org/10.5666/KMJ.2018.58.4.761
pISSN 1225-6951 eISSN 0454-8124

On Weakly Z Symmetric Spacetimes

Uday Chand De
Department of Pure Mathematics, University of Calcutta 35, Ballygunge Circular Road, Kolkata 700019, West Bengal, India
e-mail : uc_de@yahoo.com

Abstract. The object of the present paper is to study weakly Z symmetric spacetimes $(WZS)_4$. At first we prove that a weakly Z symmetric spacetime is a quasi-Einstein spacetime and hence a perfect fluid spacetime. Next, we consider conformally flat $(WZS)_4$ spacetimes and prove that such a spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ. We also study $(WZS)_4$ spacetimes with divergence free conformal curvature tensor. Moreover, we characterize dust fluid and viscous fluid $(WZS)_4$ spacetimes. Finally, we construct an example of a $(WZS)_4$ spacetime.

1. Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional pseudo-Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-, +, +, +)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity and cosmology.

The Einstein's equation [21] imply that the energy momentum tensor is of vanishing divergence. This requirement is satisfied if the energy momentum tensor is covariant constant. Chaki and Roy [7] proved that a general relativistic spacetime with covariant constant energy momentum tensor is Ricci symmetric, that is, $\nabla S = 0$, where S is the Ricci tensor of the spacetime and ∇ denotes the covariant differentiation with respect to the metric tensor g. However if, $\nabla S \neq 0$, then such a spacetime may be called weakly Ricci symmetric spacetime [31]. We may say that the Ricci symmetric condition is only a special case of weakly Ricci symmet-

Received February 2, 2018; accepted June 4, 2018.
2010 Mathematics Subject Classification: 53C25, 53C35, 53C50, 53B30.
Key words and phrases: weakly Z symmetric manifolds, weakly Z symmetric spacetimes, dust fluid and viscous fluid spacetimes, Robertson-Walker spacetime, Weyl conformal curvature tensor.

ric manifold. Recently, Mantica and Molinari [14] introduced weakly Z symmetric manifolds which generalizes the notion of weakly Ricci symmetric manifolds. As a special case, Mantica and Suh [15, 17] studied pseudo Z symmetric manifolds and pseudo Z symmetric spacetimes. It is therefore meaningful to study the properties of weakly Z symmetric spacetimes in general theory of relativity and cosmology.

In 1993 Tamássy and Binh [31] introduced the notion of weakly Ricci symmetric manifolds. A non-flat pseudo-Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly Ricci symmetric* if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(Y,X), \tag{1.1}$$

where ∇ denotes the Levi-Civita connection and A, B, D are 1-forms which are non-zero simultaneously. Such an n-dimensional manifold is denoted by $(WRS)_n$. If $A = B = D = 0$, then the manifold reduces to a Ricci symmetric ($\nabla S = 0$) manifold. Several authors studied spacetimes in several ways such as conformally flat almost pseudo Ricci symmetric spacetimes by De, Özgür and De [10], m-projectively flat spacetimes by Zengin [34], pseudo Z symmetric spacetimes by Mantica and Suh [17, 18] and many others.

According to Yano [33] a vector field V is torse-forming if

$$\nabla_X V = fX + \omega(X)V,$$

where f is a scalar function, ω is a 1-form. Its properties in pseudo-Riemannian manifolds were studied by Mikes and Rachunek [19]. The vector is named concircular if ω is closed.

In a pseudo-Riemannian manifold (M^n, g), $(n > 2)$, a (0,2) symmetric tensor is a generalized Z tensor [16, 17] if

$$Z(X,Y) = S(X,Y) + \phi g(X,Y), \tag{1.2}$$

where ϕ is an arbitrary scalar function. The scalar Z is obtained by contracting (1.2) over X and Y as follows:

$$Z = r + n\phi, \tag{1.3}$$

where the scalar curvature $r = \Sigma_{i=1}^{n} \epsilon_i S(e_i, e_i)$, $g(e_i, e_i) = \epsilon_i$, $\epsilon_i = \pm 1$, $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold.

A pseudo-Riemannian manifold is said to be *weakly Z symmetric* [14], denoted by $(WZS)_n$, if the generalized Z tensor satisfies the condition

$$(\nabla_X Z)(U,V) = A(X)Z(U,V) + B(U)Z(X,V) + D(V)Z(U,X), \tag{1.4}$$

where A, B and D are 1-forms not simultaneously zero. If $\phi = 0$ we recover from (1.4) a $(WRS)_n$ and its particular case pseudo Ricci symmetric manifolds $(PRS)_n$.

If $\phi = -\frac{r}{n}$ (classical Z tensor) and if A is replaced by $2A$ and B and D are replaced by A, then

$$Z(U, V) = \frac{n-1}{n} P(U, V),$$

where $P(U, V)$ is the projective Ricci tensor introduced by Chaki and Saha [8] which is obtained by a contraction of the projective curvature tensor. It was a generalization of the notion of weakly Ricci symmetric manifolds [31], pseudo Ricci symmetric manifolds [5], pseudo projective Ricci symmetric manifolds [8].

Recently, Mantica and Suh studied pseudo Z symmetric Riemannian manifolds [15] and recurrent Z forms on Riemannian manifolds [16], that is, Riemannian manifolds on which the form $\Lambda_{(Z)l} = Z_{kl}dx^k$ satisfies the condition $D\Lambda_{(Z)l} = \beta \wedge \Lambda_{(Z)l}$, D being the exterior covariant derivative and $\beta = \beta_i dx^i$, the associated one-form. It should be noted that the concept of Z recurrent form embraces both pseudo Z symmetric and weakly Z symmetric manifolds.

On the otherhand, Lorentzian manifolds with Ricci tensor S of the form

$$S(X, Y) = ag(X, Y) + bA(X)A(Y), \tag{1.5}$$

where a and b are scalars and the vector field ρ metrically equivalent to the 1-form A, that is, $g(X, \rho) = A(X)$ for all X is a unit timelike vector field, that is, $g(\rho, \rho) = -1$, are often named perfect fluid spacetimes. It is well known that any Robertson-Walker spacetime is a perfect fluid spacetime [21]. The form (1.5) of the Ricci tensor is implied by Einstein's equation if the energy -matter content of the spacetime is a perfect fluid with velocity vector ρ. The scalars a and b are linearly related to the pressure p and the energy density σ measured in the locally comoving initial frame.

Geometers identify the special form (1.5) of the Ricci tensor as the defining property of quasi-Einstein manifolds [6]. Pseudo-Riemannian quasi-Einstein manifolds arose in the study of exact solutions of Einstein's equations. Robertson-Walker spacetimes are quasi-Einstein [3]. The importance of the study of the quasi-Einstein spacetime lies in the fact that this spacetime represents the present state of the universe when the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be considered as a perfect fluid.

Shepley and Taub [28] studied perfect fluid spacetimes with equation of state $p = p(\sigma)$, p is the isotropic pressure and σ is the energy density and the additional condition $divC = 0$, where C is the conformal curvature tensor and 'div' denotes divergence. A related result was obtained by Sharma [27]. De et al [10] proved that conformally flat almost pseudo-Ricci symmetric spacetimes, that is,

$$(\nabla_X S)(Y, U) = (A(X) + B(X))S(Y, U) + A(Y)S(X, U) + A(Z)S(X, Y),$$

are Robertson-Walker spacetimes.

Motivated by the above works, in the present paper we study $(WZS)_4$ spacetimes. Study of such a spacetime partly deals with the physical structure of the universe at a large scale and describes physical processes occurring throughout its evolution and having observable consequences in the present time.

The paper is organized as follows:
After introduction in Section 2, we prove that a $(WZS)_4$ spacetime is a quasi Einstein spacetime and hence $(WZS)_4$ can be taken as the model of the perfect fluid spacetimes. In this section we prove that under certain condition the Lie derivative of the energy momentum tensor of such a spacetime is zero. Section 3 is devoted to study conformally flat $(WZS)_4$ spacetimes. Section 4 is concerned with the study of $(WZS)_4$ spacetimes satisfying the condition $divC = 0$, where "*div*" denotes divergence. In this section we first show that such a spacetime satisfying the condition $divC = 0$ under certain assumption, the integral curves of the vector field ρ are geodesic and the vector field ρ is irrotational. Next we prove that such a spacetime is locally a product space. Also, we show that a $(WZS)_4$ spacetime under certain condition is the generalized Robertson-Walker spacetime. Moreover, in this section it is shown that such a spacetime has vanishing vorticity tensor and shear tensor and local cosmological structure of the spacetimes are of Petrov type I, D or O. Section 5 deals with the study of dust fluid and viscous fluid $(WZS)_4$ spacetimes. Here we prove an interesting result which states that under certain condition a dust fluid $(WZS)_4$ spacetime satisfying Einstein's field equation with cosmological constant is devoid of matter. Finally, we construct an example of a $(WZS)_4$ spacetime.

2. Weakly Z Symmetric Spacetimes

In this Section we prove that a $(WZS)_4$ spacetime is a quasi-Einstein spacetime. Interchanging U and V in (1.4) we obtain

$$(\nabla_X Z)(V,U) = A(X)Z(V,U) + B(V)Z(U,X) + D(U)Z(X,V). \tag{2.1}$$

Substracting (2.1) from (1.4) we get

$$[B(U) - D(U)]Z(V,X) + [D(V) - B(V)]Z(X,U) = 0,$$

which implies

$$[B(U) - D(U)]Z(V,X) = [B(V) - D(V)]Z(X,U). \tag{2.2}$$

If possible, let $E(X) = g(X,\rho) = B(X) - D(X)$, for all vector fields X, where ρ is a unit timelike vector field associated with the 1-form E, that is, $g(\rho,\rho) = -1$. Then from (2.2) we have

$$E(U)Z(V,X) = E(V)Z(X,U). \tag{2.3}$$

Taking a frame field and contracting (2.3) over X and V, we obtain

$$E(U)[r + 4\phi] = S(U,\rho) + \phi E(U),$$

which implies

$$S(U,\rho) = [r + 3\phi]E(U). \tag{2.4}$$

Putting $V = \rho$ in (2.3) yields

$$E(U)Z(X,\rho) = -Z(X,U), \tag{2.5}$$

since $E(\rho) = g(\rho,\rho) = -1$. Using (1.2) in (2.5) we obtain

$$E(U)\{S(X,\rho) + \phi g(X,\rho)\} = -\{S(X,U) + \phi g(X,U)\}. \tag{2.6}$$

In virtue of (2.4) and (2.6) we have

$$E(U)[\{r + 3\phi\}E(X) + \phi E(X)] = -[S(X,U) + \phi g(X,U)],$$

which implies

$$S(X,U) = -\phi g(X,U) - (r + 4\phi)E(X)E(U),$$

that is,

$$S(X,U) = ag(X,U) + bE(X)E(U), \tag{2.7}$$

where $a = -\phi$ and $b = -(r + 4\phi)$.

Thus we can state the following:

Theorem 2.1. *A $(WZS)_4$ spacetime is a quasi-Einstein spacetime and hence a perfect fluid spacetime.*

We now consider a $(WZS)_4$ spacetime with constant scalar curvature and the associated scalar ϕ of the Z tensor is constant. Then a and b are constant.
The Einstein's field equation with cosmological constant can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{2.8}$$

where λ is the cosmological constant, κ is the gravitational conatant and T is the energy-momentum tensor of type (0,2).

Using the equations (2.7) and (2.8) we obtain that

$$(\lambda - \frac{r}{2} + a)g(X,Y) + bE(X)E(Y) = \kappa T(X,Y). \tag{2.9}$$

Let us suppose that the generator ρ of the spacetime is a Killing vector field.
Then

$$(\pounds_\rho g)(X,Y) = 0, \tag{2.10}$$

where $\pounds$ denotes the Lie derivative.

Now

$$\begin{aligned}(\pounds_\rho E)X &= \pounds_\rho E(X) - E(\pounds_\rho X)\\ &= \pounds_\rho g(X,\rho) - g(\pounds_\rho X,\rho)\\ &= (\pounds_\rho g)(X,\rho),\end{aligned}$$

since $\pounds_\rho \rho = 0$, by using (2.10).
Now from (2.9) we obtain that

$$(\lambda - \frac{r}{2} + a)(\pounds_\rho g)(X,Y) = \kappa(\pounds_\rho T)(X,Y). \tag{2.11}$$

Since $\kappa \neq 0$, equations (2.10) and (2.11) yield

$$(\pounds_\rho T)(X,Y) = 0. \tag{2.12}$$

Thus we can state the following:

Theorem 2.2. *In $(WZS)_4$ spacetime with constant scalar curvature and constant scalar Z obeying Einstein's field equation, the energy-momentum tensor is invariant along the vector field ρ, provided the vector field ρ is a Killing vector field.*

3. Conformally Flat $(WZS)_4$ Spacetimes

This section is devoted to study conformally flat $(WZS)_4$ spacetimes. In a conformally flat 4-dimensional Lorentzian manifold the curvature tensor R is of the form

$$\begin{aligned}R(X,Y)U = \frac{1}{2}[S(Y,U)X - S(X,U)Y + g(Y,U)QX - g(X,U)QY]\\ -\frac{r}{6}[g(Y,U)X - g(X,U)Y],\end{aligned} \tag{3.1}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$.

Using (2.7) in (3.1) yields

$$\begin{aligned}R(X,Y)U = \frac{1}{2}[ag(Y,U) + bE(Y)E(U) - ag(X,U)Y - bE(X)E(U)Y\\ +ag(Y,U)X + bg(Y,U)E(X)\rho - ag(X,U)Y - bg(X,U)E(Y)\rho]\\ -\frac{r}{6}[g(Y,U)X - g(X,U)Y].\end{aligned}$$

Let $\rho^\perp$ denote the 3-dimensional distribution in a conformally flat $(WZS)_4$ spacetimes orthogonal to ρ, then

$$R(X,Y)U = (a - \frac{r}{6})[g(Y,U)X - g(X,U)Y] \tag{3.2}$$

for all $X, Y \in \rho^\perp$ and

$$R(X,\rho)\rho = -(a - \frac{r}{6})X, \tag{3.3}$$

for every $X \in \rho^{\perp}$. According to Karchar [13] a Lorentzian manifold is called infinitesimal spatially isotropic relative to timelike unit vector field ρ if its curvature tensor R satisfies the relations

$$R(X,Y)U = l[g(Y,U)X - g(X,U)Y]$$

for all $X, Y, U \in \rho^{\perp}$ and

$$R(X,\rho)\rho = mX$$

for all $X \in \rho^{\perp}$, where l, m are real valued functions on the manifold. So by virtue of (3.2) and (3.3) we can state the following:

Theorem 3.1. *A conformally flat $(WZS)_4$ spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ρ.*

4. $(WZS)_4$ Spacetime Satisfying the Condition $divC = 0$

Suppose (M^n, g) is a pseudo-Riemannian manifold of dimension n and X is any vector field on M. Then the divergence of the vector field X, denoted by $divX$, is defined as

$$divX = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i),$$

where $\{e_i\}$ is an orthonormal basis of the tangent space T_pM at any point $p \in M$ and $\epsilon_i = \pm 1$. Again, if K is a tensor field of type (1,3), then its divergence $divK$ is a tensor field of type (0,3) defined as

$$(divK)(X_1, ..., X_3) = \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} K)(X_1, ..., X_3), e_i).$$

In this section we assume that the $(WZS)_4$ spacetimes satisfy the condition $divC = 0$, where C denotes the Weyl conformal curvature tensor and "div" denotes divergence. Hence we have [11]

$$(\nabla_X S)(Y,U) - (\nabla_U S)(Y,X) = \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \tag{4.1}$$

Using (2.7) in (4.1) we obtain

$$\begin{aligned} da(X)g(Y,U) + db(X)E(Y)E(U) + b[(\nabla_X E)(Y)E(U) \\ +(\nabla_X E)(U)E(Y)] - da(U)g(Y,X) - db(U)E(Y)E(X) \\ -b[(\nabla_U E)(Y)E(X) + (\nabla_U E)(X)E(Y)] \\ = \frac{1}{6}[g(Y,U)dr(X) - g(X,Y)dr(U)]. \end{aligned} \tag{4.2}$$

Taking a frame field and contracting over X and Y we get

$$\begin{aligned} -3da(U) + db(\rho)E(U) + bE(U)(\delta E) \\ +b(\nabla_\rho E)(U) + db(U) = -\frac{1}{2}dr(U), \end{aligned} \tag{4.3}$$

where

$$\delta E = \sum_{i=1}^{n} \epsilon_i (\nabla_{e_i} E)(e_i).$$

Putting $X = Y = \rho$ in (4.2) yields

$$\begin{aligned} b(\nabla_\rho E)(U) &= da(\rho)E)(U) - db(\rho)E(U) \\ &+ da(U) - db(U) - \frac{1}{6}[dr(\rho)E(U) + dr(U)]. \end{aligned} \tag{4.4}$$

Using (4.4) in (4.3) we obtain

$$\begin{aligned} -2da(U) + da(\rho)E(U) + bE(U)(\delta E) \\ -\frac{1}{6}dr(\rho)E(U) = -\frac{1}{3}dr(U). \end{aligned} \tag{4.5}$$

Putting $U = \rho$ in (4.5) we obtain

$$-3da(\rho) - b(\delta E) = -\frac{1}{2}dr(\rho). \tag{4.6}$$

Using (4.6) in (4.5) we get

$$-2da(U) - 2da(\rho)E(U) + \frac{1}{3}dr(\rho)E(U) = -\frac{1}{3}dr(U). \tag{4.7}$$

If possible, let $r = a$, then

$$dr(U) = da(U). \tag{4.8}$$

and

$$db(U) = 3da(U). \tag{4.9}$$

Again using (4.8) in (4.7), yields

$$da(U) = -da(\rho)E(U). \tag{4.10}$$

Using (4.8) in (4.10) we get

$$dr(U) = -dr(\rho)E(U). \tag{4.11}$$

Putting $Y = \rho$ in (4.2) and using (4.10) we have

$$(\nabla_U E)(X) - (\nabla_X E)(U) = 0, \tag{4.12}$$

since $b \neq 0$. This means that the 1-form E is closed, that is,

$$dE(X, Y) = 0.$$

Hence it follows that

$$g(\nabla_X \rho, Y) = g(\nabla_Y \rho, X), \tag{4.13}$$

for all X, Y.
Now using $Y = \rho$ in (4.13) we get

$$g(\nabla_X \rho, \rho) = g(\nabla_\rho \rho, X). \tag{4.14}$$

Since ρ is a unit timelike vector field, therefore $g(\nabla_X \rho, \rho) = 0$, from (4.14) it follows that $g(\nabla_\rho \rho, X) = 0$ for all X. Hence $\nabla_\rho \rho = 0$. This means that the integral curves of the vector field ρ are geodesic and the vector field ρ is irrotational.

Therefore we can state the following:

Theorem 4.1. *In a $(WZS)_4$ spacetime with divergence free Weyl conformal curvature tensor under the assumption $r = a$, the integral curves of the vector field ρ are geodesic and the vector field ρ is irrotational.*

Using (4.10) and (4.11) in (4.4) we obtain

$$(\nabla_\rho E)(U) = 0, \tag{4.15}$$

since $b \neq 0$. Now we consider the scalar function

$$f = \frac{1}{6} \frac{dr(\rho)}{b}. \tag{4.16}$$

Then using (4.9) we get

$$\nabla_X f = \frac{1}{2} \frac{dr(\rho)}{b^2} dr(X) + \frac{1}{6b} d^2 r(\rho, X). \tag{4.17}$$

On the other hand, (4.11) implies that

$$d^2 r(Y, X) = -d^2 r(\rho, Y) E(X) - dr(\rho)(\nabla_Y E)(X),$$

from which we get

$$d^2 r(\rho, Y) E(X) = d^2 r(\rho, X) E(Y), \tag{4.18}$$

since $(\nabla_X E)(Y) = (\nabla_Y E)(X)$ and $d^2 r(Y, X) = d^2 r(X, Y)$.
Putting $X = \rho$ in (4.18), it follows that

$$d^2 r(Y, \rho) = -d^2 r(\rho, \rho) E(Y). \tag{4.19}$$

Then using (4.19) in (4.17) we obtain

$$\nabla_X f = -\frac{dr(\rho)}{2b^2} dr(\rho) E(X) - \frac{1}{6b} d^2 r(\rho, \rho) E(X),$$

which implies that

$$\nabla_X f = \mu E(X), \tag{4.20}$$

where $\mu = \frac{1}{6b}[-d^2r(\rho,\rho) - 3dr(\rho)dr(\rho)]$.
Using (4.20), it is easy to show that

$$\omega(X) = \frac{1}{6}\frac{dr(\rho)}{b}E(X) = fE(X)$$

is closed. In fact, $d\omega(X,Y) = 0$.

Using (4.10), (4.11),(4.12) in (4.2) we obtain that

$$\begin{aligned} &-dr(\rho)E(X)g(Y,U) + b[(\nabla_X E)(Y)E(U) + (\nabla_X E)(U)E(Y)] \\ &+dr(\rho)E(U)g(Y,X) - b[(\nabla_U E)(Y)E(X) + (\nabla_U E)(X)E(Y)] \\ &= \frac{1}{6}[-g(Y,U)dr(\rho)E(X) + g(X,Y)dr(\rho)E(U)]. \end{aligned} \tag{4.21}$$

Putting $U = \rho$ in (4.21) and using (4.15) we get

$$(\nabla_X E)(Y) = (f - \frac{dr(\rho)}{b})g(X,Y) + (\omega(X) - \frac{dr(\rho)}{b}E(X))E(Y). \tag{4.22}$$

From (4.22) it follows that

$$\nabla_X \rho = (f - \frac{dr(\rho)}{b})X + (\omega(X) - \frac{dr(\rho)}{b}E(X))\rho. \tag{4.23}$$

Let $\rho^{\perp}$ denote the 3-dimensional distribution in a $(WZS)_4$ spacetime orthogonal to ρ. If X and Y belong to $\rho^{\perp}$, then

$$g(X,\rho) = 0. \tag{4.24}$$

and

$$g(Y,\rho) = 0. \tag{4.25}$$

Since $(\nabla_X g)(Y,\rho) = 0$, it follows from (4.23) and (4.25) that

$$g(\nabla_X Y,\rho) = g(\nabla_X \rho, Y) = (f - \frac{dr(\rho)}{b})g(X,Y).$$

Similarly, we get

$$g(\nabla_Y X,\rho) = g(\nabla_Y \rho, X) = (f - \frac{dr(\rho)}{b})g(X,Y).$$

Hence

$$g(\nabla_X Y,\rho) = g(\nabla_Y X,\rho). \tag{4.26}$$

Now $[X, Y] = \nabla_X Y - \nabla_Y X$ and therefore by (4.26) we have

$$g([X, Y], \rho) = g(\nabla_X Y - \nabla_Y X, \rho) = 0.$$

Hence $[X, Y]$ is orthogonal to ρ. That is, $[X, Y]$ belongs to $\rho^{\perp}$. Thus the distribution $\rho^{\perp}$ is involutive [4]. Hence from Frobenius' theorem [4] it follows that $\rho^{\perp}$ is integrable. This implies that if a $(WZS)_4$ spacetime satisfies $divC = 0$, then it is locally a product space. Hence we have the following:

Theorem 4.2. *If a $(WZS)_4$ spacetime satisfies $divC = 0$ and fulfilling the condition $r = a$, then it is locally a product space.*

From (4.22) we can write

$$(\nabla_X E)(Y) = \beta g(X, Y) + F(X)E(Y), \tag{4.27}$$

where $\beta = (f - \frac{dr(\rho)}{b})$ and $F(X) = (\omega(X) - \frac{dr(\rho)}{b}E(X))$. Obviously the 1-form F is closed. In local components this reads as $\nabla_k E_j = F_k E_j + \beta g_{kj}$.

Therefore the vector field ρ corresponding to the 1-form E defined by $g(X, \rho) = E(X)$ is a concircular vector field [26, 32]. Since ρ is a unit timelike vector field, that is, $g(\rho, \rho) = -1$, (4.27) can be written as

$$(\nabla_X E)(Y) = \beta\{g(X, Y) + E(X)E(Y)\}. \tag{4.28}$$

Thus we obtain the following:

Theorem 4.3. *If a $(WZS)_4$ spacetime satisfies $divC = 0$ and fulfilling the condition $r = a$, then ρ is a concircular vector field.*

Remark 4.4. In Theorem 3.2 of [16] the authors prove the above Theorem under a restriction on ϕ.

Since F_j is closed it is locally a gradient of a suitable scalar function, that is, $F_j = \nabla_j \sigma$ (see [12] page 242-243); setting $X_j = E_j e^{-\sigma}$ we have (see [18])

$$\nabla_k X_j = e^{-\sigma}(\nabla_k E_j - E_j \nabla_k \sigma) = e^{-\sigma}\{F_k E_j - E_j F_k + \beta g_{kj}\} = (e^{-\sigma}\beta)g_{kj}$$

and consequently

$$\nabla_k X_j = \theta g_{kj},$$

being $\theta = e^{-\sigma}\beta$ a scalar function and $X_j X^j = -e^{-2\sigma} < 0$ is a time-like vector. The previous equation can be written in the form $\nabla_k X_j + \nabla_j X_k = 2\theta g_{kj}$, that is, X_j is a conformal Killing vector [30]. We recall now the definition of a generalized Robertson-Walker spacetime [1, 24, 25]

Definition 4.5. An $n(n \geq 3)$-dimensional Lorentzian manifold is named *generalized Robertson-Walker spacetime* if the metric takes the local shape:

$$ds^2 = -(dt)^2 + q(t)^2 g^*_{\alpha\beta} dx^\alpha dx^\beta, \tag{4.29}$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, ..., n)$ and q is a function of t only.

The generalized Robertson Walker spacetime is thus the warped product $-1 \times q^2 M^*$ [1, 24, 25] where M^* is a $n-1$ dimensional Riemannian manifold. If M^* is a 3-dimensional Riemannian manifold of constant curvature, the spacetime is called Robertson-Walker spacetime. The following deep result was recently proved in the paper [9].

Theorem 4.6.([9]) *Let M be an n $(n \geq 3)$ dimensional Lorentzian manifold. Then the spacetime is a generalized Robertson-Walker spacetime if and only if it admits a time-like vector of the form $\nabla_k X_j = \theta g_{kj}$.*

In view of these results if a $(WZS)_4$ spacetime satisfies $divC = 0$ and fulfills the condition $r = a$, then it admits a concircular vector field rescalable to a timelike vector of the form $\nabla_k X_j = \theta g_{kj}$ and so becomes a generalized Robertson-Walker spacetime. Hence we conclude the following:

Theorem 4.7. *If a $(WZS)_4$ spacetime satisfies $divC = 0$ and fulfills the condition $r = a$, then the spacetime is the generalized Robertson-Walker spacetime.*

Finally, we consider $(WZS)_4$ spacetime with $divC = 0$. From Theorem 2.1 it follows that $(WZS)_4$ is a perfect fluid spacetime. Then the energymomentum tensor T is of the form [20, 21]

$$T(X,Y) = (p + \sigma)E(X)E(Y) + pg(X,Y),$$

where σ is the energy density and p is the isotropic pressure of the fluid. The velocity vector field ρ of the fluid corresponding to the 1-form E is a timelike vector field. We assume that the velocity vector field of the fluid is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to ρ. From Theorem we obtain the integral curves of the vector field ρ in a spacetime with $divC = 0$ are geodesics, the Roy Choudhury equation [23] for the fluid can be written as

$$(\nabla_X E)(Y) = \tilde{\omega}(X,Y) + \tau(X,Y) + \beta\{g(X,Y) + E(X)E(Y)\}, \tag{4.30}$$

where $\tilde{\omega}$ is the vorticity tensor and τ is the shear tensor.
Comparing (4.28) and (4.30) we get

$$\tilde{\omega}(X,Y) + \tau(X,Y) = 0. \tag{4.31}$$

Again from Theorem it follows that ρ is irrotational. Thus the spacetime under consideration is vorticity-free. Therefore $\tilde{\omega}(X,Y) = 0$ and consequently (4.31) implies that $\tau(X,Y) = 0$. Thus we can state the following:

Theorem 4.8. *If a $(WZS)_4$ spacetime satisfies $divC = 0$ and fulfilling the condition $r = a$, then the fluid has vanishing vorticity and vanishing shear.*

According to Petrov classification a spacetime can be devided into six types denoted by I, II, III, D, N and O [22]. Again Barnes [2] has proved that if a perfect fluid spacetime is shear free, vorticity free and the velocity vector field is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to the velocity vector field, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O. Thus from Theorem we can state the following:

Theorem 4.9. *If in a* $(WZS)_4$ *spacetime satisfying* $divC = 0$ *and fulfilling the condition* $r = a$, *the velocity vector field is always hypersurface orthogonal, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.*

5. Dust Fluid and Viscous Fluid $(WZS)_4$ Spacetimes

In a dust or pressureless fluid spacetime, the energy momentum tensor T is of the form [29]

$$T(X,Y) = \sigma E(X)E(Y), \tag{5.1}$$

where σ is the energy density of the dust-like matter and E is a non-zero 1-form such that $g(X, \rho) = E(X)$, for all X, ρ being the velocity vector field of the flow, that is, $g(\rho, \rho) = -1$. In Theorem 2.1, it is proved that a $(WZS)_4$ spacetime is a quasi Einstein spacetime, that is,

$$S(X,Y) = ag(X,Y) + bE(X)E(Y), \tag{5.2}$$

where $a = -\phi$, $b = -(r + 4\phi)$. Einstein's field equation with cosmological constant is

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{5.3}$$

where λ is the cosmological constant and κ is the gravitational constant.

Using (5.1) and (5.2) in (5.3), we obtain

$$(a - \frac{r}{2} + \lambda)g(X,Y) + bE(X)E(Y) = \kappa\sigma E(X)E(Y). \tag{5.4}$$

Taking a frame field after contraction over X and Y we have

$$4(a - \frac{r}{2} + \lambda) - b = -\kappa\sigma,$$

which implies

$$\lambda = \frac{1}{4}(2r - 4a + b - \kappa\sigma). \tag{5.5}$$

Again, if we put $X = Y = \rho$ in (5.4), we get

$$-(a - \frac{r}{2} + \lambda) + b = \kappa\sigma,$$

which implies that

$$\lambda = \frac{r}{2} - a + b - \kappa\sigma. \tag{5.6}$$

Combining equation (5.5) and (5.6), we obtain that

$$\sigma = -\frac{(r + 4\phi)}{\kappa}.$$

Therefore

$$\sigma = -\frac{Z}{\kappa},$$

using (1.3). Thus we can state the following:

Theorem 5.1. *A dust fluid $(WZS)_4$ spacetime satisfying Einstein's field equation with cosmological constant is vacuum, provided the scalar Z vanishes.*

Let us consider the energy momentum tensor T of a viscous fluid spacetime in the following form [20, 21]:

$$T(X,Y) = pg(X,Y) + (\sigma + p)E(X)E(Y) + P(X,Y), \tag{5.7}$$

where σ, p are the energy density and isotropic pressure respectively and P denotes the anisotropic pressure tensor of the fluid.

Using (5.2) and (5.3) in (5.7), we get

$$(a-\frac{r}{2}+\lambda)g(X,Y)+bE(X)E(Y) = \kappa[pg(X,Y)+(\sigma+p)E(X)E(Y)+P(X,Y)]. \tag{5.8}$$

Putting $X = Y = \rho$ in (5.8), yields

$$-(a - \frac{r}{2} + \lambda) + b = \kappa[-p + (\sigma + p) + I],$$

where $I = P(\rho, \rho)$, which implies

$$\sigma = -\frac{1}{\kappa}[\frac{r}{2} + \lambda + 3\phi + I\kappa]. \tag{5.9}$$

Again contracting (5.8) over X and Y, we get

$$4(a - \frac{r}{2} + \lambda) - b = \kappa[4p - (\sigma + p) + J],$$

where J =Trace of P, which implies

$$p = \frac{1}{\kappa}[\lambda - \frac{r}{2} - \phi - \frac{\kappa(I + J)}{3}] \tag{5.10}$$

Hence we can state the following:

Theorem 5.2. *In a viscous fluid $(WZS)_4$ spacetime obeying Einstein's equation with cosmological constant, the energy density and isotropic pressure are given by the relations (5.9) and (5.10).*

We now discuss whether a viscous fluid $(WZS)_4$ spacetime can admit heat flux or not. If possible, let the energy momentum tensor T be of the following form [20, 21]:

$$T(X,Y) = pg(X,Y) + (\sigma + p)E(X)E(Y) + E(X)F(Y) + E(Y)F(X), \tag{5.11}$$

where $F(X) = g(X, \xi)$ for all vector fields X; ξ being the heat flux vector field. Thus we have $g(\rho, \xi) = 0$, that is, $F(\rho) = 0$.

Using (5.2) and (5.3) in (5.11) we obtain that

$$\begin{aligned}(a - \frac{r}{2} + \lambda)g(X,Y) + bE(X)E(Y) = \kappa[&pg(X,Y) + (\sigma + p)E(X)E(Y) \\ &+E(X)F(Y) + E(Y)F(X)].\end{aligned} \tag{5.12}$$

Putting $Y = \rho$ in (5.12), yields

$$(a - \frac{r}{2} + \lambda - b + \sigma\kappa)E(X) + \kappa F(X) = 0,$$

which implies

$$F(X) = -\frac{1}{\kappa}(3\phi + \frac{r}{2} + \lambda + \kappa\sigma)E(X), \tag{5.13}$$

where $a = -\phi$, and $b = -(r + 4\phi)$.

Thus we are in a position to state the following:

Theorem 5.3. *A viscous fluid $(WZS)_4$ spacetime obeying Einstein's field equation with cosmological constant admits heat flux, provided $3\phi + \frac{r}{2} + \lambda + \kappa\sigma \neq 0$.*

6. Example of a $(WZS)_4$ Spacetime

In this section we prove the existence of a $(WZS)_4$ spacetime by constructing a non-trivial concrete example.

We consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{6.1}$$

where $i, j = 1, 2, 3, 4$.

The only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor, the Z tensor and the derivatives of the components of Z tensors are

$$\Gamma^1_{22} = -x^1,\ \ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2},\ \ \Gamma^2_{12} = \frac{1}{x^1},\ \ \Gamma^3_{23} = \frac{1}{x^2},\ \ R_{1332} = -\frac{x^2}{x^1},$$

$$S_{12} = -\frac{1}{x^1x^2}, \ \ Z_{12} = -\frac{1}{x^1x^2}, \ \ Z_{12,1} = \frac{2}{(x^1)^2x^2}, \ \ Z_{12,2} = \frac{1}{(x^2)^2x^1}.$$

We shall now show that this M^4 is a $(WZS)_4$ spacetime i.e., it satisfies the defining relation (1.4).

We choose the associated 1-form as follows:

$$A_i(x) = \begin{cases} -\frac{3}{x^1}, & \text{for i=1} \\ 0, & \text{otherwise} \end{cases}$$

$$B_i(x) = \begin{cases} \frac{1}{x^1}, & \text{for i=1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$D_i(x) = \begin{cases} -\frac{1}{x^2}, & \text{for i=2} \\ 0, & \text{otherwise} \end{cases}$$

at any point x $\in \mathbb{R}^4$.

Now equation (1.4) reduces to

$$Z_{12,1} = A_1Z_{12} + B_1Z_{12} + D_2Z_{11}, \tag{6.2}$$

$$Z_{12,2} = A_2Z_{12} + B_1Z_{22} + D_2Z_{12}. \tag{6.3}$$

Clearly, equations (6.2) and (6.3) satisfy the defining condition of $(WZS)_4$. So the manifold under consideration is a $(WZS)_4$ spacetime.

7. Conclusion

In general relativity the matter content of the spacetime is described by the energy momentum tensor T which is to be determined from physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein's equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat. Relativistic fluid models are of considerable interest in several areas of astrophysics, plasma physics and nuclear physics. Theories of relativistic stars (which would be models for supermassive stars) are also based on relativistic fluid models. The problem of accretion onto a neutron stars or a black hole is usually set in the framework of relativistic fluid models.

The physical motivation for studying various types of spacetime models in cosmology is to obtain the information of different phases in the evolution of the universe, which may be classified into three phases, namely, the initial phase, the intermediate phase and the final phase. The initial phase is just after the Big Bang when the effects of both viscosity and heat flux were quite pronounced. The intermediate phase is that when the effect of viscosity was no longer significant but the heat flux was till not negligible. The final phase, which extends to the present state of the universe when both the effects of viscosity and heat flux have become negligible and the matter content of the universe may be assumed to be perfect fluid. The study of $(WZS)_4$ spacetimes is important because such a spacetime represents the third phase in the evaluation of the universe. Consequently, the investigations of $(WZS)_4$ spacetime helps us to have a deeper understanding of the global character of the universe including the topology, because the nature of the singularities can be defined from a differential geometric stand point.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations. It is proved that a $(WZS)_4$ spacetime is a quasi-Einstein spactime. So a $(WZS)_4$ spacetime can be taken as a model of the perfect fluid spacetime in general relativity. Also it is shown that a $(WZS)_4$ spacetime satisfying divergence free conformal curvature tensor under certain condition is a generalized Robertson-Walker spacetime and the nature of the spacetime is of vanishing vorticity and vanishing shear. Moreover a $(WZS)_4$ spacetime satisfying divergence free conformal curvature tensor under certain condition, if the velocity vector field is always hypersurface orthogonal, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.

References

[1] L. Alías, A. Romero and M. Sánchez, *Compact spacelike hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Geometry and Topology of Submanifolds VII, World Sci. Publ., River Edge, NJ, (1995), 67–70.

[2] A. Barnes, *On shear free normal flows of a perfect fluid*, Gen. Relativity Gravitation, **4**(1973), 105–129.

[3] J. K. Beem, P. E. Ehrlich and K. L. Easley, *Global Lorentzian Geometry*, 2nd Edition, Monographs and Textbooks in Pure and Applied Mathematics **202**, Marcel Dekker, Inc., New York, 1996.

[4] F. Brickell and R. S. Clark, *Differentiable manifolds*, Van Nostrand Reinhold Company, London,1970.

[5] M. C. Chaki, *On pseudo Ricci symmetric manifolds*, Bulgar. J. Phys., **15**(1988), 526–531.

[6] M. C. Chaki and R. K. Maity, *On quasi Einstein manifolds*, Publ. Math. Debrecen, **57**(2000), 297–306.

[7] M. C. Chaki and S. Roy, *Spacetimes with covariant-constant energy-momentum tensor*, Int. J. Theor. Phys., **35**(1996), 1027–1032.

[8] M. C. Chaki and S. K. Saha, *On pseudo-projective Ricci symmetric manifolds*, Bulgar. J. Phys., **21**(1994), 1–7.

[9] B.-Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation, **46**(2014), Art. 1833, 5 pp.

[10] A. De, C. Özgür and U. C. De, *On conformally flat almost pseudo-Ricci symmetric spacetimes*, Internat. J. Theoret. Phys., **51**(2012), 2878–2887.

[11] L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1949.

[12] G. S. Hall, *Symmetries and Curvature Structure in general Relativity*, World Scientific Lecture Notes in Physics **46**, World Scientific, Singapore, 2004.

[13] H. Karchar, *Infinitesimal characterization of Friedmann Universe*, Arch. Math. (Basel), **38**(1982), 58–64.

[14] C. A. Mantica and L. G. Molinari, *Weakly Z Symmetric manifolds*, Acta Math. Hungar., **135(1-2)**(2012), 80–96.

[15] C. A. Mantica and Y. J. Suh, *Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors*, Int. J. Geom. Methods Mod. Phys., **9(1)**(2012), 1250004, 21 pp.

[16] C. A. Mantica and Y. J. Suh, *Recurrent Z forms on Riemannian and Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys., **9(7)**(2012), 1250059, 26 pp.

[17] C. A. Mantica and Y. J. Suh, *Pseudo-Z symmetric space-times*, J. Math. Phys., **55**(2014), 042502 12 pp.

[18] C. A. Mantica and Y. J. Suh, *Pseudo-Z symmetric space-times with divergence-free Weyl tensor and pp-waves*, Int. J. Geom. Methods Mod. Phys., **13**(2016), 1650015, 34 pp.

[19] J. Mikeš and L. Rachunek, *Torse-forming vector fields in T-symmetric Riemannian spaces*, Steps in Differential Geometry, Proc. of the Colloq. on Diff. Geometry, 25-30 July, (2000), 219-229, Debrecen.

[20] M. Novello and M. J. Reboucas, *The stability of a rotating universe*, Astrophys. J., **225**(1978), 719–724.

[21] B. O'Neill, *Semi-Riemannian Geometry with application to the Relativity*, Academic Press, New York-London, 1983.

[22] A. Z. Petrov, *Einstein spaces*, Pergamon press, Oxford, 1949.

[23] A. K. Roychaudhury, S. Banerji and A. Banerjee, *General relativity, astrophysics and cosmology*, Springer-Verlag, 1992.

[24] M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: geodesics*, Gen. Relativity Gravitation, **30**(1998), 915–932.

[25] M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: curvature and Killing fields*, J. Geom. Phys., **31**(1999), 1–15.

[26] J. A. Schouten, *Ricci-Calculus*, Springer, Berlin, 1954.

[27] R. Sharma, *Proper conformal symmetries of space-times with divergence-free Weyl conformal tensor*, J. Math. Phys., **34**(1993), 3582–3587.

[28] L. C. Shepley and A. H. Taub, *Space-times containing perfect fluids and having a vanishing conformal divergence*, Commun. Math. Phys., **5**(1967), 237–256.

[29] S. K. Srivastava, *General Relativity and Cosmology*, Prentice-Hall of India Private Limited, New Delhi, 2008.

[30] H. Sthepani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Hertl, *Exact solutions of Einstein's Field Equations*, Cambridge Monographs on Mathematical Physics, Cambridge University Press 2nd ed., 2003.

[31] L. Tamássy and T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N.S.), **53**(1993), 140–148.

[32] K. Yano, *Concircular geometry I, Concircular transformations*, Proc. Imp. Acad. Tokyo, **16**(1940), 195–200.

[33] K. Yano, *On the torse-forming directions in Riemannian spaces*, Proc. Imp. Acad. Tokyo, **20**(1944), 340–345.

[34] F. O. Zengin, *M-Projectively flat spacetimes*, Math. Rep., **14(64)/4**(2012), 363–370.