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Odd Harmonious and Strongly Odd Harmonious Graphs

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Abstract. A graph $G = (V(G), E(G)$ of order $n = |V(G)|$ and size $m = |E(G)|$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, ..., 2m-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, 5, ..., 2m-1\}$ defined by $f^*(uv) = f(u)+f(v)$ is bijection. While a bipartite graph G with partite sets A and B is said to be bigraceful if there exist a pair of injective functions $f_A : A \to \{0, 1, ..., m-1\}$ and$f_B : B \to \{0, 1, ..., m-1\}$ such that the induced labeling on the edges $f_{E(G)} : E(G) \to \{0, 1, ..., m-1\}$ defined by $f_{E(G)}(uv) = f_A(u)-f_B(v)$ (with respect to the ordered partition (A, B)), is also injective. In this paper we prove that odd harmonious graphs and bigraceful graphs are equivalent. We also prove that the number of distinct odd harmonious labeled graphs on m edges is $m!$ and the number of distinct strongly odd harmonious labeled graphs on m edges is $\lceil m/2 \rceil ! \lfloor m/2 \rfloor !$. We prove that the Cartesian product of strongly odd harmonious trees is strongly odd harmonious. We find some new disconnected odd harmonious graphs.

1. Introduction

A graph that has order n and size m is called a (n, m)-graph. Acharya and Hedge in [2] introduced arithmetic graphs. Let $G = (V, E)$ be a finite simple (n, m)-graph, D be a non-negative integer set, and let k and d be positive integers. A labeling f from V to D is said to be (k, d)-arithmetic if the vertex labels are distinct non-negative integers and the edge labels induced by $f^+(xy) = f(x) + f(y)$ for each edge xy are $k, k+d, k+2d, ..., k+(m-1)d$. Then G is said to be a (k, d)-arithmetic

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graph. Liang in [10] called the case where $k = 1, d = 2$ and $D = \{0, 1, 2, .., 2m - 1\}$ odd arithmetic labeling. Liang and Bai in [11] called odd arithmetic graphs odd harmonious. In [11], they called the case when $D = \{0, 1, 2, .., m\}$ strongly odd harmonious graph. Graceful labeling was introduced by Rosa [12] as a means of attacking the problem of cyclically decomposing the complete graph into the trees. An injective vertex function $f : V(G) \to \{0, 1, 2, ..., m\}$ is said to be a graceful if $f^*(uv) =| f(u) - f(v) |$ from $E(G)$ to $\{1, 2, 3, ..., m\}$ is injective. A graph that admits a graceful labeling is called graceful graph. A graceful graph G is said to be α-valuable if it has a graceful labeling f such that for some positive integer λ either $f(u) \leq \lambda$ and $f(v) > \lambda$ or $f(u) > \lambda$ and $f(v) \leq \lambda$ for every edge $uv \in E(G)$. λ is said to be the characteristic of f. As in [9], a bipartite graph $G(n, m)$ with partite sets A and B is bigraceful if there exist a pair of injective functions $f_A : A \to \{0, 1, ..., m - 1\}$ and$f_B : B \to \{0, 1, ..., m - 1\}$ such that the induced labeling on the edges $f_{E(G)} : E(G) \to \{0, 1, ..., m - 1\}$ defined by $f_{E(G)}(uv) = f_A(u) - f_B(v)$ (with respect to the ordered partition (A, B)), is also injective. For a dynamic survey of graph labeling, we refer to [5]. In [3], authors defined (k, d)-arithmetic as it defined above except they assumed that $D = R$, i.e $f : V(G) \to R$ and $f^+(E(G)) = \{k, k + d, ..., k + (m - 1)d\}$ such that f and f^+ are both injective, and then proved that every α-labeled graph is (k, d)-arithmetic as follows:

Lemma 1.1. *If G has an α-labeling f of characteristic λ. Let $V_1 = \{u \in V(G) : f(u) \leq \lambda\}$ and $V_2 = \{u \in V(G) : f(u) > \lambda\}$. Define $g : V(G) \to R$ by*

$$g(u) = \begin{cases} -d(f(u) + k) & u \in V_1 \\ k + df(u) & u \in V_2 \end{cases} \tag{1.1}$$

Then g is a (k, d)-arithmetic vertex function of G and G is (k, d)-arithmetic for all $k, d \in Z^+$.

Let $[0, 2, 1, 4]$ be an α-labeling of C_4. According to (1.1), the $(1, 2)$-arithmetic labeling of C_4 is $[-2, 5, -4, 9]$. To ensure that vertex labels belong to $\{0, 1, 2, ..., 2m-1\}$, We can replace (1.1) with

$$g(u) = \begin{cases} 2(\lambda - f(u)) & u \in V_1 \\ 2(f(u) - \lambda) - 1 & u \in V_2 \end{cases} \tag{1.2}$$

implies that every α-valuable graph is odd harmonious. Gallian in [5] mentioned that Koppendrayer had proved that if G has an α-labeling, then it is odd harmonious.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. The Cartesian product $G \times H$ has its vertex set $V(G) \times V(H)$ and $u = (x_1, y_1)$ is adjacent to $v = (x_2, y_2)$ whenever $[x_1 = x_2$ and $y_1y_2 \in E(H)]$ or $[y_1 = y_2$ and $x_1x_2 \in E(G)]$. For a graph G the subdivision of graph $S(G)$ is the graph obtained by subdividing every edge of G exactly once. We denote the path on n vertices by P_n.

2. Main Result

In this section, we prove that odd harmonious graphs and bigraceful graphs are equivalent. Then we find the number of odd harmonious labeled graphs.

2.1. Odd harmonious graphs and other labelings

Theorem 2.1. *A graph G is odd harmonious if and only if it is bigraceful.*

Proof. Let $G(n,m)$ be an odd harmonious graph with labeling f. The set $V(G)$ has a bipartition (A,B), where $A=\{u \in V(G) : f(u)=2i, i \geq 0\}$ and $B=\{u \in V(G) : f(u)=2i-1, i \geq 1\}$,i.e A is the subset of $V(G)$ with even labels and B is the subset of $V(G)$ with odd labels. Define the bigraceful labeling functions g_A and g_B as follows: $g_A : A \to \{0,1,\dots,m-1\}$, where $g_A(u)=\frac{f(u)}{2}$ and $g_B : B \to \{0,1,\dots,m-1\}$, where $g_B(u)=m-1-\frac{(f(u)-1)}{2}$.

- It is obvious that both g_A and g_B are injective.
- For each edge $uv \in E(G)$ with $u \in A$ and $v \in B$ we must get $g_B(v) \geq g_A(u)$: Assume on contradiction that $g_B(v) < g_A(u)$. Then $m-1-\frac{(f(v)-1)}{2} < \frac{f(u)}{2}$ and $f(u)+f(v) > 2m-1$ a contradiction.
- Edge uv, where $u \in A$ and $v \in B$, has label $g_B(v)-g_A(u)=m-1-\frac{(f(v)-1)}{2}-\frac{f(u)}{2}=m-1-\frac{f(u)+f(v)-1}{2}=m-1-\frac{k-1}{2}$,$k=f(u)+f(v)$. Since $1 \leq k \leq 2m-1$, we have $0 \leq m-1-\frac{k-1}{2} \leq m-1$, and the edge labels induced by g are $\{0,1,2,\dots,m-1\}$.

Conversely, let $G(n,m)$ be a bigraceful bipartite graph with partite sets A and B with labeling functions $g_A : A \to \{0,1,..,m-1\}$ and $g_B : B \to \{0,1,..,m-1\}$ such that $g_B(u) \geq g_A(v)$ for each edge uv with $u \in B$ and $v \in A$. Define the odd harmonious labeling function f as follows:

$$f(u)=\begin{cases} 2m-2g_B(u)-1 & u \in B \\ 2g_A(u) & u \in A \end{cases} \tag{2.1}$$

- Since both g_A and g_B are injective, f is injective and $f(V(G)) \subseteq \{0,1,\dots,2m-1\}$.
- Edge uv with $u \in A$ and $v \in B$ has label $f(u)+f(v)=2g_A(u)+2m-2g_B(v)-1=2m-2(g_B(v)-g_A(u))-1=2m-2k-1$,$k=g_B(v)-g_A(u)$. Since $0 \leq k \leq m-1$, we must have $f^+(E(G))=\{1,3,...,2m-1\}$. □

It is shown in [9] that if $G(n,m)$ is bigracful, then the complete bipartite graph $K_{m,m}$ has a cyclic decomposition into isomorphic copies of G.

Corollary 2.2. *If $G(n,m)$ is odd harmonious, then $K_{m,m}$ has a cyclic decomposition into copies isomorphic to G.*

Corollary 2.3. *If G is a strongly odd harmonious graph, then G has an $\alpha-$labeling.*

Proof. Assume $G(n, m)$ be a strongly odd harmonious graph with labeling f and the bipartition of $V(G)$ as in the proof of theorem 2.1. Define the α-labeling function $g : V(G) \to \{0, 1, 2, ..., 2m-1\}$ as follows:

$$g(u) = \begin{cases} \frac{f(u)}{2} & u \in A \\ m - \frac{f(u)-1}{2} & u \in B \end{cases} \tag{2.2}$$

It is not difficult to prove that g is an α-labeling for G with characteristic $\frac{m}{2}$ when m even and $\frac{m-1}{2}$ when m odd. □

Hence, if $G(n, m)$ is strongly odd harmonious then it decomposes both K_{2m+1} and $K_{m,m}$.

Remark 2.4.

(a) The converse of corollary 2.2 is not true, since $k_{m,n}$, with mn odd, is α-labeled but not strongly odd harmonious.

(b) If G is odd harmonious graph, then the maximum label of all vertices is at most $2m - 2\delta(G) + 1$, where $\delta(G)$ is the minimum degree of the vertices of G.

(c) Let T be a tree that is α-labeled. Then T is strongly odd harmonious iff $V(T)$ has a bipartition (A, B) such that $||A| - |B|| \leq 1$.

2.2. Counting labeled graphs

Theorem 2.5. *There are $m!$ distinct odd harmonious labeled graphs on m edges.*

Proof. An odd harmonious graph on m edges must have edge labeled 1, edge labeled 3,$\cdots$,and edge labeled $2m-1$. If an edge e has label $2i-1$,for $1 \leq i \leq 2m-1$, then we must have vertex labeled r adjacent to one labeled s, where $r + s = 2i - 1$. Since we have the following partition of odd numbers:

$$\begin{aligned}
&1 = 0 + 1, \\
&3 = 0 + 3 = 1 + 2, \\
&5 = 0 + 5 = 1 + 4 = 2 + 3, \\
&7 = 0 + 7 = 1 + 6 = 2 + 5 = 3 + 4, \\
&\quad \vdots \\
&2m - 1 = 0 + (2m-1) = 1 + (2m-2) = 2 + (2m-3) = \ldots = (m-1) + m.
\end{aligned}$$

Since, in an odd harmonious graph with m edges, we must select exactly one representation for every odd number in $\{1, 3, 5, \ldots, 2m-1\}$ from the above partition of odd numbers, the number of distinct odd harmonious labeled graphs on m edges is $m!$. □

Theorem 2.6. *There are $\lceil \frac{m}{2} \rceil ! \lfloor \frac{m}{2} \rfloor !$ distinct strongly odd harmonious labeled graphs on m edges.*

Proof. Assume without loss of generality that m is odd. In strongly odd harmonious graphs, the maximum vertex label is m and we have the following partition of the odd numbers:

$$
\begin{aligned}
&2m-1 = m + m - 1,\\
&2m-3 = [m + (m-3)] = [(m-1) + (m-2)],\\
&2m-5 = m + (m-5) = (m-1) + (m-4) = (m-2) + (m-3),\\
&\quad\vdots\\
&m = [(m-1)+0] = [(m-2)+1] = \ldots = [\lceil m/2 \rceil + \lfloor m/2 \rfloor],\\
&m-2 = [(m-2)+0] = [(m-3)+1] = \ldots = [\lceil (m-2)/2 \rceil + \lfloor (m-2)/2 \rfloor],\\
&\quad\vdots\\
&3 = [3+0] = [1+2],\\
&1 = 1 + 0,
\end{aligned}
$$

which completes the proof. □

By the same technique used in the proof of theorem 2.5 and theorem 2.6 we can prove that the number of distinct odd graceful labeled graphs [7], distinct even harmonious labeled graphs [13] and distinct strongly even harmonious labeled graphs [6] on m edges is $\prod_{i=1}^{m}(2i-1)$, m^m and $m!$, respectively.

3. Some Odd Harmonious Graphs

We will use the $\triangle_{+1}$-construction defined in [4, 8] to produce new strongly odd harmonious trees and prove that the Cartesian product of strongly odd harmonious trees is strongly odd harmonious. For two trees T_1 and T_2, let $T_1 \triangle T_2$ be the tree obtained by identifying a distinguished vertex v^* from T_2 to each vertex of T_1. First we prove that $T_1 \triangle T_2$ is strongly odd harmonious provided that T_1 and T_2 are strongly odd harmonious.

Theorem 3.1. *If T_1 and T_2 are strongly odd harmonious trees, then the graph $T_1 \triangle T_2$ is strongly odd harmonious.*

Proof. Let T_1 and T_2 be two strongly odd harmonious trees on n and m vertices,respectively. Let f be a strongly odd harmonious labeling of T_2 and $f`$ be the strongly odd harmonious labeling defined by $f`(x) = m - 1 - f(x)$, for $x \in V(T_2)$. Denote the vertices of T_2 by $x_0, x_1, \ldots, x_{m-1}$ such that $f(x_i) = i$. Let T^0, T^1, ..., T^{n-1} be n distinct copies of T_2 and $V(T^i) = \{x_{ij}, 1 \leq j \leq m\}$ for $0 \leq i \leq n-1$, where x_{ij} is the corresponding vertex to x_j. Let x_j be an arbitrary fixed vertex in T_2 and denote it by x_j^*. Let x_{ij}^* be the corresponding to x_j^* in the copy T^i . Based upon the strongly odd harmonious labeling of the tree T_1, we adjoin the copy T^i

of T_2 to vertex labeled i of T_1 in such a manner that x^*_{ij} and the vertex labeled i are identified (See Figure 1). Note that $V(T_1 \triangle T_2) = \{x_{01}, x_{02}, \ldots, x_{0(m-1)}\} \cup \{x_{11}, x_{12}, \ldots, x_{1(m-1)}\} \ldots \cup \{x_{(n-1)1}, x_{(n-1)2}, \ldots, x_{(n-1)(m-1)}\}$. Label the vertices of $T_1 \triangle T_2$ in the following manner:$g(x_{0j}) = f(x_{0j}) = f(x_j) = j, j = 0, 1, 2, \ldots, m-1$, $g(x_{ij}) = f(x_{0j}) + im = f(x_j) + im = j + im$, when i is even and $2 \leq i \leq n-1$, $g(x_{ij}) = f^{\backprime}(x_{0j}) + im = f^{\backprime}(x_j) + im = m-1-j+im$, when i is odd and $1 \leq i \leq n-1$. We prove that this labeling function g is strongly odd harmonious.

Obviously g is injective. Edges of $T_1 \triangle T_2$ are those either joining the copies T^i of T_2 or an edge in a copy T^i. Edge joining T^i and T^j has label $m(i+j+1)-1$, where $i+j$ is an edge label in T_1. Edges in T^i have labels $\{2mi+1, 2mi+3, \ldots, 2mi+2m-3\}$ for $0 \leq i \leq n-1$. What remains is to prove that edge labels are distinct. For this, assume that an edge joining T^i and T^j has the same label as an edge in T^k, i.e $m(i+j+1)-1 = 2mk+l$, for $i+j = s$ is an edge label in T_1 and l is an edge label in T_2 for $0 \leq k \leq n-1$. Which implies that $l = m(s+1) - 2mk - 1 = m(s+1-2k) - 1$. If $s = 2r-1$ for some positive integer r, then $l = 2m(r-k) - 1$ which can't happen. $\square$

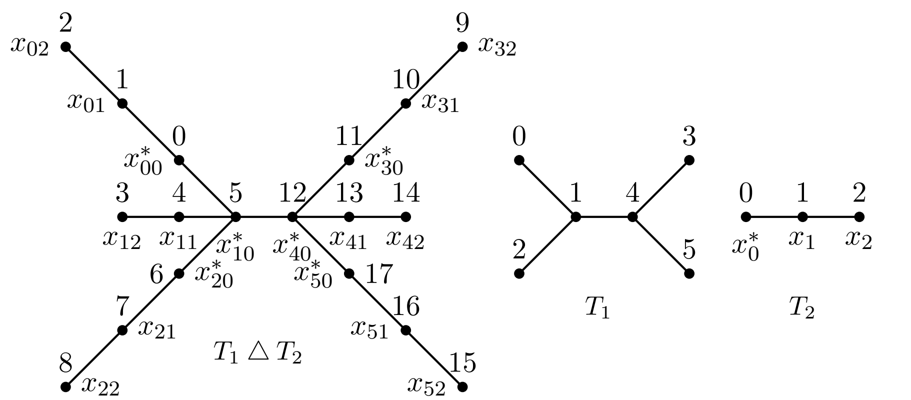

Figure 1: $T_1 \triangle T_2$ with T_1 is the bistar $B_{2,2}$ and T_2 is the path on 3 vertices

Note that if we replace the tree T_2 in theorem 3.1 by any other strongly odd harmonious graph, the result still true. Moreover, if we replace the edge $x^*_{ij}x^*_{kj}$ in $T_1 \triangle T_2$ with any other edge joining corresponding vertices(other than x^*_{ij} and x^*_{kj}) in T^i and T^k we get the generalized $T_1 \triangle T_2$ construction. $\triangle_{+1}$-construction depends upon making the $\triangle$-construction on $T_1 - v$, where v is the vertex with maximum label in T_1, then adding a vertex to the new graph. We prove the following theorem using $\triangle_{+1}$-construction.

Theorem 3.2. *If T is a strongly odd harmonious tree, then the subdivision graph of T is strongly odd harmonious.*

Proof. Let $T_1 = T$ and $T_2 = P_2$, the path on two vertices. Let f_1, f_2 be a strongly odd harmonious labelings of T_1 and T_2, respectively. Let x be the vertex in T_1 with $f_1(x) = n - 1$, where n is the number of vertices of T_1. Remove x from T_1 and construct the generalized $(T_1 - x) \triangle T_2$ construction in the following manner. Denote the vertices of P_2 by v_1 and v_2. Assume without any loss of generality that $f_2(v_1) = 0$ and $f_2(v_2) = 1$. Let T^0,T^1,...,T^{n-2} be $n - 1$ distinct copies of T_2 and $V(T^i) = \{v_{ij}, 1 \le j \le 2\}$ for $0 \le i \le n-2$, where the vertex v_{ij} is the corresponding vertex to v_j, $j = 1, 2$. Based upon the strongly odd harmonious labeling of the tree T_1, we replace the vertex labeled i of T_1 by the copy T^i of T_2. Label the vertices of T^i with $g(v_{0j}) = f_2(v_{0j}) = f_2(v_j)$ and $g(v_{ij}) = f(v_{0j}) + 2i = f_2(v_j) + 2i$. Now we add a vertex u and label it with $g(u) = 2n - 2$. If $f_1(N(x)) = \{x_1, x_2, ..., x_k\}$, we join the vertex u to $\{v_{x_1 2}, v_{x_2 2}, ..., v_{x_k 2}\}$. Denote by u_i the vertex labeled i in T_1, $i = 0, 1, 2, ..., n - 2$. If $u_i u_j$ is an edge in T_1 and $d_{T_1}(u_i, x) = d_{T_1}(u_j, x) + 1$, join v_{i2} to v_{j1}. The resulting graph is $T_1 \triangle_{+1} T_2$. Note that the difference between labeling function here and that in the proof of Theorem 3.1 is that we didn't use f\` of $T_2 = P_2$, this is to make the joining of the copies of T_2 easy. Moreover, the edges incident with u have labels in the form $2s + 1$, where s is a label of an edge incident with x in T_1. Hence g is strongly odd harmonious labeling of $S(T) = T_1 \triangle_{+1} T_2$.(See Figure 2) □

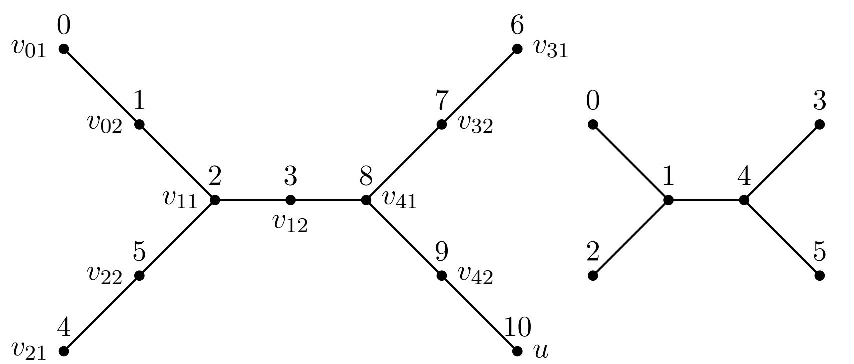

Figure 2: The subdivision graph (left) of the bistar $B_{2,2}$ (right)

Moreover, if we replace the tree $T_2 = P_2$ with a path on l vertices we would obtain, following the same proof, $S_l(T)$ is strongly odd harmonious, where $S_l(T)$ is the tree obtained by inserting l new vertices into each edge of T. The following lemma generalizes the result that $P_n \times P_m$ is odd harmonious proved by Liang and Bai [11].

Lemma 3.3. *If T_n and T_m are strongly odd harmonious trees on n and m vertices respectively, then the graph $T_n \times T_m$ is strongly odd harmonious.*

Proof. Let f_1 and f_2 be strongly odd harmonious labelings of T_n and T_m, respectively. Denote the vertices of T_n by $x_1, x_2, ..., x_n$. Let $T^0, T^1, T^2, ..., T^{m-1}$ be m distinct copies of T_n. Let $V(T^i) = \{x_{ij} | j = 1, 2, \dots, n\}$, $0 \le i \le m-1$, where x_{ij} is the corresponding vertex to x_j. Based upon the strongly odd harmonious labeling of T_m, we replace vertex labeled i by T^i and join corresponding vertices in the distinct copies of T_n to obtain the graph $T_n \times T_m$, i.e. if uv is an edge in T_m such that $f_1(u) = i$ and $f_1(v) = j$, join the corresponding vertices in T^i and T^j. Label the vertices of T^0 with f_1 and the vertices of T^i with $g(x_{ij}) = f_1(x_{0j}) + i(2n-1)$, for $1 \le i \le m-1$. In what remains we prove that the described labeling of $T_n \times T_m$ is strongly odd harmonious. Because f_1 is injective, g is injective and the maximum label assigned to the vertices is $n-1+(m-1)(2n-1) = 2mn-(m+n) =| E(T_n \times T_m) |$. Edges in T^i have labels $\{2i(2n-1)+1, 2i(2n-1)+3, \dots, 2i(2n-1)+2n-3\}$. Edge labels in between that in T^i and T^{i+1}, are the labels assigned to the edges joining T^l and T^s, where l and s are vertex labels assigned by f_2 in T_m to produce the edge label $i + (i+1)$, e.g edge labels in between that in T^1 and T^2 are labels assigned to the edges joining either T^0 and T^3 or T^1 and T^2 according to f_2, because in the strongly odd harmonious labeling of T_m, we must have either vertex labeled 0 is adjacent to one labeled 3 or vertex labeled 1 is adjacent to one labeled 2. $\square$

Lemma 3.3 would be generalized by replace T_1, T_2 or both with any strongly odd harmonious graph $G(n, n-1)$. Following the same technique in the proof of lemma 3.3, the proof of the following lemma is direct.

Lemma 3.4. *If $G_1(n_1, n_1-1)$ and $G_2(n_2, n_2-1)$ are a strongly odd harmonious graphs, then the graph $G_1 \times G_2$ is strongly odd harmonious.*

Lemma 3.5. *If T is a strongly odd harmonious tree, then the subdivision graph of $T \times P_m$ is odd harmonious.*

Proof. Let T has n vertices. It is obvious from theorem 3.2 that $S(T)$ has $2n-1$ vertices and has a strongly odd harmonious labeling that assigns the vertices of T the even labels $\{0, 2, 4, ..., 2n-2\}$. Describe $T \times P_m$ as in lemma 3.3, where we assume the strongly odd harmonious labeling of P_m, $f_2(x_i) = i-1$, for $1 \le i \le m$. Let w_{ij} be the newly added vertices to the edges of T^i and y_{ij} be the newly added vertex between x_{ij} and $x_{(i+1)j}$ (see Figure 3). Let f be the strongly odd harmonious labeling of $S(T_n)$. Label the vertices of $S(T^0)$ with f. Hence $f(V(T^0)) = \{0, 2, 4, ..., 2n-2\}$ and $f(\{w_{0j}, j = 1, 2, ..., n-1\}) = \{1, 3, 5, ..., 2n-3\}$. Let g be the following labeling of the vertices of $S(T \times P_m)$:

$g(x_{ij}) = f(x_{0j}) + 2in$.

$g(w_{ij}) = 6ni + f(w_{0j}) - 4i$.

$g(y_{ij}) = 6n(i+1) - 4j - 4i - 1$.

- g is injective: It is obvious that $g(x_{ij})$ is even, while $g(w_{ij})$ and $g(y_{ij})$ are odd, for all i and j. Hence it is sufficient to show that $g(w_{ij}) \ne g(y_{st})$ for some i,j,s and t. If it happen $g(w_{ij}) = g(y_{st})$, it would imply that $6ni + f(w_{0j}) - 4i = 6n(s+1) - 4t - 4s - 1$ and we get $f(w_{0j}) = (6n-4)(s-i) + 6n - 4t - 1$. We distinguish between two cases:

i. If $i \leq s$: we must have $f(w_{0j}) > (6n-4)+6n-4t-1 > 2n-3$. Note that $f(w_{0j}) \in \{1,3,5,...,2n-3\}$ and $1 \leq t \leq n$.

ii. If $i > s$: we must have $f(w_{0j}) < 0$.

- For all $u \in V(S(T \times P_n))$, we have $g(u) \leq 6n(m-1)+2n-3-4(m-1) < 2(4nm-2(n+m))-1$.
- Edges in T^i have labels $\{8in-4i+1, 8in-4i+3, \ldots, 8in+4n-4i-5\}$ and edges joining T^i and T^{i+1} have labels $\{8ni+4n-4i-3, 8ni+4n-4i-1, \ldots, 8in+8n-4-4i-1\}$ which all are distinct. □

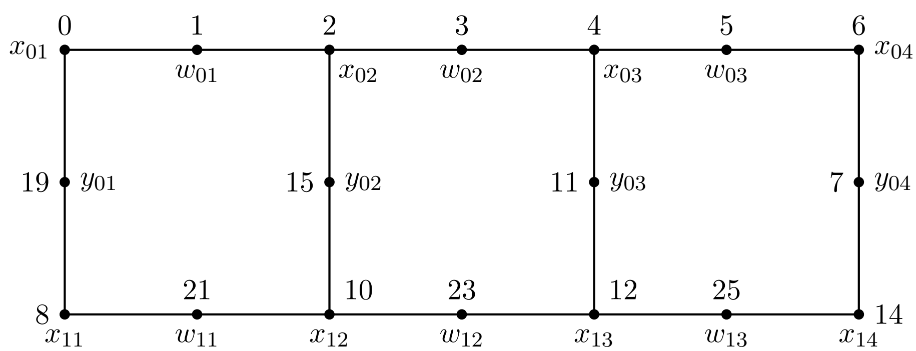

Figure 3: $S(P_4 \times P_2)$ odd harmonious labeling

Corollary 3.6. *If T is a strongly odd harmonious tree, then the graph $S(T \times P_m)$ is α-labeled.*

Proof. According to lemma 3.5, $S(T \times P_m)$ has an odd harmonious labeling g. Note that $S(T_n \times T_m)$ has $3nm-(n+m)$ vertices and $4nm-2(n+m)$ edges. $V(S(T \times P_m))$ has a bipartition (A, B), where $A = \{x_{ij}\}$ and $B = \{w_{ij}, y_{ij}\}$ for $0 \leq i \leq m-1$ and $1 \leq j \leq n$. We Define the labeling function h as follows:

$$h(u) = \begin{cases} \frac{g(u)}{2} & u \in A \\ 4nm-2(n+m)-\frac{g(u)-1}{2} & u \in B \end{cases} \tag{2.1}$$

Since $\max\limits_{u \in A} h(u) = \frac{g(x_{(m-1)n})}{2} = nm-1$ and $\min\limits_{u \in B} h(u) = 4nm-2(n+m)-\frac{g(w_{(m-1)(n-1)})-1}{2} = nm$. Hence h is an α-labeling with characteristic $nm-1$. □

The following lemma generalizes result in [14] that $C_{4m} \times P_n$ is odd harmonious.

Lemma 3.7. *If T is a strongly odd harmonious tree, then the graph $C_{4m} \times T$ is strongly odd harmonious.*

Proof. We follow the same technique in the proof lemma 3.3, let T be a tree that is strongly odd harmonious with n vertices and C_0,C_1,...,C_{n-1} be n distinct copies of C_{4m}. Denote the vertices of C_{4m} by x_1,x_2,...,x_{4m}. Assume the following two labelings of C_{4m}: f_1:$[0, 1, 2, 3, ..., 2m-1, 2m+2, 2m+1, 2m+4, 2m+3, ..., 4m, 4m-1]$ and f_2:$[4m-1, 0, 1, 2, ..., 2m-1, 2m+2, 2m+3, 2m+4, ..., 4m]$ for $[x_1, x_2, ..., x_{4m}]$ respectively. Let f be a strongly odd harmonious labeling of T. Remark that f uses all the labels in $\{0, 1, 2, ..., n-1\}$. In the strongly odd harmonious labeling of T, we replace vertex labeled i by C_i. Based upon the strongly odd harmonious labeling of T join corresponding vertices in the distinct copies of C_{4m} to obtain the graph $C_{4m} \times T$. Label the vertices of C_{2l} with $g(x_i) = f_1(x_i) + 8m(2l)$ and the vertices of C_{2l-1} with $g(x_i) = f_2(x_i)+8m(2l-1)$ for l such that $2l$ and $2l-1 \in \{0, 1, 2, ..., n-1\}$. It is not difficult to show that g is strongly odd harmonious labeling. □

Let$(P_2 \times P_m)^{P_n}$ denote the graph obtained by adding a pendant path P_n to each vertex of $P_2 \times P_m$.

Lemma 3.8. *All graphs of the form $(P_2 \times P_m)^{P_n}$ are strongly odd harmonious.*

Proof. Let P^1,P^2,...,P^m be distinct m copies of P_{2n+2}, the path with $2n+2$ vertices. Denote the vertices of P_{2n+2} by x_1,x_2,...,x_{2n+2}. Assume the labeling of P^1 :$f_1(x_j) = j-1$, for $1 \leq j \leq 2n+2$. Label the vertices of P^i by $f(x_j) = (i-1)(2n+3) + f_1(x_j)$, for $2 \leq i \leq m$. Now joining the two vertices x_{n+1} and x_{n+2} in P^i to their correspondences in P^{i+1} for $1 \leq i \leq m-1$, we get the strongly odd harmonious labeling of $(P_2 \times P_m)^{P_n}$. □

4. Disconnected Odd Harmonious Graphs

There is no strongly odd harmonious forest because for any forest on n vertices and m edges we must have $m \leq n-2$. Since all odd harmonious graphs on 3 edge are P_4 and $K_{1,3}$, any odd harmonious graph on $m \geq 4$ edges must contain at least one of them as a subgraph. According to remark 2.4, for P_n any α-labeling is equivalent to a strongly odd harmonious labeling. It is shown in [1] that:

Lemma 4.1.([1])

(i) *If $1 \leq r \leq n-1$, then there exists an α-labeling of P_{2n-1} that assigns the end vertices labels r and $r+n$.*

(ii) *If $1 \leq r \leq n$, then there exists an α-labeling of P_{2n} that assigns the end vertices labels r and $n-r$.*

In strongly odd harmonious labeling of P_n, we have the following lemma.

Lemma 4.2.

(i) *If r is even and $1 \leq r \leq n-1$, then there exists a strongly odd harmonious labeling of P_{2n-1} that assigns the end vertices labels r and $2n-2-r$.*

(ii) *If $1 \leq r \leq n$, then there exists a strongly odd harmonious labeling of P_{2n} that assigns the end vertices labels r and $2n-1-r$.*

We will use lemma 4.2 to prove the following theorem.

Theorem 4.3. *If G is (strongly) odd harmonious graph and k is the smallest label that is not assigned to any vertex of G, then $G \cup P_l$ is (strongly) odd harmonious, when $l \geq k+2$.*

Proof. Let $G(n,m)$ be an (a strongly) odd harmonious graph with labeling f and k be the smallest label that is not assigned by f to any vertex of G. Denote the vertices of G by $[x_1, x_2, \ldots, x_n]$ and the vertices of P_l by $[y_1, y_2, \ldots, y_l]$. We construct the strongly odd harmonious labeling of P_{l-1} that assigns end vertices labels $l-k-2$ and $l-2-(l-k-2) = k$ by lemma 4.2. In the strongly odd harmonious labeling of P_{l-1}, we join a vertex labeled $l-1+k$ to the end vertex labeled $l-k-2$ to obtain the odd harmonious labeling of P_l:$[l-1+k, l-k-2, \ldots, k]$. Label the vertices of G with g, where $g(x_i) = f(x_i) + l - 1$. Obviously g is (strongly) odd harmonious labeling function. □

Corollary 4.4.

(1) *The graph $C_{4m} \cup P_n$ is strongly odd harmonious for $n \geq 2m+2$.*

(2) *The graph $(C_{4m} \times T) \cup P_n$ is strongly odd harmonious when $n \geq 2m+2$, where T is a strongly odd harmonious tree.*

(3) *The graph $(C_{4m} \cup P_n) \times T$ is strongly odd harmonious, where T is a strongly odd harmonious tree, when $n \geq 2m+2$.*

(4) *The graph $K_{1,t} \cup P_n$ is odd harmonious iff $n \geq 4$.*

(5) *The graph $T_n \times T_m \cup P_l$ is strongly odd harmonious when $l \geq n+2$, where T_n and T_m are strongly odd harmonious trees on n and m vertices respectively.*

Proof.

(1) Denote the vertices of C_{4m} by $[x_1, x_2, \ldots, x_{4m}]$. Assume the strongly odd harmonious labeling of C_{4m}: $f_1 = [0, 1, 2, \ldots, 2m-1, 2m+2, 2m+1, 2m+4, 2m+3, \ldots, 4m, 4m-1]$ for $[x_1, x_2, \ldots, x_{4m}]$. Remark that f_1 does not assign any vertex the label $2m$. Hence in theorem 4.3 by letting $G = C_{4m}$ the result holds.

(2) By lemma 3.7, the strongly odd harmonious labeling described there not assign the label $2m$ to any vertex in $C_{4m} \times T_n$.

(3) The same as the proof of lemma 3.3 by setting $T_1 = C_{4m} \cup P_n$ with labeling in corollary 4.4(1) and the result holds immediately.

(4) Denote the vertices of $K_{1,t}$ by $[x, x_1, x_2, \ldots, x_t]$, where x is the center vertex. Since $K_{1,t}$ has the labeling $f : [0, 1, 3, 5, \ldots, 2t-3]$ which not assign the label 2 to any vertex, theorem 4.3 would imply that $K_{1,t} \cup P_n$ is odd harmonious when $n \geq 4$. When $n \leq 3$ we prove a more general result that the union of two bistars is not odd harmonious. Let $K_{1,n}$ and $K_{1,m}$ be two bistars. It is known that $K_{1,i}$ has only the two odd harmonious labelings $f_i : [0, 1, 3, 5, \ldots, 2i-3]$ and $g_i : [1, 0, 2, 4, \ldots, 2i-2]$ for $i = n, m$. Because of the partition of odd numbers in the proof of theorem 2.5 one of the two bistars should has one of the two labelings f_i and g_i. Assume without loss of generality that $K_{1,n}$ has the labeling $f_n : [0, 1, 3, 5, \ldots, 2n-3]$ then the edge label $2n-1$ can't be obtained.

(5) By lemma 3.3, the strongly odd harmonious labeling described there not assign the label n to any vertex in $T_n \times T_m$. □

We note here that the graph $C_{4m} \cup P_n$ is odd harmonious for all n, m. Actually it is proven in [15] that $C_{4m} \cup P_n$ has an α-labeling, for $n \leq 4m-4$ and $m \geq 2$, and so is odd harmonious. According to corollary 4.4 the remaining cases are $C_8 \cup P_5, C_4 \cup P_2$ and $C_4 \cup P_3$. If the vertices of $C_{4m} \cup P_n$ are denoted by $[x_1, x_2, \ldots, x_{4m}] \cup [y_1, y_2, \ldots, y_n]$, then the odd harmonious labeling of $C_8 \cup P_5, C_4 \cup P_3$ and $C_4 \cup P_2$ is $[0, 1, 2, 3, 4, 7, 6, 9] \cup [10, 11, 12, 5, 14]$,$[0, 1, 4, 3] \cup [9, 2, 7]$ and $[0, 1, 4, 3] \cup [2, 7]$ respectively. Also, we note that the graph $C_4 \cup C_4 \cup C_4$ has no α-labeling, but it is odd harmonious. Label the three cycles [5,2,1,0], [8,7,4,15],[10,11,6,3].

Lemma 4.5. *If G be a strongly odd harmonious (n, m)-graph that is not a tree, then the graph $K_{1,t} \cup G$ is odd harmonious.*

Proof. Let f be a strongly odd harmonious labeling of G, define the labeling g of the vertices of $K_{1,t} \cup G$ in the following manner: $g(u) = f(u)$ if $u \in V(G)$. Let x be the largest number in $\{0, 1, 2, \ldots, m\}$ that is not assigned by f to any vertex of G. Label the center of $K_{1,t}$ with x. Let $y = 2m+1-x$. Label the vertices of $K_{1,t}$ with $[y,\ y+2, y+4, \ldots, y+2t-2]$. Note that $m+3 \leq y \leq 2m-1$ and the maximum label assigned to a vertex in $K_{1,t}$ is at most $2m+2t-3$. Edges in G have labels $\{1,$ $3,\ 5, \ldots, 2m-1\}$ and edges in $K_{1,t}$ have labels $\{2m+1, 2m+3, \ldots, 2m+2t-1\}$. □

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