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Multivariable Recursively Generated Weighted Shifts and the 2-variable Subnormal Completion Problem

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Abstract. In this paper, we give a new approach to solving the 2-variable subnormal completion problem (SCP for short). To this aim, we extend the notion of recursively generated weighted shifts, introduced by R. Curto and L. Fialkow, to 2-variable case. We next provide "concrete" necessary and sufficient conditions for the existence of solutions to the 2-variable SCP with minimal Berger measure. Furthermore, a short alternative proof to the propagation phenomena, for the subnormal weighted shifts in 2-variable, is given.

1. Introduction and Results

The notion of recursively generated weighted shifts, largely studied in the literature, is employed to solve various questions in operator theory. R. Curto and L. Fialkow [3, 4] have used this concept to solve the Subnormal Completion Problem (SCP for short) in one variable. In this paper, we extend this notion to 2-variables, and we use it to provide a new approach to solve the open problem of 2-variable SCP. A concrete solution to the minimal 2-variable SCP (see Section 4) is given as well as an alternative proof for the propagation phenomena for subnormal 2-variable weighted shifts.

First we recall some definitions and notations. A bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ is normal if $TT^* = T^*T$, subnormal if $T = N\mid_{\mathcal{H}}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and hyponormal if $T^*T - TT^* \geq 0$. The n-tuple $\mathbf{T} \equiv (T_1, \dots, T_n)$ is said to be normal if $\mathbf{T}$ is commuting and each T_i is normal, and $\mathbf{T}$ is subnormal if $\mathbf{T}$ is the restriction of a normal n-tuple to a common

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invariant subspace.

For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called weights), let $W_\alpha : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ be the associated unilateral weighted shift, defined by $W_\alpha e_n := \alpha_n e_{n+1}$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $l^2(\mathbb{Z}_+)$. Given $k \in \mathbb{Z}_+$, the moments of α of order k is given by

$$\gamma_k \equiv \gamma_k(\alpha) := \|w_\alpha^k e_0\|^2 = \begin{cases} 1 & \text{if } k = 0, \\ \alpha_0^2 \dots \alpha_{k-1}^2 & \text{if } k \geq 1. \end{cases}$$

It is easy to see that W_α is never normal. In the case where W_α is subnormal, Stampfli showed in [13] that a propagation phenomenon occurs which forces the flatness of W_α, that is, if $\alpha_k = \alpha_{k+1}$ for some $k \geq 0$, then $\alpha_n = \alpha_{n+1}$ for every $n \geq 0$.

Consider double-indexed positive bounded sequences $\alpha \equiv \{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$. We define in a similar way the 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ acting on the Hilbert space $l^2(\mathbb{Z}_+^2)$, associated with, $\alpha \equiv \{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ and $\beta \equiv \{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ by

$$T_1 e_{\mathbf{k}} := \alpha_{\mathbf{k}} e_{\mathbf{k}+\epsilon_1} \text{ and } T_2 e_{\mathbf{k}} := \beta_{\mathbf{k}} e_{\mathbf{k}+\epsilon_2},$$

where $\epsilon_1 := (1, 0)$ and $\epsilon_2 := (0, 1)$. We recall here that, the vector space $l^2(\mathbb{Z}_+^2)$ is canonically isometrically isomorphic to $l^2(\mathbb{Z}_+) \otimes l^2(\mathbb{Z}_+)$, equipped with its canonical orthonormal basis $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$.

Clearly,

$$T_1 T_2 = T_2 T_1 \Leftrightarrow \beta_{\mathbf{k}+\epsilon_1}\alpha_{\mathbf{k}} = \alpha_{\mathbf{k}+\epsilon_2}\beta_{\mathbf{k}} (\text{ for all } \mathbf{k} \in \mathbb{Z}_+^2). \tag{1.1}$$

Given $\mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2$, the moment of (α, β) of order $\mathbf{k}$ is

$$\gamma_{\mathbf{k}} := \|T_1^{k_1} T_2^{k_2} e_{\mathbf{0}}\|^2 = \begin{cases} 1 & \text{if } k = 0, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 = 0, \\ \beta_{(0,0)}^2 \cdots \beta_{(0,k_2-1)}^2 & \text{if } k_1 = 0 \text{ and } k_2 \geq 1, \\ \alpha_{(0,0)}^2 \cdots \alpha_{(k_1-1,0)}^2 . \beta_{(k_1,0)}^2 \cdots \beta_{(k_1,k_2-1)}^2 & \text{if } k_1 \geq 1 \text{ and } k_2 \geq 1. \end{cases} \tag{1.2}$$

Conversely, one can recover the weights from the moments, by using the following relations:

$$\alpha_{\mathbf{k}} = \sqrt{\frac{\gamma_{\mathbf{k}+\epsilon_1}}{\gamma_{\mathbf{k}}}} \text{ and } \beta_{\mathbf{k}} = \sqrt{\frac{\gamma_{\mathbf{k}+\epsilon_2}}{\gamma_{\mathbf{k}}}}. \tag{1.3}$$

The presence of consecutive equal weights, of a 2-variable subnormal weighted shift, leads to horizontal or vertical flatness (see Definition 3.4). Explicitly, if, for some $k_1, k_2 \geq 1$, $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ (resp. $\beta_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}$), then $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$) for all $k_1, k_2 \geq 1$. This result is known as propagation phenomena. We give new short proof of this fact in Theorem 3.4.

A characterization of subnormality for multivariable weighted shifts is given in [10]. More precisely, $\mathbf{T}$ admits a commuting normal extension if and only if there is a probability measure μ, which we call the Berger measure of $\mathbf{T}$, defined on the 2-dimensional rectangle $R = [0, \|T_1\|^2] \times [0, \|T_2\|^2]$ such that

$$\gamma_{\mathbf{k}} = \int_R x^{k_1} y^{k_2} d\mu(x,y), \qquad \text{for all } \mathbf{k} \equiv (k_1, k_2) \in \mathbb{Z}_+^2. \tag{1.4}$$

A measure μ satisfies (1.4) is also known as representing measure for $\{\gamma_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$.

With a given bi-sequence $\gamma^{(2n)} \equiv \{\gamma_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}_+^2, |\mathbf{i}|\leq 2n} \equiv \{\gamma_{ij}\}_{i+j\leq 2n}$, we associate the moment matrix $M(n) \equiv M(n)(\gamma^{2n})$, introduced by R. Curto and L. Fialkow [5, 6], build as follows.

$$M(n) = \begin{pmatrix} M[0,0] & M[0,1] & \dots & M[0,n] \\ M[1,0] & M[1,1] & \dots & M[1,n] \\ \vdots & \vdots & \ddots & \vdots \\ M[n,0] & M[n,1] & \dots & M[n,n] \end{pmatrix}, \tag{1.5}$$

where

$$M[i,j] = \begin{pmatrix} \gamma_{i+j,0} & \gamma_{i+j-1,1} & \cdots & \gamma_{i,j} \\ \gamma_{i+j-1,1} & \gamma_{i+j-2,2} & \cdots & \gamma_{i-1,j-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{i,j} & \gamma_{i-1,j+1} & \cdots & \gamma_{0,i+j} \end{pmatrix}. \tag{1.6}$$

Considering the following lexicographic order, to denote rows and columns of the moment matrix $M(n)$,

$$1, X, Y, X^2, XY, Y^2, \dots, X^n, X^{n-1}Y, \dots, XY^{n-1}, Y^n. \tag{1.7}$$

The matrix $M(n)$ detects the positivity of the Riesz functional

$$\Lambda_{\gamma^{(2n)}} : p(x,y) \equiv \sum_{0\leq i+j\leq 2n} a_{ij} x^i y^j \longrightarrow \sum_{0\leq i+j\leq 2n} a_{ij}\gamma_{ij}$$

on the cone generated by $\{p^2 : p \in \mathbb{R}_n[x,y]\}$, the sum of squares of polynomials (sometimes abbreviated as SOS), where $\mathbb{R}_n[x,y]$ is the vector space of polynomials in two variables with real coefficients and total degree less than or equal to n.

For reason of simplicity, we identify a polynomial $p(x,y) \equiv \sum a_{ij}x^i y^j$ with its coefficient vector $p = (a_{ij})$ with respect to the basis of monomials of $\mathbb{R}_n[x,y]$ in degree-lexicographic order (see (1.7)). Clearly $M(n)$ acts on these coefficient vectors as follows:

$$< M(n)p, q >= q^T M(n) p = \Lambda_{\gamma^{(2n)}}(pq), \qquad p, q \in \mathbb{R}_n[x,y]. \tag{1.8}$$

Furthermore, let $g \in \mathbb{R}[x,y]$ be with coefficient vector $\{g_\beta\}$ and let $g * \gamma$ denote the shifted vector in $\mathbb{R}^{\mathbb{Z}_+^2}$ whose α-th entry is $(g*\gamma)_\alpha := \sum_\beta g_\beta \gamma_{\beta+\alpha}$. The moment matrix $M(n)(g*\gamma) = M_g(n + [\frac{1+\deg g}{2}])$ is called the localizing matrix with respect to γ and g. For example, given $\gamma^{(4)} \equiv \{\gamma_{ij}\}_{i+j\leq 4}$, then

$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} \end{pmatrix}, M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{10} & \gamma_{01} & \gamma_{20} & \gamma_{11} & \gamma_{02} \\ \gamma_{10} & \gamma_{20} & \gamma_{11} & \gamma_{30} & \gamma_{21} & \gamma_{12} \\ \gamma_{01} & \gamma_{11} & \gamma_{02} & \gamma_{21} & \gamma_{12} & \gamma_{03} \\ \gamma_{20} & \gamma_{30} & \gamma_{21} & \gamma_{40} & \gamma_{31} & \gamma_{22} \\ \gamma_{11} & \gamma_{21} & \gamma_{12} & \gamma_{31} & \gamma_{22} & \gamma_{13} \\ \gamma_{02} & \gamma_{12} & \gamma_{03} & \gamma_{22} & \gamma_{13} & \gamma_{04} \end{pmatrix},$$

$$M_x(2) = \begin{pmatrix} \gamma_{10} & \gamma_{20} & \gamma_{11} \\ \gamma_{20} & \gamma_{30} & \gamma_{21} \\ \gamma_{11} & \gamma_{21} & \gamma_{12} \end{pmatrix} \text{ and } M_y(2) = \begin{pmatrix} \gamma_{01} & \gamma_{11} & \gamma_{02} \\ \gamma_{11} & \gamma_{21} & \gamma_{12} \\ \gamma_{02} & \gamma_{12} & \gamma_{03} \end{pmatrix}.$$

In the one variable case, the SCP was stated and solved abstractly by J. Stampfli in [13]:

Problem 1.(One-Variable Subnormal Completion Problem)
Given $m \geq 0$ and a finite collection of positive numbers $\Omega_m = \{\alpha_k\}_{k=0}^m$, find necessary and sufficient conditions on Ω_m to guarantee the existence of a subnormal weighted shift whose initial weights are given by Ω_m.

When $m = 0$ or $m = 1$ the solution can be given by the canonical completion $\alpha_0, \alpha_0, \alpha_0, \dots$ and $\alpha_0 < \alpha_1, \alpha_1, \dots$, with Berger measure $\mu := \delta_{\alpha_0^2}$ and $\mu := \frac{\alpha_0^2}{\alpha_1^2}\delta_{\alpha_1^2}$, respectively. In [13], J. Stampfli showed that given $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}$ with $a < b < c$, there always exists a subnormal completion of α (this solves the case $m = 2$), but for $\alpha : \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}$, with $a < b < c < d$, such a subnormal completion may not exist. The complete solution of the SCP in one variable was given by R. curto and L. Fialkow [3], the explicit calculation requires recursively generated weighted shifts (such shifts have finite atomic Berger measures).

We now state the 2-variable SCP:

Problem 2.(2-Variable Subnormal Completion Problem)
Given $m \geq 0$ and a finite collection of pairs of positive numbers $\Omega_m = \{(\alpha_{\boldsymbol{k}}; \beta_{\boldsymbol{k}})\}_{|\boldsymbol{k}|\leq m}$ satisfying (1.1) for all $|\boldsymbol{k}| \leq m$ (where $|\boldsymbol{k}| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a subnormal 2-variable weighted shift whose initial weights are given by Ω_m.

In [8], R. Curto, S. H. Lee and J. Yoon have introduced an approach to the 2-variable SCP based on positivity and rank-preserving extension of the moment matrix, developed in [5, 6, 7]. Although this lead to an explicit criterion for SCP with quadratic moment data (i.e., $m = 1$), the 2-Variable Subnormal Completion Problem remains open. Recently, S. H. Lee and J. Yoon [11] found, by using the recursiveness, a necessary and sufficient conditions for the existence of 2-Variable subnormal completion with minimal Berger measure for the case $m = 2$ (i.e., rank $M(1)$-atomic Berger measure). We will provide , in Theorem 4.2, a complete (and concrete) solution for the 2-Variable SCP with minimal Berger measure.

In the present paper, we will show that if a given, finite, collection of weights Ω_m admits a subnormal completion in 2-variable, then there exists a 2-variable subnormal weighted shift (solution of the SCP for Ω_m) with moment sequence obey to some recurrence relations, we shall refer to such shifts as 2-variable recursively generated weighted shifts (see Definition 2.5). In Theorem 3.1, we will provide necessary and sufficient conditions for the existence of 2-variable recursively generated weighted shift completion, and thus a solution to the SCP in 2-variable. As application, we provide a generalization of a recent result of S. H. Lee and J. Yoon [11, Theorem 2.2]. More precisely, in Theorem 4.2 we give a concrete necessary and sufficient conditions for the existence of a subnormal completion with minimal Berger measure.

This paper is organized as follows. In Section 2, we introduce the recursively generated weighted shifts and we exhibit some useful results. We devote Section 3 to provide a solution to the 2-Variable Subnormal Completion Problem (Theorem 3.1) and the phenomena propagation for subnormal weighted shifts. In section 4, we solve the minimal SCP in 2-variable.

2. The 2-variable Recursively Weighted Shifts

We introduce below the notion of 2-variable weighted shifts, which will play a central role in this paper, and we give some useful properties.

Definition 2.1. Let $\mathbf{T} \equiv (T_1, T_2)$ be a 2-variable weighted shift and let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be its associated moment sequence. A polynomial $p(x,y) = \sum\limits_{i,j} p_{ij}x^iy^j \in \mathbb{R}[x,y]$ is said to be *characteristic polynomial* associated with $\mathbf{T}$, or with γ, if

$$\sum_{i,j} p_{i,j}\gamma_{i+n,j+m} = 0, \text{ for all } n,m \in \mathbb{Z}_+. \tag{2.1}$$

Remark 2.2. If p is a characteristic polynomial associated with γ, then, for every $q \in \mathbb{R}[x,y]$, the polynomial pq is also a characteristic polynomial. In particular, the set of all characteristic polynomials associated with γ is an ideal in $\mathbb{R}[x,y]$.

The following proposition is an immediate consequence of relations (1.8) and (2.1).

Proposition 2.3. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a bi-indexed sequence and let $p(x,y) \in \mathbb{R}[x,y]$. Then p is a characteristic polynomial of γ if and only if $M(\infty)(\gamma)p = 0$.*

Definition 2.4. A sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ is said to be *recursive double indexed sequence* (*RDIS* in short) if it has two characteristic polynomials $p_1, p_2 \in \mathbb{R}[x,y]$, with

$$(2.2)\qquad \begin{cases} p_1(x,y) = & x^{r+1} - \sum\limits_{i+j\le r} a_{ij}x^iy^j \\ p_2(x,y) = & y^{s+1} - \sum\limits_{i+j\le s+1, j\ne s+1} b_{i,j}x^iy^j, \end{cases}$$

or in the symetric form

$$(2.3)\qquad \begin{cases} p_1(x,y) = & x^{r+1} - \sum\limits_{i+j\le r+1, i\ne r+1} a_{ij}x^iy^j \\ p_2(x,y) = & y^{s+1} - \sum\limits_{i+j\le s} b_{i,j}x^iy^j. \end{cases}$$

Without loss of generality, throughout this paper, we assume that the pair of characteristic polynomials of the $RDIS$ $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ is given in the form (2.2).

Definition 2.5. A 2-variable weighted shift is said to be *recursively generated* if its associated moment sequence is a $RDIS$.

Let us show that $RDIS$ are well defined. Consider a $RDIS$ $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ associated with a pair of characteristic polynomials as in (2.2), that is, for all $n,m,e \in \mathbb{Z}_+$ with $n \ge r$ and $m \ge s$,

$$(2.4)\qquad \gamma_{n+1,e} = \sum_{i+j\le r} a_{ij}\gamma_{n-r+i,e+j} \text{ and } \gamma_{e,m+1} = \sum_{l+k\le s} b_{lk}\gamma_{e+l,n-s+k}.$$

A direct computation shows that

$$(2.5)\qquad \begin{aligned} \gamma_{n+1,m+1} &= \sum_{i+j\le r} a_{ij}\gamma_{n-r+i,m+1+j} \quad \Big(= \sum_{l+k\le s} b_{lk}\gamma_{n+1+l,m-s+k}\Big) \\ &= \sum_{i+j\le r, l+k\le s} a_{ij}b_{lk}\gamma_{n-r+i+l,m-s+j+k}. \end{aligned}$$

Equation (2.5) gives the compatibility condition of the two relations in (2.4). Hence the sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\ge 0}$ is well defined. The other case (i.e., when the characteristic polynomial are defined as in (2.3)) is treated similarly.

Different pair of characteristic polynomials can be associated with the same $RDIS$, as shown in the following example.

Example 1. Let us consider the bi-sequence $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ defined by $\gamma_{n,m} = a2^n + b3^m$, where a and b are real numbers. The polynomials $x^2 - 4x - \frac{1}{2}y + \frac{9}{2}$ and $y + 2x - 5$ are two characteristic polynomials of γ. Indeed,

$$\begin{aligned} \gamma_{n+2,m} - 4\gamma_{n+1,m} - \frac{1}{2}\gamma_{n,m+1} + \frac{9}{2}\gamma_{n,m} &= a2^{n+2} + b3^m - 4a2^{n+1} - 4b3^m \\ &\quad - \frac{1}{2}a2^n - \frac{1}{2}b3^{m+1} + \frac{9}{2}a2^n + \frac{9}{2}b3^m \\ &= 0, \end{aligned}$$

and

$$\gamma_{n,m+1} + 2\gamma_{n+1,m} - 5\gamma_{m,n} = a2^n + b3^{m+1} + 2a2^{n+1} + 2b3^m - 5a2^n - 5b3^m = 0.$$

On the other hand, we have

$$\begin{aligned}\gamma_{n+2,m} - 3\gamma_{n+1,m} + 2\gamma_{n,m} &= a2^{n+2} + b3^m - 3a2^{n+1} - 3b3^m + 2a2^n + 2b3^m \\ &= 0, \\ \gamma_{n,m+2} - 4\gamma_{n,m+1} + 3\gamma_{n,m} &= a2^n + b3^{m+2} - 4a2^n - 4b3^{m+1} + 3a2^n + 3b3^m \\ &= 0.\end{aligned}$$

Hence $(x^2-3x+2; y^2-4y+3)$ is, also, a pair of characteristic polynomials associated with γ. (The last characteristic polynomials are analytic, i.e., $(x^2 - 3x + 2; y^2 - 4y + 3) \in \mathbb{R}[X] \times \mathbb{R}[Y]$).

Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a $RDIS$, we denote by $\mathcal{P}_\gamma$ $(\subseteq \mathbb{R}[x,y] \times \mathbb{R}[x,y])$ the set of pair of characteristic polynomials associated with γ and we denote by $\mathcal{A}_\gamma$ $(\subseteq \mathbb{R}[X] \times \mathbb{R}[Y])$ the family of analytic, monic, characteristic polynomials associated with γ.

Remark 2.6. The pair of characteristic polynomials $(p_1, p_2) \in \mathcal{P}_\gamma$, together with the initial conditions $\{\gamma_{ij}\}_{\substack{0 \le i \le \deg p_1 - 1, \\ 0 \le j \le \deg p_2 - 1.}}$, are said to define the sequence γ.

We use structural properties of moment matrices to get the following interesting lemma.

Lemma 2.7. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a bi-indexed sequence and let $f, g, h \in \mathbb{R}[x,y]$, then*

$$f^T M(\infty)(\gamma)(gh) = (fg)^T M(\infty)(\gamma)h. \tag{2.6}$$

Proof. We write $f = \sum_{i_1,i_2} f_{(i_1,i_2)} x^{i_1} y^{i_2}$, $g = \sum_{j_1,j_2} g_{(j_1,j_2)} x^{j_1} y^{j_2}$ and $h = \sum_{k_1,k_2} h_{(k_1,k_2)} \mathbf{x}^{k_1} y^{k_2}$. As the entry, of the infinite moment matrix $M(\infty)(\gamma)$, corresponding to the column $X^n Y^m$ and the line $X^l Y^k$ is $\gamma_{n+l,m+k}$, we obtain

$$\begin{aligned} f^T M(\gamma)(gh) &= \Big(\sum_{i_1,i_2} f_{(i_1,i_2)} x^{i_1} y^{i_2}\Big)^T M(\infty)(\gamma)\Big(\sum_{j_1,j_2,k_1,k_2} g_{(j_1,j_2)} h_{(k_1,k_2)} x^{j_1+k_1} y^{j_2+k_2}\Big) \\ &= \sum_{i_1,i_2,j_1,j_2,k_1,k_2} f_{(i_1,i_2)} g_{(j_1,j_2)} h_{(k_1,k_2)} \gamma_{i_1+j_1+k_1, i_2+j_2+k_2} \\ &= (fg)^T M(\gamma) h, \end{aligned}$$

which is the required result. □

It follows that

Proposition 2.8. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be a bi-indexed sequence and let $M(\infty)(\gamma)$ be the associated infinite moment matrix. If $M(\infty)(\gamma) \geq 0$, Then, for any polynomial $p \in \mathbb{R}[x,y]$ and any integer $n \geq 1$, we have*

$$M(\infty)(\gamma)p^n = 0 \Longrightarrow M(\infty)(\gamma)p = 0. \tag{2.7}$$

Proof. If $M(\infty)(\gamma)p^2 = 0$, then $0 = M(\infty)(\gamma)p^2 = 1^T M(\infty)(\gamma)p^2 = p^T M(\infty)(\gamma)p$, from (2.6); since $M(\infty)(\gamma) \geq 0$, we obtain $M(\infty)(\gamma)p = 0$ and hence (2.7) holds for $n = 2$. By induction, (2.7) remains valid for any power of 2. Now, if $M(\infty)(\gamma)p^n = 0$ we choose r in such a way that $r + k$ is a power of 2 to ensure that

$$M(\infty)(\gamma)p^{n+r} = (p^r)^T M(\infty)(\gamma)p^n = 0.$$

Which gives $M(\infty)(\gamma)p = 0$. □

Before continuing our investigations on $RDIS$, we recall the next result on weighted r-generalized Fibonacci sequences $\{y_A(n)\}_{n=0}^{+\infty}$ where the initial conditions $A \equiv \{\alpha_n\}_{n=0}^{r-1} \subset \mathbb{C}$ are given and the sequence is associated with the characteristic polynomial $p(x) = x^r - a_1x^{r-1} - \ldots - a_r \in \mathbb{C}[x]$. It is also is the sequence generated by the difference equation with initial values:

$$\begin{cases} y_A(n) & = \alpha_n; \quad n = 0,1,\ldots,r-1, \\ y_A(n+r) & = a_1 y_A(n+r-1) + \ldots + a_r y_A(n); \quad n = r, r+1, \ldots. \end{cases} \tag{2.8}$$

We have

Theorem 2.9.([9, Theorem 1]) *Let $\{y_A(n)\}_{n=0}^{+\infty}$ be a $RDIS$ and $p(x)$ be the associated polynomial as above, with $p(x) = \prod_{i=1}^{k}(x-\lambda_i)^{m_i}$, ($m_1 + \ldots + m_k = r$). Then the difference equation* (2.8) *has r independent solution $n^j\lambda_l^n$ ($j = 0,\ldots,m_l - 1; l = 1,\ldots,k$). Moreover, any solution of* (2.8) *is of the form*

$$y_A(n) = \sum_{l=1}^{k}\sum_{j=0}^{m_l-1} e_{l,j} n^j \lambda_l^n,$$

where $e_{l,j}$ are solutions of the following system of r-equations

$$\sum_{l=1}^{k}\sum_{j=0}^{m_l-1} e_{l,j} n^j \lambda_l^n = y_A(n), \qquad n = 0,\ldots,r-1.$$

As observed in [2, Proposition 2.1], among all characteristic polynomials defining S, there exists a unique monic characteristic polynomial p_0 of minimal degree, called the minimal characteristic polynomial, and which, moreover, divides every characteristic polynomial. The next proposition gives a generalization of this result to the 2-variable case.

Proposition 2.10. *For every RDIS $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, with $\mathcal{A}_\gamma \neq \emptyset$, there exists a unique pair of characteristic polynomials $(p_1^\gamma, p_2^\gamma) \in \mathcal{A}_\gamma$ with minimal degree. Moreover, for every $(Q_1, Q_2) \in \mathcal{A}_\gamma$, the polynomials p_1^γ and p_2^γ divide Q_1 and Q_2, respectively.*

Proof. Let $(Q_1, Q_2) \in \mathcal{A}_\gamma$, with $Q_1(x) = x^{r+1} - a_0 x^r - \ldots - a_r$ and $Q_2(y) = y^{s+1} - b_0 y^s - \ldots - b_s$.

Given an integer $j \in \mathbb{Z}_+$. Since, for all $i \in \mathbb{Z}_+$,

$$\gamma_{i+r+1,j} = a_0\gamma_{i+r,j} + \ldots + a_r\gamma_{i,j},$$

then Q_1 is a characteristic polynomial of the Fibonacci sequence $\gamma_j \equiv \{\gamma_{ij}\}_{i\in\mathbb{Z}_+} : i \to \gamma_{ij}$, hence there exists a minimal characteristic polynomial $Q_1^{(j)}$, which divides Q_1. Thus $Q_1^\gamma = \bigwedge\limits_{j\geq 0} Q_1^{(j)}$, the smallest common multiple of $\{Q_1^{(j)}; j \in \mathbb{Z}_+\}$, divides Q_1 and is a characteristic polynomial of γ.

Similarly, Given an integer $i \geq 0$. Q_2 is a characteristic polynomial for the Fibonacci sequence $\gamma_i \equiv \{\gamma_{ij}\}_{j\in\mathbb{Z}_+} : j \to \gamma_{ij}$, then there exists a minimal characteristic polynomial $Q_2^{(i)}$ of γ_i, and hence $Q_2^\gamma = \bigwedge\limits_{i\geq 0} Q_2^{(i)}$ is a characteristic polynomial of γ, which divides Q_2. We conclude that the pair of analytic characteristic polynomials (Q_1^γ, Q_2^γ) provides a positive answer to the proposition. □

In the following proposition, as well as in the remainder of this paper, we associate every *RDIS* γ (with $\mathcal{A}_\gamma \neq \emptyset$) with its pair of minimal polynomials. The next theorem, of independent interest, is a crucial point in our approach.

Theorem 2.11. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a RDIS and let $(p_1, p_2) = (p_1^\gamma, p_2^\gamma) \in \mathcal{A}_\gamma$. If $M(\infty)(\gamma) \geq 0$, then the polynomials p_1 and p_2 have distinct roots.*

Proof. Let $l = 1, 2$ and let $p_l(x) = \prod\limits_{i=1}^{m}(x-\lambda_i)^{d_i}$, where $\lambda_i \in \mathbb{C}$. We notice that since $p_l(x) \in \mathbb{R}[x, y]$, then if $\lambda_i \notin \mathbb{R}$, for some i, hence there exists $j \neq i$ such that $\lambda_i = \overline{\lambda_j}$ and $d_i = d_j$. Setting $M = \max\limits_{i=1}^{m} d_i$. Since p_l is a characteristic polynomial of γ, then, from Proposition 2.3, $M(\infty)(\gamma)\prod\limits_{i=1}^{m}(x-\lambda_i)^{d_i} = 0$ and hence $M(\infty)(\gamma)\prod\limits_{i=1}^{m}(x-\lambda_i)^{M} = 0$. By using Proposition 2.8, it follows that $M(\infty)(\gamma)\prod\limits_{i=1}^{m}(x-\lambda_i) = 0$. Hence, via Proposition 2.3, the polynomial $\prod\limits_{i=1}^{m}(x-\lambda_i)$ is a characteristic polynomial of γ, which divides p_l. As p_l is minimal, we deduce that $p_l(x) = \prod\limits_{i=1}^{m}(x-\lambda_i)$, as desired. □

In the next, we give an extension of Theorem 2.9 to 2-variables.

Lemma 2.12. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a RDIS and let $(P_1, P_2) \in \mathcal{A}_\gamma$, with $P_1(x) = \prod_{l=1}^{k_1}(x-\lambda_l)^{m_l}$ and $P_2(y) = \prod_{l=1}^{k_2}(y-\beta_l)^{n_l}$, ($\lambda_l, \beta_l \in \mathbb{C}$). Then*

$$\gamma_{ij} = \sum_{a=1}^{k_1}\sum_{b=0}^{m_a-1}\sum_{c=1}^{k_2}\sum_{d=0}^{n_c-1} e_{a,b,c,d}\, i^b j^d \lambda_a^i \beta_c^j, \tag{2.9}$$

where $e_{a,b,c,d}$ are determined by using the initial condition $\{\gamma_{ij}\}_{0\le i\le r-1, 0\le j\le s-1}$.

Proof. Given an integer $j \in \mathbb{Z}_+$, the single sequence $\gamma_j : i \to \gamma_{ij}$ is a general Fibonacci sequence associated with P_1, setting $r := \deg P_1$. This implies, by virtue of Theorem 2.9, that

$$\gamma_{ij} = \sum_{a=1}^{k_1}\sum_{b=0}^{m_a-1} e_{a,b}(j)\, i^b \lambda_a^i, \tag{2.10}$$

where $e_{a,b}$ are determined uniquely by the initial conditions $\{\gamma_{ij}\}_{0\le i\le r-1}$.

Since, for every i, the sequence $j \to \gamma_{ij}$ is a general Fibonacci sequence associated with $P_2(x) = x^s - p_1 x^{s-1} - \ldots - p_s$, then

$$\gamma_{i,j+s} - p_1\gamma_{i,j+s-1} - \ldots - p_s\gamma_{i,j} = 0.$$

Hence, using (2.10), we obtain

$$\sum_{a=1}^{k_1}\sum_{b=0}^{m_a-1} (e_{a,b}(j+s) - p_1 e_{a,b}(j+s-1) - \ldots - p_s e_{a,b}(j))\, i^b \lambda_a^i = 0. \tag{2.11}$$

As $i \to i^b\lambda_a^i$ (with $b = 0, \ldots, m_a - 1; a = 1, \ldots, k_1$) are linearly independent, see Theorem 2.9, then

$$e_{a,b}(j+s) - p_1 e_{a,b}(j+s-1) - \ldots - p_s e_{a,b}(j) = 0.$$

Since j is arbitrary, the sequences $j \to e_{a,b}(j)$ ($b = 0, \ldots, m_a - 1; a = 1, \ldots, k_1$) are general Fibonacci sequences, associated with the characteristic polynomial P_2. Hence, for all $b = 0, \ldots, m_a - 1$ and $a = 1, \ldots, k_1$, we obtain

$$e_{a,b}(j) = \sum_{c=1}^{k_2}\sum_{d=0}^{n_c-1} e_{a,b,c,d}\, j^d \beta_c^j, \tag{2.12}$$

where $e_{a,b,c,d}$ are determined by the initial conditions $\{e_{a,b}(j)\}_{0\le j\le s-1}$. We conclude, from (2.11) and (2.12), that

$$\gamma_{ij} = \sum_{a=1}^{k_1}\sum_{b=0}^{m_a-1}\sum_{c=1}^{k_2}\sum_{d=0}^{n_c-1} e_{a,b,c,d}\, i^b j^d \lambda_a^i \beta_c^j. \qquad \square$$

By virtue of Theorem 2.11 and Lemma 2.12, we have the next corollary.

Corollarly 2.13. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a RDIS, with $M(\infty)(\gamma) \geq 0$, and let $(P_1, P_2) \in \mathcal{A}_\gamma$. Then*

$$\gamma_{ij} = \sum_{a=1}^{r+1}\sum_{c=1}^{s+1} e_{a,c}\lambda_a^i\beta_c^j, \tag{2.13}$$

where $Z(P_1) := \{z \in \mathbb{C}$ such that $P(z,\overline{z}) = 0\} = \{\lambda_1, \ldots, \lambda_{r+1}\}$ and $Z(P_2) = \{\beta_1, \ldots, \beta_{s+1}\}$.

The next lemma establishes, for a *RDIS* γ, a link between the positivity of infinite moment matrix and that of the finite one.

Lemma 2.14. *Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a RDIS and let $(p_1, p_2) \in \mathcal{P}_\gamma$. Then*

$$M(\infty)(\gamma) \geq 0 \Longleftrightarrow M(\deg p_1 + \deg p_2 - 2)(\gamma) \geq 0.$$

Moreover, $rankM(\infty)(\gamma) = rankM(\deg p_1 + \deg p_2 - 2)(\gamma)$.

Proof. Let (p_1, p_2) be as in (2.2) and let $H \in \mathbb{R}_{v+1}[x,y]$, with $v = r+s$. First, we will show that, for every $e = 0, \ldots, v+1$, there exist some real numbers $\{\alpha_{lk}^{(e)}; l,k \in \mathbb{Z}_+$ with $l+k \leq v\}$ such that

$$\Lambda_\gamma(x^e y^{v+1-e}H) = \sum_{l+k\leq v}\alpha_{lk}^{(e)}\Lambda_\gamma(x^l y^k H).$$

To this end we distinguish two cases.

(i) When $e \geq r+1$. We have, from Remark 2.2, $\Lambda_\gamma(p_1 x^{e-r-1}y^{v+1-e}H) = 0$, then $\Lambda_\gamma(x^e y^{v+1-e}H - \sum_{i+j\leq r} a_{ij}x^{i+e-r-1}y^{j+v+1-e}H) = 0$.

Hence
$$\begin{aligned}\Lambda_\gamma(x^e y^{v+1-e}H) &= \sum_{i+j\leq r} a_{ij}\Lambda_\gamma(x^{i+e-r-1}y^{j+v+1-e}H)\\ &= \sum_{l+k\leq v}\alpha_{lk}^{(e)}\Lambda_\gamma(x^l y^k H),\end{aligned}$$

with $l = i+e-r-1$, $k = j+v+1-e$ and $a_{l-e+r+1,k-v-1+e} = \alpha_{lk}^{(e)}$.

(ii) When $e \leq r$. It follows, from Remark 2.2, that $\Lambda(p_2 x^e y^{v-s-e}H) = 0$. Since $p_2 = y^{s+1} - \sum_{f=1}^{s+1} b_{f,s+1-f}x^f y^{s+1-f} - \sum_{i+j\leq s} b_{ij}x^i y^j$, then

$$\Lambda_\gamma(x^e y^{v+1-e}H - \sum_{f=1}^{s+1} b_{f,s+1-f}x^{f+e}y^{v+1-e-f}H - \sum_{i+j\leq s} b_{ij}x^{i+e}y^{j+v+1-e}H) = 0.$$

Hence,

$$\Lambda_\gamma(x^e y^{v+1-e}H) = \Lambda_\gamma(\sum_{f=1}^{s+1} b_{f,s+1-f}x^{f+e}y^{v+1-e-f}H) + \Lambda_\gamma(\sum_{i+j\leq s} b_{ij}x^{i+e}y^{j+v+1-e}H). \tag{2.14}$$

In the first term of the right-hand expression of equation (2.14), the Riesz functional Λ_γ is applied to a sum of monomials of total degree $v+1$ times H. With the aim of lowering the degree of the associated polynomial, we employ the method used in the case (i), for the monomials with power of x greater than $r+1$ (i.e., $f+e \geq r+1$). When $f+e \leq r+1$, we reapply the technic of the case (ii) in order to increase strictly the power of x. It follows that each time when applying case (ii) the minimum power of x, of all monomials, increases strictly. Since we can decrease the total degree of each monomial with power in x greater than or equal $r+1$, by applying the case (i), we write

$$\Lambda_\gamma(x^e y^{v+1-e}H) = \sum_{l+k\leq v} \alpha_{lk}^{(e)} \Lambda_\gamma(x^l y^k H).$$

Now we construct a $m(v) \times v$-matrix W_v with successive rows defined by the relation,

$$P_{x^e y^{v+1-e}} = \sum_{l+k\leq v} \alpha_{lk}^{(e)} e_{(l,k)},$$

where $e = 0, \dots, v+1$ and $\{e_{(l,k)}; k+l \leq v\}$ the canonical basis of $\mathbb{R}^{m(n)}$.

Therefore, it is easy to show that:

$$M(v+1)(\gamma) = \begin{pmatrix} M(v)(\gamma) & B \\ B^* & C \end{pmatrix}$$

with $B = M(v)(\gamma)W_v$ and $C = B^*W_v$. Since $M(v)(\gamma) \geq 0$, then, by using Smul'jan's theorem [12] (see also Section 4), $M(v+1)(\gamma) \geq 0$ and $rankM(v) = rankM(v+1)$. In the same way one can show that $M(v+2)(\gamma) \geq 0$ and $rank(v+2) = rankM(v+1) = rankM(v)$. And thus, by induction, we conclude that $M(\infty)(\gamma) \geq 0$ and $rankM(v)(\gamma) = rankM(\infty)(\gamma)$, as desired. □

3. The 2-variable SCP and the propagation phenomena

In this section we involve the 2-variable recursively generated weighted shifts to obtain a solution to the SCP in 2-variable. Also, a simple and new proof to the propagation phenomena, for the 2-variable subnormal weighted shift, is given.

Theorem 3.1. *Let* $\Omega_n := \{(\alpha_{(k_1,k_2)}, \beta_{(k_1,k_2)}); k_1+k_2 \leq 2n\}$ *be a given collection of weights obeying* (1.1) *and* $\gamma^{(2n+1)}$ *be the associated moment sequence, given by* (1.2)*. The following statements are equivalent.*

i) Ω_n *admits a 2-variable subnormal completion,*

ii) Ω_n *admits a recursively generated 2-variable subnormal completion,*

iii) *There exists a RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ *such that* $\gamma^{(2n+1)} \subset \gamma$ *and the matrices* $M(\deg p_1 + \deg p_2 - 2)(\gamma)$, $M_x(\deg p_1 + \deg p_2 - 1)(\gamma)$ *and* $M_y(\deg p_1 + \deg p_2 - 1)(\gamma)$ *are positive, where* $(p_1, p_2) \in \mathcal{P}_\gamma$.

Proof. First, let us show that i) $\Rightarrow$ ii). If Ω_n admits a 2-variable subnormal completion, then there is a Berger's measure ν, supported in $\mathbb{R}^2_+$, such that

$$\gamma_{ij} = \int x^i y^j d\nu, \qquad i+j \leq 2n+1. \tag{3.1}$$

A result of C. Bayer and J. Teichmann in [1] states that if a finite bi-sequence of positive numbers $\{\gamma_{ij}\}_{0\leq i+j\leq 2n}$ has a probability measure verifies (3.1), supported in $\mathbb{R}^2_+$, then it has a finitely atomic positive measure μ verify the same relation and supported, also, in $\mathbb{R}^2_+$. Write supp$\mu \subset \{\lambda_1, \lambda_2, \ldots, \lambda_r\} \times \{\beta_1, \beta_2, \ldots, \beta_s\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_r$ and $\beta_1, \beta_2, \ldots, \beta_s$ are real numbers. It is easy to see that $\int x^{i+1-r} y^j \prod_{k=1}^{r} (x-\lambda_k) d\mu = \int x^i y^{j+1-s} \prod_{k=1}^{s} (x-\beta_k) d\mu = 0$, for all $i \geq r-1$ and $j \geq s-1$. Then μ is a representing measure (that is, μ satisfies the relation (1.4)) for the *RDIS* $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$, defined by the initial conditions $\{\gamma_{ij}\}_{\substack{0 \leq i \leq r-1, \\ 0 \leq j \leq s-1.}}$ and by the following linear recurrence relations:

$$\gamma_{i+1,j} = \sum_{n=0}^{r-1} a_n \gamma_{i-n,j} \text{ and } \gamma_{i,j+1} = \sum_{m=0}^{s-1} b_m \gamma_{i,j-m}, \tag{3.2}$$

for all $i \geq r-1$ and $j \geq s-1$, where

$$\begin{cases} a_{k-1} = (-1)^{k-1} \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_k\leq r} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}, \\ b_{k-1} = (-1)^{k-1} \displaystyle\sum_{1\leq i_1<i_2<\ldots<i_k\leq s} \beta_{i_1}\beta_{i_2}\ldots\beta_{i_k}. \end{cases} \tag{3.3}$$

Remark that $x^r - \sum_{n=0}^{r-1} a_n x^{r-1-n} = \prod_{k=1}^{r} (x-\lambda_k)$ and $x^s - \sum_{n=0}^{s-1} b_n x^{s-1-n} = \prod_{k=1}^{s} (x-\beta_k)$. Hence γ is a *RDIS* admitting a Berger's measure, and thus the recursively generated 2-variable subnormal completion associated with γ gives a positive answer to the SCP associated with Ω_n.

To show that ii) $\Rightarrow$ iii), it suffices to remark that if Ω_n admits a recursively generated 2-variable subnormal completion, then there exists a finite Berger's measure μ and a *RDIS* $\gamma \equiv \{\gamma_{ij}\}$ such that $\gamma^{(2n)} \subset \gamma$ and $\int x^i y^j d\mu = \gamma_{ij}$, for all $i, j \in \mathbb{Z}_+$. Hence, for every $Q \in \mathbb{R}[x,y]$ with $\deg Q \leq \deg p_1 + \deg p_2 - 2$, we have

$$Q^T M(p_1 + \deg p_2 - 2)(\gamma) Q = \int Q^2 d\mu \geq 0.$$

Therefore $M(p_1 + \deg p_2 - 2)(\gamma) \geq 0$. Similarly, let $H \in \mathbb{R}[x,y]$ be with $\deg H \leq \deg p_1 + \deg p_2 - 2$, since μ is a positive measure supported in $\mathbb{R}^2_+$, we get

$$H^T M_x(p_1 + \deg p_2 - 1)(\gamma)H = \int xH^2 d\mu \geq 0$$

and

$$H^T M_y(p_1 + \deg p_2 - 1)(\gamma)H = \int yH^2 d\mu \geq 0.$$

Hence $M_x(p_1 + \deg p_2 - 1)(\gamma)$ and $M_y(p_1 + \deg p_2 - 1)(\gamma)$ are positive.

It remains to prove the implication iii) $\Rightarrow$ i). From Lemma 2.14 it follows that $M(\infty)(\gamma)$ has finite rank, then there exists an integer n (resp., an integer m) such that, in the matrix $M(\infty)(\gamma)$, the column X^{n+1} (resp., the column Y^{m+1}) is a linearly combination of the columns $1, X, \ldots, X^n$ (resp., the columns $1, Y, \ldots, Y^m$). So we write

$$X^{n+1} = \alpha_0 X^n + \ldots + a_n 1 \text{ and } Y^{m+1} = \beta_0 Y^m + \ldots + b_m.$$

Hence, for every $i, j \in \mathbb{Z}_+$, we have

$$\begin{cases} (X^iY^j)^T M(\infty)(\gamma) X^{n+1} & = (X^iY^j) M(\infty)(\gamma)(a_0 X^n + \ldots + a_n 1) \\ (X^iY^j)^T M(\infty)(\gamma) Y^{m+1} & = (X^iY^j) M(\infty)(\gamma)(b_0 Y^m + \ldots + b_m 1), \end{cases}$$

that is,

$$\begin{cases} \gamma_{i+n+1,j} & = a_0\gamma_{i+n,j} + \ldots + a_n\gamma_{i,j} \\ \gamma_{i,j+m+1} & = b_0\gamma_{i,j+m} + \ldots + b_m\gamma_{i,j}. \end{cases}$$

Setting $Q_1(x) = a_0x^n + \ldots + a_n$ and $Q_2(y) = b_0y^m + \ldots + b_m$, we have $(Q_1, Q_2) \in \mathcal{A}_\gamma$. Since $M(\deg p_1 + \deg p_2 - 2)(\gamma) \geq 0$, then, via Lemma 2.14, $M(\infty)(\gamma) \geq 0$ and thus there exits, by using Theorem 2.11, a pair of minimal analytic characteristic polynomials $(H_1, H_2) \in \mathcal{A}_\gamma$ with distinct roots, writing $H_1(x) = \prod_{l=1}^{r+1}(x - \lambda_l)$ and $H_2(x) = \prod_{l=1}^{s+1}(y - \beta_l)$, where $\lambda_l, \beta_l \in \mathbb{C}$.

Hence, via Corollary 2.13,

$$\gamma_{ij} = \sum_{a=1}^{r+1}\sum_{c=1}^{s+1} e_{a,c}\lambda_a^i\beta_c^j, \text{ for all } i, j \in \mathbb{Z}_+,$$

and thus the measure $\mu = \sum_{a=1}^{r+1}\sum_{c=1}^{s+1} e_{a,c}\delta_{(\lambda_a,\beta_c)}$ (Here, $\delta_{(\lambda_a,\beta_c)}$ is the Dirac measure at $(\lambda_a, \beta_c) \in \mathbb{R}^2$, having mass (λ_a, β_c) at x and mass 0 elsewhere) verifies

$$\gamma_{ij} = \int x^iy^j d\mu, \text{ for all } i, j \in \mathbb{Z}_+.$$

We will show that μ is a Berger measure, that is, μ is a positive measure supported in $\mathbb{R}^2_+$, which implies that γ is a moment sequence of some 2-variable subnormal weighted shifts, so that Ω_n admits a 2-variable subnormal completion.

Let $I_\mu = \{(a,b)$ such that $e_{a,c} \neq 0\}$ and let

$$L_{(\lambda_i,\beta_j)}(x,y) = \prod_{\substack{0 \le l \le r+1 \\ l \neq i}} (\frac{x-\lambda_i}{\lambda_l - \lambda_i}) \prod_{\substack{0 \le k \le s+1 \\ k \neq j}} (\frac{y-\beta_j}{\lambda_k - \lambda_j}) \tag{3.4}$$

Clearly

$$L_{(\lambda_i,\beta_j)}(x,y) = \begin{cases} 1 & \text{if } (i,j) = (l,k), \\ 0 & \text{if } (i,j) \neq (l,k). \end{cases} \tag{3.5}$$

Thus, for every $(a,b) \in I_\mu$, we have

$$e_{a,c} = \int \mid L_{(\lambda_a,\beta_c)} \mid^2 d\mu = L^T_{(\lambda_a,\beta_c)} M(\infty)(\gamma) L_{(\lambda_a,\beta_c)} \geq 0.$$

and then $e_{a,c} > 0$, because $e_{a,c} \in I_\mu$. Therefore, μ is a positive measure.

It remains to show that $(\lambda_a, \beta_b) \in \mathbb{R}^2_+$, for all $(a,b) \in I_\mu$. To this aim, let $(a,b) \in I_\mu$, we have

$$e_{a,c}\lambda_a = \int \mid L_{(\lambda_a,\beta_c)} \mid^2 x d\mu = L^T_{(\lambda_a,\beta_c)} M_x(\infty)(\gamma) L_{(\lambda_a,\beta_c)} \geq 0$$

and

$$e_{a,c}\beta_c = \int \mid L_{(\lambda_a,\beta_c)} \mid^2 y d\mu = L^T_{(\lambda_a,\beta_c)} M_y(\infty)(\gamma) L_{(\lambda_a,\beta_c)} \geq 0.$$

Since $e_{a,c} > 0$, we deduce that $(\lambda_a, \beta_c) \in \mathbb{R}^2_+$, as desired. □

The following corollary, of Theorem 3.1, characterizes subnormal recursively generated 2-variables weighted shifts.

Crollary 3.2. *Let $T \equiv (T_1, T_2)$ be a recursively generated 2-variable weighted shift and let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be the associated moment sequence, with $(P_1, P_2) \in \mathcal{A}_\gamma$. Then T is subnormal if and only if $M(deg(P_1) + deg(P_2) - 2)(\gamma) \geq 0$.*

In particular, we have

Crollary 3.3. Every collection of positive numbers $\alpha_{(0,0)}, \beta_{(0,0)}, \alpha_{(0,1)}, \beta_{(1,0)}$, such that $\beta_{(1,0)}\alpha_{(0,0)} = \alpha_{(0,1)}\beta_{(0,0)}$, admits a subnormal completion.

Proof. Let $\{\gamma_{ij}\}_{0\leq i,j\leq 1}$ be the collection of moments associated with $\omega \equiv \{\alpha_{(0,0)}, \beta_{(0,0)}, \alpha_{(0,1)}, \beta_{(1,0)}\}$, given by (1.2), and let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be the $RDIS$ defined by the initial conditions $\{\gamma_{ij}\}_{0\leq i,j\leq 1}$ and by the pair of, minimal, analytic characteristic polynomials (P_1, P_2). We set $P_1(X) = (X - \lambda_0)(X - \lambda_1)$ and $P_2(X) = (X - \beta_0)(X - \beta_1)$, with $0 \leq \lambda_0 \leq \lambda_1$ and $0 \leq \beta_0 \leq \beta_1$.

The measure $\mu = C_{00}\delta_{(\lambda_0,\beta_0)} + C_{10}\delta_{(\lambda_1,\beta_0)} + C_{01}\delta_{(\lambda_0,\beta_1)} + C_{11}\delta_{(\lambda_1,\beta_1)}$ is a representing measure for γ if and only if $(C_{ij})_{0\le i,j\le 1}$ are nonnegative and satisfy the following linear system of 4 equations

$$\begin{cases} C_{00} + C_{10} + C_{01} + C_{11} & = \gamma_{00}, \\ C_{00}\beta_0 + C_{10}\beta_0 + C_{01}\beta_1 + C_{11}\beta_1 & = \gamma_{01}, \\ C_{00}\lambda_0 + C_{10}\lambda_1 + C_{01}\lambda_0 + C_{11}\lambda_1 & = \gamma_{10}, \\ C_{00}\beta_0\lambda_0 + C_{10}\beta_0\lambda_1 + C_{01}\beta_1\lambda_0 + C_{11}\beta_1\lambda_1 & = \gamma_{11}. \end{cases}$$

Since the determinant of the preceding system is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \beta_0 & \beta_0 & \beta_1 & \beta_1 \\ \lambda_0 & \lambda_1 & \lambda_0 & \lambda_1 \\ \beta_0\lambda_0 & \beta_0\lambda_1 & \beta_1\lambda_0 & \beta_1\lambda_1 \end{vmatrix} = -((\lambda_1-\lambda_0)(\beta_1-\beta_0))^2 \neq 0,$$

we obtain the existence of $\{C_{00}, C_{10}, C_{01}, C_{11}\}$. Thus for the existence of $\mathbf{T} \equiv (T_1, T_2)$ a subnormal completion of $\{\gamma_{ij}\}_{0\le i,j\le 1}$ it suffices show that $\{C_{ij}\}_{0\le i,j\le 1}$ can be positive.

The numbers C_{00}, C_{10}, C_{01} and C_{11} are given by

$$\begin{cases} C_{00} = \frac{\gamma_{11}-\lambda_1\gamma_{01}-\beta_1\gamma_{10}+\lambda_1\beta_1}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}, \\ C_{01} = \frac{-\gamma_{11}+\lambda_1\gamma_{01}+\beta_0\gamma_{10}-\lambda_1\beta_0}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}, \\ C_{10} = \frac{-\gamma_{11}+\lambda_0\gamma_{01}+\beta_1\gamma_{10}-\lambda_0\beta_1}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}, \\ C_{11} = \frac{\gamma_{11}-\lambda_0\gamma_{01}-\beta_0\gamma_{10}+\lambda_0\beta_0}{(\lambda_1-\lambda_0)(\beta_1-\beta_0)}. \end{cases}$$

Direct computations show that C_{00}, C_{10}, C_{01} and C_{11} are positive numbers precisely when,

$$\begin{cases} \max\{2\gamma_{10}, \frac{2\gamma_{11}}{\gamma_{01}}\} \le \lambda_1, \\ \max\{2\gamma_{01}, \frac{2\gamma_{11}}{\gamma_{10}}\} \le \beta_1, \\ 0 \le \lambda_0 \le \min\{\frac{\gamma_{11}}{2\gamma_{01}}, \frac{\gamma_{10}}{2}\}, \\ 0 \le \beta_0 \le \min\{\frac{\gamma_{01}}{2}, \frac{\gamma_{11}}{2\gamma_{10}}\}. \end{cases} \tag{6}$$

Therefore it suffices to choose the roots of the polynomials p_1 and p_2 obeying (6). $\square$

We employ Berger's Theorem and Equality (1.3) to give a simple proof to the propagation phenomena for 2-variable subnormal weighted shifts.

Definition 3.4. A 2-variable weighted shift $\mathbf{T} \equiv (T_1, T_2)$ is *horizontally flat* (resp. *vertically flat*) if $\alpha_{(k_1,k_2)} = \alpha_{(1,1)}$, for all $k_1, k_2 \ge 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(1,1)}$).

Theorem 3.5. *Let* $\mathbf{T} \equiv (T_1, T_2)$ *be a subnormal 2-variable weighted shift associated with the weight sequences* $\{\alpha_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$ *and* $\{\beta_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^2}$. *If* $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$ *for some*

$k_1, k_2 \geq 1$ (resp. $\beta_{(k_1,k_2)} = \beta_{(k_1,k_2+1)}$), then $\mathbf{T}$ is horizontally flat (resp. vertically flat).

Proof. Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\geq 0}$ be the moment sequence of $\mathbf{T}$, see (1.2). Given an arbitrary number $n_0 \geq \max(k_1+1, k_2)$, let $\gamma^{(n_0)} \equiv \{\gamma_{ij}\}_{i,j\leq n_0}$ be a truncated subsequence of γ. Since γ admits a Berger measure, then, via Tchakaloff's Theorem [1], there exists a finite supported Berger measure μ of $\gamma^{(n_0)}$, write $\mu = \sum\limits_{\substack{0\leq i\leq p,\\ 0\leq j\leq q.}} \rho_{i,j}\delta_{(\lambda_i,\vartheta_j)}$. Notice that $\{\vartheta_j; 0 \leq j \leq q\} \neq \{0\}$ because the moments of the subnormal weighted shift are strictly positive.

Since $\alpha_{(k_1,k_2)} = \alpha_{(k_1+1,k_2)}$, then it follows from (1.3) that

$$\frac{\gamma_{(k_1+1,k_2)}}{\gamma_{(k_1,k_2)}} = \frac{\gamma_{(k_1+2,k_2)}}{\gamma_{(k_1+1,k_2)}},$$

hence

$$\Big(\sum_{\substack{0\leq i\leq p,\\ 0\leq j\leq q.}} \rho_{ij}\lambda_i^{k_1+1}\vartheta_j^{k_2}\Big)^2 = \Big(\sum_{\substack{0\leq i\leq p,\\ 0\leq j\leq q.}} \rho_{ij}\lambda_i^{k_1}\vartheta_j^{k_2}\Big)\Big(\sum_{\substack{0\leq i\leq p,\\ 0\leq j\leq q.}} \rho_{ij}\lambda_i^{k_1+2}\vartheta_j^{k_2}\Big),$$

that is,

$$\sum_{\substack{0\leq i<k\leq p,\\ 0\leq j,h\leq q.}} \rho_{ij}\rho_{hk}(\lambda_i-\lambda_k)^2\lambda_i^{k_1}\lambda_k^{k_1}\vartheta_j^{k_2}\vartheta_h^{k_2} = 0.$$

We deduce that $\lambda_i = \lambda_k$ whenever $\lambda_i.\lambda_k \neq 0$, and thus

$$\gamma_{nl} = \lambda_1^n\Big(\sum_{0\leq j\leq q} \rho_{jl}\vartheta_j^l\Big), \text{ for all } n, l \leq n_0.$$

Since n_0 is arbitrary, then

$$\alpha_{(n,m)} = \sqrt{\frac{\gamma_{(n+1,m)}}{\gamma_{(n,m)}}} = \sqrt{\frac{\gamma_{(2,1)}}{\gamma_{(1,1)}}} = \alpha_{(1,1)}, \text{ for all } n, m \in \mathbb{N}^*.$$

The vertically flatness can be proved analogously. $\square$

4. The Minimal 2-variable SCP

Given an integer $n \geq 0$, let $\Omega_n := \{(\alpha_{(k_1,k_2)}, \beta_{(k_1,k_2)}); k_1 + k_2 \leq 2n\}$ be a given collection of weight obeying (1.1) and let $\gamma^{(2n+1)} \equiv \{\gamma_{ij}\}_{i+j\leq 2n+1}$ be the corresponding moment sequence given by (1.2). We say that Ω_n admits a minimal 2-variable subnormal completion if it has a subnormal completion with $rankM(n)$-atomic Berger measure.

For $n = 1$, a characterization of weights admitting minimal 2-variable subnormal completion is given by S. H. Lee and J. Yoon [11]. In this section we give the complete solution of this problem.

Recall that, an extension of $M(n)(\gamma^{(2n)})$ is a block matrix of the form

$$M = \begin{pmatrix} M(n)(\gamma^{(2n)}) & B \\ B^* & C \end{pmatrix}. \tag{4.1}$$

A theorem of Smul'jan [12] shows that $M(n)(\gamma^{(2n)})$ admits a positive extension in the form of moment matrix if, and only if,

i) $M(n) \geq 0$,

ii) there exists a matrix W such that $B = AW$,

iii) $C \geq B^*W$,

iv) C is a Hankel $(n+2) \times (n+2)$ matrix.

Moreover, $M(n)(\gamma^{(2n)})$ admits a flat extension, or M is flat, if, and only if, $C = B^*W$. Notice that the condition iv) serves only to ensure that M is a moment matrix, see (1.5) and (1.6). If the conditions i)–iv) are satisfied, we set $M \equiv M(n+1)(\widehat{\gamma}^{(2n+2)})$, where $\gamma^{(2n)} \subseteq \widehat{\gamma}^{(2n+2)}$, i.e., $\gamma_{ij} = \widehat{\gamma}_{ij}$ for all $i+j \leq 2n$. To simplify our notations, we set $\gamma^{(2n+2)} = \widehat{\gamma}^{(2n+2)}$.

Observe that if $M(n+1)(\gamma^{(2n+2)}) \equiv M(n+1)$ is flat (that is, $rankM(n+1) = rankM(n)$), then every column of $M(n+1)$, indexed by a monomial of degree $n+1$, is a linear combination of columns indexed by monomials of degree less than or equal to n. Explicitly, the columns in $M(n+1)$ satisfies

$$X^{n+1-e}Y^e = \sum_{i+j\leq n} a_{ij}^{(e)} X^i Y^j \qquad \text{with } a_{ij}^{(e)} \in \mathbb{R} \text{ and } e = 0, \dots, n+1. \tag{4.2}$$

Hence

$$(X^lY^k)^T M(n+1) X^{n+1-e}Y^e = \sum_{i+j\leq n} a_{ij}^{(e)} (X^lY^k)^T M(n+1) X^iY^j,$$

for $(l+k \leq n+1)$, that is

$$\gamma_{n+1-e+l,e+k} = \sum_{i+j\leq n} a_{ij}^{(e)} \gamma_{i+l,j+k} \qquad (e = 0, \dots, n+1 \text{ and } l+k \leq n+1). \tag{4.3}$$

Setting $P_{X^{n+1-e}Y^e}(x,y) = \sum_{i+j\leq n} a_{ij}^{(e)} x^i y^j$.

Lemma 4.1. *Let $\gamma^{(2n+2)} \equiv \{\gamma_{ij}\}_{i+j\leq 2n+2}$ be a finite collection of non negative numbers such that $rankM(n+1) = rankM(n)$ and let $P_{X^{n+1-e}Y^e}$ be as above. Then, for all $e \in \{0, \dots, n+1\}$, the polynomial $x^{n+1-e}y^e - P_{X^{n+1-e}Y^e}$ is a characteristic one of the RDIS defined by the initial conditions $\{\gamma_{ij}\}_{i,j\leq n}$ and by the pair of characteristic polynomials $(x^{n+1} - P_{X^{n+1}}, y^{n+1} - P_{Y^{n+1}})$.*

Proof. Let $\gamma \equiv \{\gamma_{ij}\}_{i,j\in\mathbb{Z}_+}$ be a *RDIS* defined by the initial conditions $\{\gamma_{ij}\}_{i,j\leq n}$

and by the pair of characteristic polynomials $(x^{n+1}-P_{X^{n+1}}, y^{n+1}-P_{Y^{n+1}})$. Corresponding to the sequence γ, the Riesz functional $\Lambda : \mathbb{R}[x,y] \to \mathbb{R}$ defined by $\Lambda(\sum\limits_{i,j} p_{ij}x^iy^j) = \sum\limits_{i,j} p_{ij}\gamma_{ij}$. Then , for all $a,b \in \mathbb{Z}_+$, we have

$$\Lambda(x^ay^b(x^{n+1}-P_{X^{n+1}})) = \Lambda(x^ay^b(y^{n+1}-P_{Y^{n+1}})) = 0. \tag{4.4}$$

Let $e \in \{0,\ldots,n+1\}$ be a fixed integer, we will show that $x^{n+1-e}y^e - P_{X^{n+1-e}Y^e}$ is a characteristic polynomial of γ, that is, for all $a,b \in \mathbb{Z}_+$,

$$\Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})) = 0. \tag{4.5}$$

We distinguish two cases;

1) $a+b \le n+1$, then (4.3) implies that $\Lambda(x^{a+n+1-e}y^{b+e}) = \Lambda(x^ay^bP_{X^{n+1-e}Y^e})$ and hence (4.5) is verified.

2) $a+b = n+2$, we show first that

$$\Lambda(x^fy^gQ_k) = 0 \text{ for all } f+g \le n+1, \tag{4.6}$$

where $Q_k = yP_{X^{n+1-k}Y^k} - xP_{X^{n-k}Y^{k+1}}$ and $k \in \{0,\ldots,n+1\}$.

For $f+g \le n$, we have

$$\begin{aligned} \Lambda(x^fy^gQ_k) &= \Lambda(x^fy^g(yP_{X^{n+1-k}Y^k} - xP_{X^{n-k}Y^{k+1}})) \\ &= \Lambda(x^fy^{g+1}P_{X^{n+1-k}Y^k} - x^{f+1}y^gP_{X^{n-k}Y^{k+1}})) \\ &= \Lambda(x^{f+n+1-k}y^{g+1+k} - x^{f+1+n-k}y^{g+k+1}), \text{ due to (4.3)}, \\ &= 0. \end{aligned} \tag{4.7}$$

For $f+g = n+1$, since $\deg Q_k \le n+1$, and according to (4.3), we derive that $\Lambda(x^fy^gQ_k) = \Lambda(P_{X^fY^g}Q_k)$. As $\deg P_{X^fY^g} \le n$, then (4.7) implies that $\Lambda(P_{X^fY^g}Q_k) = 0$, and hence $\Lambda(x^fy^gQ_k) = 0$ (for all $f+g \le n+1$).

Now we consider the case $a+b = n+2$, which we split in two subcases.
i) If $a \ge e$: according to (4.6), we have

$$\begin{aligned} \Lambda(x^{a-e}y^{b+e}P_{X^{n+1}}) &= \Lambda(x^{a-e+1}y^{b+e-1}P_{X^nY}) \\ &= \ldots = \Lambda(x^ay^bP_{X^{n+1-e}Y^e}) \end{aligned}$$

Thus

$$\begin{aligned} 0 &= \Lambda(x^{a-e}y^{b+e}P_{X^{n+1}}) - \Lambda(x^ay^bP_{X^{n+1-e}Y^e}) \\ &= \Lambda(x^{a-e+n+1}y^{b+e}) - \Lambda(x^ay^bP_{X^{n+1-e}Y^e}), \text{ by applying (4.4)}, \\ &= \Lambda(x^{a+n+1-e}y^{b+e} - x^ay^bP_{X^{n+1-e}Y^e}) \\ &= \Lambda(x^ay^b(x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})). \end{aligned} \tag{4.8}$$

ii) If $a \le e$: since $a+b=n+2$, then $b+e \ge n+1$ and $b+e \ge n+1$. According to (4.7), we have

$$\begin{aligned}\Lambda(x^a y^b P_{X^{n+1-e}Y^e}) &= \Lambda(x^{a+1}y^{b-1}P_{X^{n-e}Y^{e+1}}) \\ &= \ldots = \Lambda(x^{a+n+1-e}y^{b-n-1+e}P_{Y^{n+1}})\end{aligned} .$$

Hence

$$\begin{aligned}0 &= \Lambda(x^{a+n+1-e}y^{b-n-1+e}P_{Y^{n+1}}) - \Lambda(x^a y^b P_{X^{n+1-e}Y^e}) \\ &= \Lambda(x^{a+n+1-e}y^{b+e}) - \Lambda(x^a y^b P_{X^{n+1-e}Y^e}), \text{ due to (4.4)}, \\ &= \Lambda(x^{a+n+1-e}y^{b+e} - x^a y^b P_{X^{n+1-e}Y^e}) \\ &= \Lambda(x^a y^b (x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})).\end{aligned} \tag{4.9}$$

Therefore, we conclude that

$$\Lambda(x^a y^b (x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})) = 0, \qquad \text{for all } a+b \le n+2. \tag{4.10}$$

By induction, we obtain $\Lambda(x^a y^b (x^{n+1-e}y^e - P_{X^{n+1-e}Y^e})) = 0$, for all $a, b \in \mathbb{Z}_+$. $\square$

Now we are able to give a concrete solution to the minimal 2-variable SCP. To this aim, let $\gamma^{(2n)} \equiv \{\gamma_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^2_+,|\mathbf{i}|\le 2n} \equiv \{\gamma_{ij}\}_{i+j\le 2n}$ be a given bi-sequence and let $M[i,j]$ $(i,j = 0,1,\ldots,n)$ be as in (1.6). In the next theorem, we denote by B the following matrix

$$B \equiv B(n+1) := \begin{pmatrix} M[0,n+1] \\ \vdots \\ M[n-1,n+1] \\ M[n,n+1] \end{pmatrix}.$$

Theorem 4.2. *Let $\Omega_n := \{(\alpha_{(k_1,k_2)}, \beta_{(k_1,k_2)}); k_1+k_2 \le 2n\}$ be a given collection of weights obeying* (1.1) *and let $\gamma^{(2n+1)}$, $M(n)$, $M_x(n)$, $M_y(n)$, B and $P_{x^{n+1-i}y^i}$ $(i = 0,\ldots,n+1)$ be as above, with $RangB \subseteq RangM(n)$ and $M(n)$, $M_x(n)$ and $M_y(n)$ are positive. Then Ω_n admits a minimal subnormal completion if and only if*

$$P^T_{x^{n+1-j}y^j} M(n) P_{x^{n+1-i}y^i} = P^T_{x^{n-j}y^{j+1}} M(n) P_{x^{n+2-i}y^{i-1}} \tag{4.11}$$

for all integers i and j with $0 \le j \le n-1$ and $2+j \le i \le n+1$.

Proof. Let $m \equiv m(n)$ denote the number of rows (or columns) in $M(n)$. Adopting the notation in (4.1), with B is a $m \times (n+1)$ matrix. Observe that, the sequence $\gamma^{(2n+1)}$ fills only the matrices $M(n)$ and B. We are looking for new numbers (entries) for the matrix C in order that M be a flat moment matrix.

Let the rows $X^{n+1-j}Y^j$, columns $X^{n+1-i}Y^i$ $(i,j = 0,\ldots,n+1)$ entry of the matrix M be equal to $P^T_{x^{n+1-j}y^j} M(n) P_{x^{n+1-i}y^i}$ (i.e., $C = (M(n)W)^*W$). With this completion, M becomes a flat completion of $M(n)$.

Let us show that M is a moment matrix, that is, C is Hankel. The relation (4.11) implies that the upper left triangular part of the matrix C is Hankel type. Since $M(n)$ is symmetric, then C is also symmetric, and thus $C=(M(n)W)^*W$ is a Hankel matrix. Setting $M=M(n+1)(\gamma^{(2n+2)})$. It follows, from Lemma 4.1, that there exists a $DIRS$ γ such that $rankM(n)=rankM(\infty)(\gamma)$ (where $M(n)=M(n)(\gamma^{(2n)})=M(n)(\gamma)$).

We show now that $rankM_x(n+1)(\gamma)=rankM_x(\infty)(\gamma)$ and $rankM_y(n+1)(\gamma)=rankM_y(\infty)(\gamma)$. We prove first that the columns in $M_x(n+2)$ verify the relation $X^eY^{n+1-e}=\sum\limits_{i+j\leq n}a_{ij}^{(e)}X^iY^j(=P_{x^ey^{n+1-e}}(X,Y))$. We have, for all $a+b\leq n+1$,

$$\begin{aligned}
&(X^aY^b)^TM_x(n+2)(\gamma)(X^eY^{n+1-e}-P_{x^ey^{n+1-e}}(X,Y))\\
&\qquad =<M_x(n+2)(\gamma)(x^ey^{n+1-e}-P_{x^ey^{n+1-e}}),x^ay^b>,\ \text{see (1.8)},\\
&\qquad =<M(n+1)(\gamma)(x^ey^{n+1-e}-P_{x^ey^{n+1-e}}),x^{a+1}y^b>\\
&\qquad =0,\ \text{since } x^ey^{n+1-e}-P_{x^ey^{n+1-e}} \text{ is a characteristic polynomial of } \gamma.
\end{aligned}$$

Hence $M_x(n+2)(\gamma)(X^eY^{n+1-e})=M_x(n+2)(\gamma)(P_{x^ey^{n+1-e}}(X,Y))$. Since $\deg P_{x^ey^{n+1-e}}\leq n$, then $rankM_x(n+1)(\gamma)=rankM_x(n+2)(\gamma)$, and thus we obtain, by induction, $rankM_x(n+1)(\gamma)=rankM_x(\infty)(\gamma)$. Similarly, one shows that $rankM_y(n+1)(\gamma)=rankM_y(\infty)(\gamma)$. Since $M_x(n+1)(\gamma)$ and $M_y(n+1)(\gamma)$ are positive, then, via Schmul'jan's theorem, $M_x(\infty)(\gamma)$ and $M_y(\infty)(\gamma)$ are positive.

Let $(p_1,p_2)\in\mathcal{A}_\gamma$. We conclude, from above, that $M(\deg p_1+\deg p_2-2)(\gamma)$, $M_y(\deg p_1+\deg p_2-1)(\gamma)$ and $M_y(\deg p_1+\deg p_2-1)(\gamma)$ are positive. Now, by applying Theorem 3.1, Ω_n admits a subnormal completion with finite Berger measure, say $\mu=\sum\limits_{(a,b)\in I_\mu}e_{(a,b)}d\delta_{(\lambda_a,\beta_b)}$ (with $e_{(a,b)}\neq 0$, for all $(a,b)\in I_\mu$).

It remain to show that $card\,supp\mu=rankM(n)(\gamma)$. Let $\zeta_{(\lambda_a,\beta_c)}:=(1,\lambda_a,\beta_c,\lambda_a^2,\lambda_a\beta_c,\beta_c^2,\ldots)=(\lambda_a^i\beta_c^j)_{(i,j)\in\mathbb{Z}_+^2}\in\mathbb{R}^{\mathbb{Z}_+^2}$. By using Corollary 2.13, the infinite moment matrix can be formulated as follows $M(\infty)(\gamma)=\sum\limits_{(a,c)\in I_\mu}e_{(a,c)}\zeta_{(\lambda_a,\beta_c)}\zeta_{(\lambda_a,\beta_c)}^T$, then

$$rankM(\infty)(\gamma)\leq card\,I_\mu=card\,supp\mu.$$

On the other hand, Let $L_{(\lambda_a,\beta_c)}$ be as in (3.5) and let

$$M(\infty)(\gamma)\sum_{(a,c)\in I_\mu}\kappa_{(a,c)}\frac{1}{\sqrt{e_{(a,c)}}}L_{(\lambda_a,\beta_c)}=0,$$

where $\{\kappa_{(a,c)}\}_{(a,c)\in I_\mu}$ are real numbers (not all zero).

Since $\frac{1}{\sqrt{e_{(i,j)}}}L_{(\lambda_i,\beta_j)}^TM(\infty)(\gamma)\frac{1}{\sqrt{e_{(n,m)}}}L_{(\lambda_n,\beta_m)}$ is 1 if $(i,j)=(n,m)$ and equal to

0 if $(i,j) \neq (n,m)$, then

$$0 = (\sum_{(a,c)\in I_\mu} \kappa_{(a,c)} \frac{1}{\sqrt{e_{(a,c)}}} L_{(\lambda_a,\beta_c)})^T M(\infty)(\gamma) \sum_{(a,c)\in I_\mu} \kappa_{(a,c)} \frac{1}{\sqrt{e_{(a,c)}}} L_{(\lambda_a,\beta_c)}$$
$$= \sum_{(a,c)\in I_\mu} \kappa^2_{(a,c)},$$

a contradiction. Hence $card\, supp\mu \leq rank M(\infty)(\gamma)$, and thus $card\, supp\mu = rank M(n)(\gamma)$, as desired. □

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