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A LOCAL MINIMIZER IN SMECTIC LIQUID CRYSTALS

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Abstract. In this paper, we consider a simplified energy functional for ferroelectric liquid crystals. With a special ansatz, we study existence of local minimizer with a certain symmetry.

1. Introduction

Let Ω be bounded domain in $\mathbf{R}^2$ which is occupied by a liquid crystal. Smectic Liquid crystal is described by the direction field $\mathbf{n}$ and a complex vector field ψ. For more details about the governing energy, we refer the reader to [2, 3] and references are therein.

Throughout this paper, we assume that $\psi = e^{iw}$ and $\Omega = (-\pi, \pi) \times (-\pi h, \pi h)$ for small $h > 0$. We further suppose that $\mathbf{n}$ is a fixed unit vector and w is $2h-$ periodic. With the above conditions, we consider the simplest energy functional

$$\mathcal{E}(w) = \int_{\Omega^+} \left\{ w_x^2 + (w_z + 1)^2 \right\} \, d\mathbf{x} \tag{1.1}$$

in a class of functions defined below. Here $\Omega^+ = (-\pi, \pi) \times (0, \pi h)$.

Let

$$\mathcal{M} = \{ w : \Omega^+ \to \mathbf{R} : \Delta w = \delta_{z_1} - \delta_{z_2} \text{ in } \Omega^+, \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega^+, z_1, z_2 \in \Omega \}.$$

From now on, we study minimization problem of the energy functional

$$\mathcal{E}(w) = \int_{\Omega^+} \left\{ w_x^2 + (w_z + 1)^2 \right\} \, d\mathbf{x} = \int_{\Omega^+} \left\{ |\nabla w|^2 + 2w_z \right\} \, d\mathbf{x} + 2\pi^2 h.$$

in $\mathcal{M}$.

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Since each function $w \in \mathcal{M}$ depends on two points $z_1, z_2 \in \Omega$, the minimization problem is equivalent to find a pair of z_1 and z_2 in Ω which minimizes $\mathcal{E}(w)$. For a given value h, let V be such that

$$\Delta V = \delta_{(0,0)} - \frac{1}{4\pi^2 h} \text{ in } \Omega,$$
$$\frac{\partial V}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

It can be shown that for $w \in \mathcal{M}$ the energy $\mathcal{E}(w)$ can be written as

$$-V(2z_2) + 2V(z_2 - z_1) - V(2z_1) + 2V(z_1 + z_2) + 2(z_1 - z_2) + 2\pi^2 h.$$

For simplicity we assume that $0 < z_1 < \frac{\pi h}{2}, \;\; z_1 < z_2 < \pi h$. Let

$$\Gamma(z_1, z_2) = -V(2z_2) + 2V(z_2 - z_1) - V(2z_1) + 2V(z_1 + z_2) + 2(z_1 - z_2).$$

Then we have the following Lemmas.

LEMMA 1. *For $h > \frac{2}{\pi^2}$, $\Gamma(d, \pi h - d)$ achieves its local minimum.*

Proof. We first write V in terms of singular and regular parts as follows:

$$V = \frac{1}{2\pi} \log|\mathbf{x}| - U, \;\; \mathbf{x} = (x, z), \tag{1.2}$$

$$\Delta U = \frac{1}{4\pi^2 h} \text{ in } \Omega, \tag{1.3}$$

$$\frac{\partial U}{\partial \nu} = \frac{\partial}{\partial \nu}\left(\frac{1}{2\pi}\log|\mathbf{x}|\right) \text{ on } \partial\Omega. \tag{1.4}$$

We get

$$\begin{aligned} \Gamma(d, \pi h - d) =& \frac{1}{\pi}\left(\log(\pi h - 2d) - \log(2d)\right) + 4d \\ & + 2\left[U(0, 2d) - U(0, \pi h - 2d)\right]. \end{aligned} \tag{1.5}$$

From the standard elliptic PDEs [4], one can show that $U(0, 2d) - U(0, \pi h - 2d)$ is a non-decreasing function. It suffices to show that $\Phi(d) := \Gamma(d, \pi h - d) - 2[U(0, 2d) - U(0, \pi h - 2d)]$ has a local minimum. Note that

$$\Phi'(d) = \frac{1}{\pi}\left(\frac{-2}{\pi h - 2d} - \frac{1}{d}\right) + 4 = 0, \tag{1.6}$$

It is clear that (1.6) has roots $d_0 = \frac{2\pi h \pm \sqrt{4\pi^2 h^2 - 8h}}{8}$ if $h > \frac{2}{\pi^2}$. Since $\lim_{d \uparrow \frac{\pi h}{2}} \Phi(d) = -\infty$ and $\lim_{d \downarrow 0} \Phi(d) = \infty$, $\Gamma(d, \pi h - d)$ has a local minimum. □

LEMMA 2. *Γ is symmetric with respect to the line $l : z_1 + z_2 = \pi h$ and the minimum point obtained in lemma 1 is also a critical point of Γ. Moreover,*

(1) *if z_1 is fixed, then*

$$\lim_{z_2 \downarrow z_1} \Gamma(z_1, z_2) = -\infty, \quad \textit{and} \quad \lim_{z_2 \uparrow \pi h} \Gamma(z_1, z_2) = \infty.$$

(2) *if z_2 is fixed, then*

$$\lim_{z_1 \uparrow z_2} \Gamma(z_1, z_2) = -\infty, \quad \textit{and} \quad \lim_{z_1 \downarrow 0} \Gamma(z_1, z_2) = \infty.$$

Proof. Let (z, w) and $(\tilde{z}, \tilde{w})$ be symmetric with respect to l. Then we get

$$\tilde{z} - \tilde{w} = z - w, \quad z + w = 2\pi h - (\tilde{z} + \tilde{w}). \tag{1.7}$$

Since V is symmetric with respect to the origin and $2\pi h$-periodic, we obtain

$$\begin{aligned}
\Gamma(\tilde{z}, \tilde{w}) &= - V(2\tilde{w}) + 2V(\tilde{w} - \tilde{z}) - V(2\tilde{z}) + 2V(\tilde{z} + \tilde{w}) + 2(\tilde{z} - \tilde{w}) \\
&= - V(2\pi h - 2z) + 2V(w - z) - V(2\pi h - 2w) \\
&\quad + 2V(2\pi h - z - w) + 2(z - w) \\
&= - V(2z) + 2V(w - z) - V(2w) + 2V(z + w) + 2(z - w).
\end{aligned}$$

It also follows that $(z_1, z_2) = (d, \pi h - d)$ is a critical point if d is a local minimum point of $\Gamma(d, \pi h - d)$. □

From Lemma 1 and Lemma 2, we conclude the following theorem

THEOREM 3. *If $h > \frac{2}{\pi^2}$ and $\Gamma(d, \pi h - d)$ has a local minimum at $d = d_0$, then $(d_0, \pi h - d_0)$ is also local minimum of Γ in the domain D, where*

$$D = \left\{ (z_1, z_2) : 0 < z_1 < \frac{\pi h}{2}, z_1 < z_2 < \pi h \right\}.$$

Proof. Using $u = z_1 + z_2,\ v = z_2 - z_1$, we express Γ in terms of u and v as

$$\Gamma(z_1, z_2) = 2V(u) + 2V(v) - V(u - v) - V(u + v) - 2v := W(u, v).$$

Since $\Gamma(d, \pi h - d)$ has a local minimum at d_0, then we get

$$\frac{\partial^2}{\partial u^2} W(d_0, \pi h - 2d_0) = -V''(2d_0) + V''(\pi h - 2d_0) > 0.$$

We compute

$$\frac{\partial^2}{\partial v^2}W(d_0, \pi h - 2d_0) = -V''(2d_0) + V''(\pi h).$$

By symmetry of V, we obtain

$$\frac{\partial^2}{\partial u \partial v}W(d_0, \pi h - 2d_0) = -V''(2\pi h - 2d_0) + V''(2d_0) = 0.$$

Now

$$\begin{aligned} V(\mathbf{x}) &= \frac{1}{2\pi}\log|\mathbf{x}| - U(\mathbf{x}), \\ \Delta U &= \frac{1}{4\pi^2 h}, \\ \frac{\partial U}{\partial \nu} &= \begin{cases} \frac{1}{2(\pi^2+z^2)} & \text{if } x = \pm\pi, \\ \frac{h}{2(\pi^2h^2+x^2)} & \text{if } z = \pm\pi h \end{cases} \end{aligned}$$

Let (z, w) be a point on the line passing through $(d_0, \pi h - d_0)$ perpendicular to the line $z_1 + z_2 = \pi h$. Then $z - d_0 = w - (\pi h - d_0)$, and thus

$$\begin{aligned} &w = z + \pi h - 2d_0, \quad w - z = \pi h - 2d_0, \\ &z + w = 2z + \pi h - 2d_0, \quad 2w = 2z + 2(\pi h - 2d_0). \end{aligned}$$

Since $U(0, \cdot)$ is $2\pi h$-periodic and symmetric with respect to $z = \pi h$, we get

$$\begin{aligned} F(z) &:= U(0, 2z) + U(0, 2w) - 2U(z + w) \\ &= U(0, 2z) + U(0, 2z + 2(\pi h - 2d_0)) - 2U(0, 2z + \pi h - 2d_0) \\ &= U(0, 2z) + U(0, 2z - 4d_0)) - 2U(0, \pi h - (2z - 2d_0)). \end{aligned}$$

Note that $d_0 < \frac{\pi h}{4}$. For any z satisfying $d_0 \le z \le \pi h$,

$$F'(z) = 2U_z(0, 2z) + 2U_z(0, 2z - 4d_0) + 4U_z(0, \pi h - (2z - 2d_0)).$$

Then

$$\begin{aligned} \lim_{z \downarrow d_0} F'(z) &= 2U_z(0, 2d_0) + 2U_z(0, -2d_0) + 4U_z(0, \pi h) \\ &= 4U_z(0, \pi h) > 0. \end{aligned}$$

This implies that F is increasing on $[d_0, d_0 + \delta_1)$ for some $\delta_1 > 0$. Similarly, F is decreasing on $(d_0 - \delta_2, d_0]$ for some $\delta_2 > 0$ by symmetry.

Now,

$$\Gamma(z,w) = F(z) - \frac{1}{2\pi}\left(\log|2z-4d_0| + \log|2z| - 2\log|\pi h - (2z-2d_0)|\right) + 2V(\pi h - 2d_0) - 2(\pi h - 2d_0).$$

It suffices to show that $G(z) = \Gamma(z,w) - F(z)$ has a local minimum at $z = d_0$. We calculate

$$G''(z) = \frac{1}{(z-2d_0)^2} + \frac{1}{z^2} - \frac{8}{(2z-2d_0-\pi h)^2}.$$

and thus using $d_0 < \frac{\pi h}{4}$ we get

$$G''(d_0) = \frac{2}{d_0^2} - \frac{8}{\pi^2 h^2} > \frac{24}{\pi^2 h^2} > 0.$$

Hence

$$\frac{\partial^2}{\partial v^2} W(d_0, \pi h - 2d_0) > 0.$$

By second derivative test, we see that Γ indeed achieves its local minimum at $(d_0, \pi h - d_0)$. This completes the proof. □

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