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On $[m, C]$-symmetric Operators

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Abstract. In this paper first we show properties of isosymmetric operators given by M. Stankus [13]. Next we introduce an $[m, C]$-symmetric operator T on a complex Hilbert space $\mathcal{H}$. We investigate properties of the spectrum of an $[m, C]$-symmetric operator and prove that if T is an $[m, C]$-symmetric operator and Q is an n-nilpotent operator, respectively, then $T + Q$ is an $[m + 2n - 2, C]$-symmetric operator. Finally, we show that if T is $[m, C]$-symmetric and S is $[n, D]$-symmetric, then $T \otimes S$ is $[m + n - 1, C \otimes D]$-symmetric.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\, ,\, \rangle$ and $B(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Let $\mathbb{N}$ be the set of all natural numbers. For the study of Jordan operators, J.W. Helton ([9] and [10]) introduced an operator

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$T \in B(\mathcal{H})$ which satisfies

$$\alpha_m(T) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^j = 0 \quad (m \in \mathbb{N}).$$

In particular, if T is normal, then $\alpha_m(T) = (T^* - T)^m$. An operator $T \in B(\mathcal{H})$ is said to be an *m-symmetric* operator if $\alpha_m(T) = 0$. Hence T is 1-symmetric if and only if T is Hermitian. It is well known that if T is m-symmetric, then T is n-symmetric for all $n \geq m$. The concept of m-symmetric operators is little strong. For example, if T is m-symmetric, then $\sigma(T) \subset \mathbb{R}$ (cf.[10]). And T is Hermitian even if T is 2-symmetric. Also if T is normal and m-symmetric, then T is Hermitian due to the fact that $T^* - T$ is normal and nilpotent, that is, $T^* - T = 0$.

Recently, C. Gu and M. Stankus ([8]) showed interesting properties of m-symmetric operators. On the other hand, for $m \in \mathbb{N}$, an operator $T \in B(\mathcal{H})$ is said to be an *m-isometric* operator if

$$\beta_m(T) := \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{m-j} = 0.$$

It is well known that if T is m-isometric, then T is n-isometric for all $n \geq m$. In 1995, J. Agler and M. Stankus [1] introduced an m-isometric operator and showed many important results of such an operator. If T is an invertible m-isometric operator and m is even, then T is $(m-1)$-isometric. But if T is m-symmetric and m is even, then T is always $(m-1)$-symmetric by Theorem 3.4 of [12]. For every odd number m, there exists an invertible m-isometric operator T which is not $(m-1)$-isometric (see Theorem 1 in [5]).

Throughout this paper, let I be the identity operator on $\mathcal{H}$ and m, n be natural numbers. An operator $Q \in B(\mathcal{H})$ is said to be a nilpotent operator of order n if $Q^n = 0$ and $Q^{n-1} \neq 0$. For a subset $A \subset \mathbb{C}$, let $A^* = \{\overline{z} : \ z \in A \}$. Let $\sigma(T)$ and $\sigma_p(T)$ be the spectrum and the point spectrum of $T \in B(\mathcal{H})$, respectively. The approximate point spectrum of T is defined by $\sigma_a(T) := \{ z \in \mathbb{C} : T - zI$ is not bounded below$\}$, and the surjective spectrum of T is defined by $\sigma_s(T) := \{ z \in \mathbb{C} : T - zI$ is not surjective$\}$. It is known that $\sigma(T) = \sigma_a(T) \cup \sigma_s(T), \sigma_a(T)^* = \sigma_s(T^*)$, and $\sigma_s(T)^* = \sigma_a(T^*)$.

2. Isosymmetric Operators

First we show the following result of m-symmetric operators.

Proposition 2.1. *Let $T \in B(\mathcal{H})$. Then the following statements hold;*

(a) *T is a 2-symmetric operator if and only if T is Hermitian.*

(b) *Let T be an m-symmetric operator. For $a \neq b$ and non-zero vectors $x, y \in \mathcal{H}$, if $Tx = ax$ and $Ty = by$, then $\langle x, y \rangle = 0$.*

(c) *Let T be an m-symmetric operator. For $a \neq b$ and sequences $\{x_k\}, \{y_k\}$ of unit vectors of $\mathcal{H}$, if $(T-a)x_k \to 0$ and $(T-b)y_k \to 0$, then $\lim_{k\to\infty} \langle x_k, y_k\rangle = 0$.*

Proof. (a) If T is Hermitian, then it is obvious that T is 2-symmetric. If T is 2-symmetric, then T is 1-symmetric from [12, Theorem 3.4] and so it is Hermitian.
(b) Since $a, b \in \sigma(T)$, it follows from [10] that a, b are real numbers. Hence it holds

$$0 = \langle \alpha_m(T)x,\, y\rangle = (b-a)^m \cdot \langle x,\, y\rangle.$$

Since $a \neq b$, we have $\langle x,\, y\rangle = 0$.
(c) By similar arguments of the proof of (b), a, b are real numbers and it holds

$$0 = \lim_{k\to\infty} \langle \alpha_m(T)x_k,\, y_k\rangle = (b-a)^m \cdot \lim_{k\to\infty} \langle x_k,\, y_k\rangle.$$

Since $a \neq b$, we have $\lim_{k\to\infty} \langle x_k,\, y_k\rangle = 0$. □

Definition 1. For an operator $T \in B(\mathcal{H})$, we define $\gamma_{m,n}(T)$ by

$$\gamma_{m,n}(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} \alpha_n(T) T^{m-j} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*n-k} \beta_m(T) T^k.$$

Then T is said to be (m, n)-isosymmetric if $\gamma_{m,n}(T) = 0$.

It is easy to see that

$$\gamma_{m+1,n}(T) = T^*\gamma_{m,n}(T)T - \gamma_{m,n}(T) \text{ and } \gamma_{m,n+1}(T) = T^*\gamma_{m,n}(T) - \gamma_{m,n}(T)T.$$

Hence if T is (m, n)-isosymmetric, then T is (m', n')-isosymmetric for all $n' \geq n$ and $n' \geq n$. M. Stankus proved the following properties.

Proposition 2.2.([13, Corollary 30]) *Let T be (m, n)-isosymmetric.*

(1) *If $\sigma(T) \subset \{x \in \mathbb{R} : |x| > 1\}$ or $\sigma(T) \subset \{x \in \mathbb{R} : |x| < 1\}$, then T is n-symmetric.*

(2) *If $\sigma(T) \subset \{e^{i\theta} : 0 < \theta < \pi\}$ or $\sigma(T) \subset \{e^{i\theta} : \pi < \theta < 2\pi\}$, then T is m-isometric.*

For $a, b \in \mathbb{C}$ and non-zero vectors $x, y \in \mathcal{H}$, if $Tx = ax, Ty = by$, then it holds that

$$\begin{aligned} \langle \gamma_{m,n}(T)x, y\rangle &= \langle (\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} \alpha_n(T) T^{m-j})x, y\rangle \\ &= (a\bar{b} - 1)^m (a - \bar{b})^n \langle x, y\rangle. \end{aligned}$$

Hence we have the following theorem.

Theorem 2.3. *Let T be (m, n)-isosymmetric and x, y be unit vectors and x_k, y_k be sequences of unit vectors of $\mathcal{H}$.*

(1) *If* $Tx = ax$, $Ty = by$, $a \neq b$ *and* $a \neq \bar{b}$, *then* $\langle x, y\rangle = 0$.

(2) *If* $(T-a)x_k \to 0$, $(T-b)y_k \to 0$ $(k \to \infty)$, $a \neq b$ *and* $a \neq \bar{b}$, *then* $\lim_{k\to\infty} \langle x_k, y_k\rangle = 0$.

Theorem 2.4. *Let* T *be* (m,n)*-isosymmetric.*

(1) *Then* T^k *is* (m,n)*-isosymmetric for any* $k \in \mathbb{N}$.

(2) *If* T *is invertible, then* T^{-1} *is* (m,n)*-isosymmetric.*

Proof. (1) Note that for $k \in \mathbb{N}$, the following equation holds;

$$\begin{aligned} & (y^k x^k - 1)^m (y^k - x^k)^n \\ = & \big((yx-1)(y^{k-1}x^{k-1} + y^{k-2}x^{k-2} + \cdots + 1)\big)^m \\ & \cdot\big((y-x)(y^{k-1} + y^{k-2}x + \cdots + x^{k-1})\big)^n \\ = & \sum_{\ell=0}^{m(k-1)} \sum_{j=0}^{n(k-1)} \lambda_\ell \mu_j y^{m(k-1)-\ell} y^{n(k-1)-j} \big(yx-1\big)^m \big(y-x\big)^n x^j x^{m(k-1)-\ell} \end{aligned}$$

where λ_ℓ and μ_j are some constants. From this, we have

$$\gamma_{m,n}(T^k) = \sum_{\ell=0}^{m(k-1)} \sum_{j=0}^{n(k-1)} \lambda_\ell \mu_j T^{*m(k-1)-\ell+n(k-1)-j}\, \gamma_{m,n}(T)\, T^{j+m(k-1)-\ell}.$$

Hence T^k is (m,n)-isosymmetric.

(2) Assume that T is invertible. Since

$$\begin{aligned} 0 &= T^{*-m-n}\gamma_{m,n}(T)T^{-m-n} \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*-m-n}T^{*m-j}\alpha_n(T)T^{m-j}T^{-m-n} \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*-n-j}\alpha_n(T)T^{-n-j} \\ &= \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*-j}\Big(T^{*-n}\alpha_n(T)T^{-n}\Big)T^{-j} \\ &= \begin{cases} \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*-j}\cdot(\alpha_n(T^{-1}))\cdot T^{-j} = \gamma_{m,n}(T^{-1}) & (m \text{ is even}) \\ \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*-j}\cdot(-\alpha_n(T^{-1}))\cdot T^{-j} = -\gamma_{m,n}(T^{-1}) & (m \text{ is odd}), \end{cases} \end{aligned}$$

it follows that T^{-1} is (m,n)-isosymmetric. □

Operators T and S are said to be *doubly commuting* if $TS = ST$ and $TS^* = S^*T$. From the equation

$$((y_1+y_2)(x_1+x_2)-1)^m((y_1+y_2)-(x_1+x_2))^n$$
$$= \sum_{j=0}^{n}\sum_{i+l+h=m}\binom{n}{j}\binom{m}{i,l,h}(y_1+y_2)^i y_2^l (y_1x_1-1)^h(y_1-x_1)^{n-j}(y_2-x_2)^j x_1^l x_2^i,$$

if T and S are doubly commuting, then it holds

(2.1)
$$\gamma_{m,n}(T+S) = \sum_{j=0}^{n}\sum_{i+l+h=m}\left(\begin{array}{c} n \\ j\end{array}\right)\left(\begin{array}{c} m \\ i,l,h\end{array}\right)\cdot (T^*+S^*)^i S^{*l}\,\gamma_{h,n-j}(T)\,\alpha_j(S)\,T^l S^i.$$

Theorem 2.5. *Let T be (m,n)-isosymmetric and let Q be a nilpotent operator of order k. If T and Q are doubly commuting, then $T+Q$ is $(m+2k-2, n+2k-1)$-isosymmetric.*

Proof. From equation (2.1), it holds

$$\gamma_{m+2k-2,n+2k-1}(T+Q) = \sum_{j=0}^{n+2k-1}\sum_{i+l+h=m+2k-2}\left(\begin{array}{c} n+2k-1 \\ j\end{array}\right)\left(\begin{array}{c} m+2k-2 \\ i,l,h\end{array}\right)$$
$$\cdot (T^*+Q^*)^i Q^{*l}\gamma_{h,n+2k-1-j}(T)\alpha_j(Q)T^lQ^i.$$

(1) If $j \geq 2k$ or $i \geq k$ or $l \geq k$, then $\alpha_j(Q) = 0$ or $Q^i = 0$ or $Q^{*l} = 0$, respectively.
(2) If $j \leq 2k-1$ and $i \leq k-1$ and $l \leq k-1$, then $h = m+2k-2-i-l \geq m$ and $n+2k-1-j \geq n+2k-1-(2k-1) = n$, i.e., $\gamma_{h,n+2k-1-j}(T) = 0$.

By (1) and (2) we have $\gamma_{m+2k-2,n+2k-1}(T+Q) = 0$. Therefore $T+Q$ is $(m+2k-2, n+2k-1)$-isosymmetric. □

Note that the equation

$$(y_1y_2x_1x_2-1)^m\cdot(y_1y_2-x_1x_2)^n$$
$$= \sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}y_1^{j+k}(y_1x_1-1)^{m-k}(y_1-x_1)^{n-j}(y_2x_2-1)^k(y_2-x_2)^j x_1^k x_2^{n-j}.$$

From this, if T and S are doubly commuting, then it holds

(2.2)
$$\gamma_{m,n}(TS) = \sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}T^{*j+k}\gamma_{m-k,n-j}(T)\cdot\gamma_{k,j}(S)T^kS^{n-j}.$$

Theorem 2.6. *Let T be (m,n)-isosymmetric and let S be m'-isometric and n'-symmetric. If T and S are doubly commuting, then TS is $(m+m'-1, n+n'-1)$-isosymmetric.*

Proof. From equation (2.2), it holds

$$\gamma_{m+m'-1,n+n'-1}(TS) = \sum_{k=0}^{m+m'-1} \sum_{j=0}^{n+n'-1} \begin{pmatrix} n+n'-1 \\ j \end{pmatrix} \begin{pmatrix} m+m'-1 \\ k \end{pmatrix} T^{*j+k}$$

$$\cdot \gamma_{m+m'-1-k,n+n'-1-j}(T) \cdot \gamma_{k,j}(S) \cdot T^k S^{n-j}.$$

(1) If $k \geq m'$ or $j \geq n'$, then $\gamma_{k,j}(S) = 0$.
(2) If $k \leq m'-1$ and $j \leq n'-1$, then $m+m'-1-k \geq m$ and $n+n'-1-j \geq n$, i.e., $\gamma_{m+m'-1-k,n+n'-1-j}(T) = 0$.

By (1) and (2) we have $\gamma_{m+m'-1,n+n'-1}(TS) = 0$. Hence it completes the proof. □

For a complex Hilbert space $\mathcal{H}$, let $\mathcal{H} \otimes \mathcal{H}$ denote the completion of the algebraic tensor product of $\mathcal{H}$ and $\mathcal{H}$ endowed a reasonable uniform cross-norm. For operators $T \in B(\mathcal{H})$ and $S \in B(\mathcal{H})$, $T \otimes S \in B(\mathcal{H} \otimes \mathcal{H})$ denote the *tensor product* operator defined by T and S. Note that $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$.

Theorem 2.7. *Let T be (m,n)-isosymmetric and let S be m'-isometric and n'-symmetric. Then $T \otimes S$ is $(m+m'-1, n+n'-1)$-isosymmetric.*

Proof. It is clear that if T is (m,n)-isosymmetric, then $T \otimes I$ is (m,n)-isosymmetric and if S is m'-isometric and n'-symmetric, then $I \otimes S$ is m'-isometric and n'-symmetric. Since $T \otimes I$ and $I \otimes S$ are doubly commuting, it follows from Theorem 2.6 that $T \otimes S$ is $(m+m'-1, n+n'-1)$-isosymmetric. Hence it completes the proof. □

3. Conjugation and Example

In this section, we introduce $[m,C]$-symmetric operators and provide several examples. An antilinear operator C on $\mathcal{H}$ is said to be a *conjugation* if C satisfies $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *complex symmetric* if $CTC = T^*$ for some conjugation C.

Definition 2. For an operator $T \in B(\mathcal{H})$ and a conjugation C, we define the operator $\alpha_m(T;C)$ by

$$\alpha_m(T;C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} CT^{m-j}C \cdot T^j.$$

An operator $T \in B(\mathcal{H})$ is said to be an $[m,C]$-symmetric operator if $\alpha_m(T;C) = 0$.

Hence if T is complex symmetric and $[m,C]$-symmetric, then T is m-symmetric. It holds that

$$CTC \cdot \alpha_m(T;C) - \alpha_m(T;C) \cdot T = \alpha_{m+1}(T;C). \tag{3.1}$$

Moreover, if T is $[m, C]$-symmetric, then T is $[n, C]$-symmetric for every natural number n $(\geq m)$ and $\ker(\alpha_{m-1}(T; C))$ $(m \geq 2)$ is an invariant subspace for T.

Example 3.1. Let $\mathcal{H} = \mathbb{C}^2$ and let C be a conjugation on $\mathcal{H}$ given by $C\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{y} \\ \overline{x} \end{pmatrix}$ for $x, y \in \mathbb{C}$.

(a) If $T = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}$ on C^2, then T is not Hermitian and $CTC = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} = T$. Hence T is $[1, C]$-symmetric.
Hence, in this case, $\sigma(T) = \{0\}$ due to the fact that T is nilpotent.

(b) Let $S = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$ on C^2. Then S is not Hermitian and $CSC = \begin{pmatrix} i & \sqrt{2} \\ \sqrt{2} & -i \end{pmatrix}$ and $CSC = S \neq S^*$. Therefore, S is $[1, C]$-symmetric. Furthermore, $\sigma(S) = \{1, -1\}$.

(c) If $R = \begin{pmatrix} 1 & \frac{1}{2}i \\ \frac{1}{2}i & 2 \end{pmatrix}$ on C^2, then

$$CR^2C - 2CRC \cdot R + R^2 = \begin{pmatrix} \frac{15}{4} & -\frac{3}{2}i \\ -\frac{3}{2}i & \frac{3}{4} \end{pmatrix} - 2\begin{pmatrix} \frac{9}{4} & 0 \\ 0 & \frac{9}{4} \end{pmatrix} + \begin{pmatrix} \frac{3}{4} & \frac{3}{2}i \\ \frac{3}{2}i & \frac{15}{4} \end{pmatrix} = 0.$$

Hence R is $[2, C]$-symmetric. It is easy to see that R is not $[1, C]$-symmetric. Moreover, $\sigma(R) = \{\frac{3}{2}\}$.

(d) Let $W = \begin{pmatrix} 2i & 1 \\ 1 & -2i \end{pmatrix}$ on C^2. Then it is easy to see that $CWC = W \neq W^*$. Hence W is $[1, C]$-symmetric and $\sigma(W) = \{\sqrt{3}\,i, -\sqrt{3}\,i\}$.

Example 3.2. Let $\mathcal{H} = \ell^2$, let $\{e_n\}_{n=1}^{\infty}$ be the natural basis of $\mathcal{H}$ and let $C : \mathcal{H} \longrightarrow \mathcal{H}$ be a conjugation given by

$$C(\sum_{n=1}^{\infty} x_n\, e_n) = \sum_{n=1}^{\infty} \overline{x_n}\, e_n$$

where $\{x_n\}$ is a sequence in C with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and $Ce_n = e_n$.

(i) If $U \in B(\mathcal{H})$ is the unilateral shift on ℓ^2, then it is easy to compute $U = CUC$ and so U is a $[1, C]$-symmetric operator with $\sigma(U) = \mathbb{D}$ (unit disk).

(ii) Let W be the weighted shift given by $We_n = \alpha_n e_{n+1}$, where $\alpha_n = \begin{cases} 2i & (n = 1) \\ \frac{n+1}{n} i & (n \geq 2). \end{cases}$ Then

$$(CW^2C - 2CWCW + W^2)e_n = [(\overline{\alpha_n} - \alpha_n)\overline{\alpha_{n+1}} - \alpha_n(\overline{\alpha_{n+1}} - \alpha_{n+1})]e_n$$

for all $n \geq 1$. Hence W is $[2, C]$-symmetric operator.

An operator $A \in \mathcal{L}(\mathcal{H})$ is *n-Jordan* if $A = T + N$ where T is self-adjoint, N is nilpotent of order $[\frac{n+1}{2}]$, and $TN = NT$ where $[k]$ denotes the integer part of k.

Example 3.3. Let C be a conjugation C on $\mathcal{H}$. Suppose that $A = T + N$ is an n-Jordan operator where $T = T^* = CTC$, N is nilpotent of order $[\frac{n+1}{2}]$, $TN = NT$, and $CN = NC$. Then A is $[n, C]$-symmetric for the conjugation C. Indeed, since $T = T^* = CTC$, $TN = NT$, and $CN = NC$, it follows that

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} CA^{n-j}C \cdot A^j = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (T + N)^{n-j} \cdot (T + N)^j = 0$$

which means that A is an n-symmetric operator from [12, Theorem 3.2]. Hence A is $[n, C]$-symmetric.

4. $[m, C]$-symmetric Operators

Let C be a conjugation on $\mathcal{H}$. Then C satisfies $\|Cx\| = \|x\|$ and $C(\alpha x) = \overline{\alpha} \cdot Cx$ for all $x \in \mathcal{H}$ and all $\alpha \in \mathbb{C}$. Moreover, since $C^2 = I$, it follows that $(CTC)^* = CT^*C$ and $(CTC)^n = CT^nC$ for every positive integer n (see [7] for more details).

We now provide properties of $[m, C]$-symmetric operators.

Theorem 4.1. *Let $T \in B(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Then the following assertions hold;*

(a) *T is an $[m, C]$-symmetric operator if and only if so is T^*.*

(b) *If T is an $[m, C]$-symmetric operator, then T^k is $[m, C]$-symmetric for any $k \in \mathbb{N}$.*

(c) *If T is an $[m, C]$-symmetric operator and invertible, then T^{-1} is $[m, C]$-symmetric.*

(d) *If T is a $[2, C]$-symmetric operator, then $\ker(T) \subset \ker(T^2) \bigcap C(\ker(T^2))$.*

Proof. (a) Since T is $[m, C]$-symmetric, it follows that $\alpha_m(T; C) = 0$. Therefore,

$$0 = C(\alpha_m(T; C))^* C = \begin{cases} \alpha_m(T^*; C) & (m \text{ is even}) \\ -\alpha_m(T^*; C) & (m \text{ is odd}). \end{cases}$$

Hence T^* is $[m, C]$-symmetric. The converse implication holds in a similar way.
(b) Note that

$$(a^k - b^k)^m = \big((a - b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1})\big)^m = \sum_{j=0}^{m(k-1)} \lambda_j a^{m(k-1)-j} (a - b)^m b^j,$$

where λ_j are some coefficients $(j = 0, ..., m(k-1))$. This implies that

$$\alpha_m(T^k; C) = \sum_{j=0}^{m(k-1)} \lambda_j C T^{m(k-1)-j} C \cdot \alpha_m(T; C) \cdot T^j = 0.$$

Hence T^k is $[m, C]$-symmetric.
(c) Since T is $[m, C]$-symmetric, it follows that $\alpha_m(T; C) = 0$ and therefore

$$0 = CT^{-m}C \cdot \alpha_m(T; C) \cdot T^{-m} = \begin{cases} \alpha_m(T^{-1}; C) & (m \text{ is even}) \\ -\alpha_m(T^{-1}; C) & (m \text{ is odd}). \end{cases}$$

Hence T^{-1} is $[m, C]$-symmetric.
(d) It is clear $\ker(T) \subset \ker(T^2)$. If T is $[2, C]$-symmetric and $x \in \ker(T)$, then

$$CT^2Cx = 2CTCTx - T^2x = 0$$

and hence $T^2Cx = 0$. Thus $Cx \in \ker(T^2)$ and so $x \in C(\ker(T^2))$. Hence we get $\ker(T) \subset \ker(T^2) \bigcap C(\ker(T^2))$. □

Lemma 4.2. *For $T \in B(\mathcal{H})$, a conjugation C, and two complex numbers λ, μ, it holds*

$$\alpha_m(T; C) = \sum_{n_1+n_2+n_3=m} (-1)^{n_2} \binom{m}{n_1, n_2, n_3} (CTC - \lambda I)^{n_1} (T - \mu I)^{n_2} (\lambda - \mu)^{n_3}.$$

In particular, for $\lambda \in \mathbb{C}$ we have

$$\alpha_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CTC - \lambda I)^{m-j} (T - \lambda I)^j. \quad (4.1)$$

Proof. Using the multinomial formula, it holds

$$\begin{aligned} \alpha_m(T; C) &= (y - x)^m \big|_{y=CTC,\, x=T} \\ &= \big([y - \lambda] - [x - \mu] + [\lambda - \mu] \big)^m \big|_{y=CTC,\, x=T} \\ &= \sum_{n_1+n_2+n_3=m} (-1)^{n_2} \binom{m}{n_1, n_2, n_3} (y - \lambda)^{n_1} (x - \mu)^{n_2} (\lambda - \mu)^{n_3} \big|_{y=CTC,\, x=T} \\ &= \sum_{n_1+n_2+n_3=m} (-1)^{n_2} \binom{m}{n_1, n_2, n_3} (CTC - \lambda I)^{n_1} (T - \mu I)^{n_2} (\lambda - \mu)^{n_3}. \end{aligned}$$

Equation (4.1) follows from $\lambda = \mu$ in the first formula. □

By Lemma 2.7 of [3], for $T \in B(\mathcal{H})$ and two complex numbers λ, μ, it holds

$$\beta_m(T) = \sum_{n_1+n_2+n_3=m} (-1)^{n_2} \binom{m}{n_1, n_2, n_3} (T^* - \overline{\mu} I)^{n_1} T^{n_1} \overline{\mu}^{n_2} (T - \lambda I)^{n_2} (\lambda \overline{\mu} - 1)^{n_3}.$$

We investigate properties of spectra of $[m, C]$-symmetric operators. In [11], S. Jung, E. Ko and J. E. Lee proved the following result.

Proposition 4.3.([11, Lemma 3.21]) *If C is a conjugation on $\mathcal{H}$ and $T \in B(\mathcal{H})$, then $\sigma(CTC) = \sigma(T)^*, \sigma_p(CTC) = \sigma_p(T)^*, \sigma_a(CTC) = \sigma_a(T)^*$ and $\sigma_s(CTC) = \sigma_s(T)^*$.*

Theorem 4.4. *Let $T \in B(\mathcal{H})$ be an $[m, C]$-symmetric operator where C is a conjugation on $\mathcal{H}$. Then $\sigma_p(T) = \sigma_p(T)^*, \sigma_a(T) = \sigma_a(T)^*, \sigma_s(T) = \sigma_s(T)^*$ and $\sigma(T) = \sigma(T)^*$.*

Proof. Let $z \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that $(T - z)x_n \to 0$ $(n \to \infty)$. By equation (4.1) it holds

$$\alpha_m(T; C) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CTC - zI)^{m-j}(T - zI)^j.$$

Hence we have $0 = \lim_{n\to\infty} \alpha_m(T; C)x_n = \lim_{n\to\infty} (CTC - z)^m x_n$. Therefore, it is easy to see $z \in \sigma_a(CTC)$. Hence $\sigma_a(T) \subset \sigma_a(CTC)$. Since $\sigma_a(CTC) = \sigma_a(T)^*$ by Proposition 4.3, this means $\sigma_a(T) \subset \sigma_a(T)^*$. Hence $\sigma_a(T)^* \subset \sigma_a(T)^{**} = \sigma_a(T)$ and so $\sigma_a(T) = \sigma_a(T)^*$.

Since T^* is also an $[m, C]$-symmetric operator by Theorem 4.1, we have $\sigma_a(T^*) = \sigma_a(T^*)^*$. Hence $\sigma_s(T) = \sigma_s(T)^*$ and $\sigma(T) = \sigma_a(T) \cup \sigma_s(T) = \sigma_a(T)^* \cup \sigma_s(T)^* = \sigma(T)^*$. From the above proof it is clear that $\sigma_p(T) = \sigma_p(T)^*$. □

For $T, S \in B(\mathcal{H})$, a pair (T, S) of operators is said to be a *C-doubly commuting pair* if $TS = ST$ and $CSC \cdot T = T \cdot CSC$ for a conjugation C.

Lemma 4.5. *Let (T, S) be a C-doubly commuting pair where C is a conjugation on $\mathcal{H}$. Then it holds*

$$\alpha_m(T + S; C) = \sum_{j=0}^{m} \binom{m}{j} \alpha_j(T; C) \cdot \alpha_{m-j}(S; C). \tag{4.2}$$

Proof. From the assumption, it holds $T \cdot CS^jC = CS^jC \cdot T$ and $S \cdot CT^jC = CT^jC \cdot S$ for every $j \in \mathbb{N}$. It is clear that equation (4.2) holds for $m = 1$. Assume that equation (4.2) holds for m. Then by (3.1) we have

$$\begin{aligned}
&\alpha_{m+1}(T + S; C) \\
= \;& C(T + S)C \cdot \alpha_m(T + S; C) - \alpha_m(T + S; C) \cdot (T + S) \\
= \;& \sum_{j=0}^{m} \binom{m}{j} (CTC + CSC) \cdot \alpha_j(T; C) \cdot \alpha_{m-j}(S; C) \\
& - \sum_{j=0}^{m} \binom{m}{j} \alpha_j(T; C) \cdot \alpha_{m-j}(S; C) \cdot (T + S)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{m} \binom{m}{j} \Big(CTC \cdot \alpha_j(T;C) - \alpha_j(T;C) \cdot T \Big) \alpha_{m-j}(S;C) \\
&\quad + \sum_{j=0}^{m} \binom{m}{j} \alpha_j(T;C) \Big(CSC \cdot \alpha_{m-j}(S;C) - \alpha_{m-j}(S;C) \cdot S \Big) \\
&= \sum_{j=0}^{m} \binom{m}{j} \alpha_{j+1}(T;C) \cdot \alpha_{m-j}(S;C) + \sum_{j=0}^{m} \binom{m}{j} \alpha_j(T;C) \cdot \alpha_{m+1-j}(S;C) \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} \alpha_j(T;C) \cdot \alpha_{m+1-j}(S;C).
\end{aligned}
$$

Hence equation (4.2) holds for any $m \in \mathbb{N}$. □

Therefore we have the following theorem.

Theorem 4.6. *Let $T \in B(\mathcal{H})$ be an $[m, C]$-symmetric operator and let $S \in B(\mathcal{H})$ be an $[n, C]$-symmetric operator where C is a conjugation on $\mathcal{H}$. If (T, S) is a C-doubly commuting pair, then $T + S$ is an $[m + n - 1, C]$-symmetric operator.*

Proof. By Lemma 4.5, it holds

$$\alpha_{m+n-1}(T+S;C) = \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} \alpha_j(T;C) \cdot \alpha_{m+n-1-j}(S;C).$$

(i) If $0 \leq j \leq m-1$, then $m+n-1-j \geq m+n-1-(m-1) = n$. Therefore we have
$\alpha_{m+n-1-j}(S;C) = 0$.
(ii) If $j \geq m$, then $\alpha_j(T;C) = 0$.
Hence we get $\alpha_{m+n-1}(T+S;C) = 0$ and so $T+S$ is $[m+n-1, C]$-symmetric. □

Theorem 4.7. *Let C be a conjugation on $\mathcal{H}$. If Q is a nilpotent operator of order n, then Q is a $[2n-1, C]$-symmetric operator.*

Proof. It holds

$$\alpha_{2n-1}(Q;C) = \sum_{j=0}^{2n-1} (-1)^j \binom{2n-1}{j} CQ^{2n-1-j}C \cdot Q^j.$$

(i) If $0 \leq j \leq n-1$, then $2n-1-j \geq 2n-1-(n-1) = n$. Hence $Q^{2n-1-j} = 0$.
(ii) If $j \geq n$, then $Q^j = 0$.

Therefore $\alpha_{2n-1}(Q;C) = 0$ and hence Q is $[2n-1, C]$-symmetric. □

Corollary 4.8. *Let $T \in B(\mathcal{H})$ be an $[m, C]$-symmetric operator and let $Q \in B(\mathcal{H})$ be a nilpotent operator of order n where C is a conjugation on $\mathcal{H}$. If (T, Q) is a C-doubly commuting pair, then $T + Q$ is an $[m + 2n - 2, C]$-symmetric operator.*

Proof. The proof follows from Theorems 4.6 and 4.7. □

Example 4.9. Let C_n be the conjugation on C^n defined by

$$C_n(z_1, z_2, \cdots, z_n) := (\overline{z_1}, \overline{z_2}, \cdots, \overline{z_n}).$$

Assume that R_n is an $n \times n$ matrix as follows;

$$R_n = aI_n + Q_n = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} + \begin{pmatrix} 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & b \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for $a, b \in C$. Since Q_n is nilpotent of order n, it follows that Corollary 4.8 that R_n is a $[2n-1, C_n]$-symmetric operator.

Lemma 4.10. *If (T, S) is a C-doubly commuting pair where C is a conjugation on $\mathcal{H}$, then it holds*

$$\alpha_m(TS; C) = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_j(T; C) \cdot T^{m-j} \cdot CS^jC \cdot \alpha_{m-j}(S; C). \tag{4.3}$$

Proof. It is easy to see that equation (4.3) holds for $m = 1$. Assume that equation (4.3) holds for m. Then by (3.1) we obtain

$$\begin{aligned}
&\alpha_{m+1}(TS; C) \\
= \;& (CTSC) \cdot \alpha_m(TS; C) - \alpha_m(TS; C) \cdot TS \\
= \;& CTC \cdot CSC \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_j(T; C) \cdot T^{m-j} \cdot CS^jC \cdot \alpha_{m-j}(S; C) \\
& - \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_j(T; C) \cdot T^{m+1-j} \cdot CS^jC \cdot \alpha_{m-j}(S; C) \cdot S \\
= \;& \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \Big(CTC \cdot \alpha_j(T; C) - \alpha_j(T; C) \cdot T \Big) T^{m-j} \cdot CS^{j+1}C \cdot \alpha_{m-j}(S; C) \\
& + \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_j(T; C) \cdot T^{m+1-j} \cdot CS^jC \cdot \Big(CSC \cdot \alpha_{m-j}(S; C) - \alpha_{m-j}(S; C) \cdot S \Big) \\
= \;& \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_{j+1}(T; C) \cdot T^{m-j} \cdot CS^{j+1}C \cdot \alpha_{m-j}(S; C) \\
& + \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} \alpha_j(T; C) \cdot T^{m+1-j} \cdot CS^jC \cdot \alpha_{m+1-j}(S; C) \\
= \;& \sum_{j=0}^{m+1} \begin{pmatrix} m+1 \\ j \end{pmatrix} \alpha_j(T; C) \cdot T^{m+1-j} \cdot CS^jC \cdot \alpha_{m+1-j}(S; C).
\end{aligned}$$

Hence equation (4.3) holds for any $m \in \mathbb{N}$. □

Theorem 4.11. *Let T be an $[m, C]$-symmetric operator and let S be an $[n, C]$-symmetric operator where C is a conjugation on $\mathcal{H}$. If (T, S) is a C-doubly commuting pair, then TS is an $[m+n-1, C]$-symmetric operator.*

Proof. Since (T, S) is a C-doubly commuting pair, it follows from equation (4.3) that

$$\alpha_{m+n-1}(TS; C) = \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} \alpha_j(T; C) \cdot T^{m+n-1-j} \cdot CS^j C \cdot \alpha_{m+n-1-j}(S; C).$$

(i) If $0 \leq j \leq m-1$, then $m+n-1-j \geq m+n-1-(m-1) = n$. Therefore we get $\alpha_{m+n-1-j}(S; C) = 0$.
(ii) If $m \leq j$, then $\alpha_j(T; C) = 0$.

Therefore $\alpha_{m+n-1}(TS; C) = 0$. Hence TS is $[m+n-1, C]$-symmetric. □

Corollary 4.12. *Let C be a conjugtion on $\mathcal{H}$. If $T = T_1 \oplus I$ and $S = I \oplus S_1$ where T_1 and S_1 are $[m, C]$-symmetric, then TS is $[2m-1, C]$-symmetric.*

Proof. Since T_1 and S_1 are $[m, C]$-symmetric, it follows that $T = T_1 \oplus I$ and $S = I \oplus S_1$ are $[m, C]$-symmetric. In addition, we know that (T, S) is a C-doubly commuting pair. Therefore, TS is $[2m-1, C]$-symmetric by Theorem 4.11. □

In [6], B. Duggal proved the following proposition.

Proposition 4.13. *Let T and S be an m-isometric operator and an n-isometric operator, respectively. Then $T \otimes S$ is an $(m+n-1)$-isometric operator.*

Similarly, we show the following result.

Theorem 4.14. *Let T be an $[m, C]$-symmetric operator and let S be an $[n, D]$-symmetric operator where C and D are conjugations on $\mathcal{H}$. Then $T \otimes S$ is an $[m+n-1, C \otimes D]$-symmetric operator on $\mathcal{H} \otimes \mathcal{H}$.*

Proof. Since C and D are conjugations on $\mathcal{H}$, it follows from [4] that $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$. If T is $[m, C]$-symmetric and S is $[n, D]$-symmetric, it is easy to see that $T \otimes I$ is $[m, C \otimes D]$-symmetric and $I \otimes S$ is $[n, C \otimes D]$-symmetric on $\mathcal{H} \otimes \mathcal{H}$, respectively. Also it is clear that $(T \otimes I, I \otimes S)$ is a $C \otimes D$-doubly commuting pair. Since $T \otimes S = (T \otimes I)(I \otimes S)$, it follows from Theorem 4.11 that $(T \otimes I)(I \otimes S) = T \otimes S$ is $[m+n-1, C \otimes D]$-symmetric. □

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