JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 1, February 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.177

APPROXIMATION OF C_0-SEQUENTIALLY EQUICONTINUOUS SEMIGROUPS

Young Seob Lee*

Abstract. The purpose of this paper is to present approximation of C_0-sequentially equicontinuous semigroups on a sequentially complete locally convex space X.

1. Introduction

One of the topics in semigroup theory is the approximation of semigroups. The approximation of C_0-semigroup on a Banach space is a classical result which can be presented in many textbooks. The generation and approximation theorems of equicontinuous semigroups on sequentially complete locally convex space were obtained in [3], which is parallel to the cases of Banach spaces. The theory of C_0-semigroups on Banach spaces are well established and has many applications (see [2]). But some semigroups, for example, the semigroup of conditional expectation are not strongly continuous with respect to the norm. So theory of semigroups on Banach spaces was extended to locally convex spaces. Recently, Federico and Rosestolato [1] introduced the notion of C_0-sequentially equicontinuous semigroups, which is a generalization of the notion of C_0-equicontinuous semigroups and obtained the generation theorem. Instead of the equicontinuity of the semigroup, they dealt with the sequential equicontinuity of the semigroup. If locally convex spaces are metrizable, then two notions coincide. One of the advantages of the notion of C_0-sequentially equicontinuous semigroups is that proving sequential equicontinuity is easier than proving equicontinuity (see Remark 3.13 in [1]). In this paper, we present convergence of C_0-sequentially equicontinuous semigroups on a sequentially complete

Received September 21, 2016; Accepted February 06, 2018.
2010 Mathematics Subject Classification: Primary 47D03, 47D99.
Key words and phrases: sequentially complete locally convex space, C_0-sequentially equicontinuous semigroup.
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locally convex space X without any assumptions on generators. And we show the equivalent conditions about the convergence of generators and the sequentially continuous inverses of generators.

2. Approximation

Let X be a sequentially complete locally convex space and let P_X be the family of seminorms inducing the topology on X. Let $L_0(X)$ be the space of all sequentially continuous linear operators on X.

DEFINITION 2.1. A family $\{T(t) : t \geq 0\}$ in $L_0(X)$ is called a C_0-sequentially equicontinuous semigroup if

(i) $T(0) = I$ and $T(t)T(s) = T(t+s)$ for all $t,\ s \geq 0$.
(ii) $\lim_{t \to 0^+} T(t)x = x$ for every $x \in X$.
(iii) $\{T(t) : t \geq 0\}$ is sequentially equicontinuous, that is, for every sequence $\{x_n\}$ in X converging to x, we have

$$\lim_{n\to\infty} \sup_{t\geq 0} p(T(t)x_n - T(t)x) = 0 \quad \text{for all} \quad p \in P_X.$$

The generator of $\{T(t) : t \geq 0\}$ is defined by

$$Ax = \lim_{h\to 0^+} \frac{T(h)x - x}{h}$$

with domain $D(A) = \{x \in X : \lim_{h\to 0^+}(T(h)x - x)/h \ \text{ exists}\}$.

The following theorem gives a characterization of the C_0-sequentially equicontinuous semigroup and its generator (see [1]).

THEOREM 2.2. *Let A be the generator of a C_0-sequentially equicontinuous semigroup $\{T(t) : t \geq 0\}$ on X. Then*

(i) $\cap_{n=1}^{\infty} D(A^n)$ *is sequentially dense in X.*
(ii) *For $x \in D(A)$ $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax \text{ for all } \ t \geq 0.$$

(iii) *For $\lambda > 0$, $(\lambda I - A)$ is one to one and onto, and $R(\lambda, A) = (\lambda I - A)^{-1}$ is sequentially continuous.*

By standard algebraic computation we have the resolvent equation

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

Thus $R(\lambda, A)$ is infinitely differentiable with respect to $\lambda > 0$ and $\frac{d^n}{d\lambda^n}R(\lambda, A)x = (-1)^n n! R(\lambda, A)^{n+1}x$ for $x \in X$. By the sequential completeness of X, it is known in [1] that

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t}T(t)xdt \quad \text{for all} \quad x \in X.$$

Hence we have

$$(\lambda R(\lambda, A))^n x = \frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-\lambda t}t^{n-1}T(t)xdt$$

and $p((\lambda R(\lambda, A))^n x) \leq \sup_{t\geq 0} p(T(t)x)$ for $n \in N$, $x \in X$ and $p \in P_X$.

Before presenting the convergence of the sequence of C_0-sequentially equicontinuous semigroups we first introduce the notion of uniformly C_0-sequentially equicontinuous semigroups.

Definition 2.3. A sequence $\{T_n(t) : t \geq 0\}$ of C_0-sequentially equicontinuous semigroups on X is called uniformly C_0-sequentially equicontinuous semigroups if for every sequence $\{x_k\}$ in X converging to x, we have

$$\lim_{k\to\infty} \sup_{t\geq 0, n\in N} p(T_n(t)x_k - T_n(t)x) = 0 \quad \text{for all} \quad p \in P_X.$$

Theorem 2.4. *Let $\{T_n(t) : t \geq 0\}$ be uniformly C_0-sequentially equicontinuous semigroups on X. Then for $x \in X$ $\{T_n(t)x : t \geq 0,\ n \in N\}$ is bounded.*

Proof. Suppose that $\{T_n(t)x\}$ is not bounded. Then there exist $p \in P_X$ and $t_{n_k} > 0$ such that $p(T_{n_k}(t_{n_k})x) \geq k$ for $k \in N$.

Since $\lim_{k\to\infty} x/k = 0$ and $\{T_n(t) : t \geq 0\}$ is uniformly C_0-sequentially equicontinuous semigroups, $\lim_{k\to\infty} p(T_{n_k}(t_{n_k})x/k) = 0$. This is a contradiction. □

Now, we show the convergence of uniformly C_0-sequentially equicontinuous semigroups by the convergence of sequentially continuous inverses of their generators.

Theorem 2.5. *Let $\{T_n(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ be uniformly C_0-sequentially equicontinuous semigroups with generators A_n and A, respectively. Suppose that $\lim_{n\to\infty} R(\lambda_0, A_n)x = R(\lambda_0, A)x$ for all $x \in X$ and some $\lambda_0 > 0$. Then $\lim_{n\to\infty} T_n(t)x = T(t)x$ for all $x \in X$ and the convergence is uniform on bounded $t-$intervals.*

Proof. Let $x \in X$, $0 \leq t \leq T$ and $p \in P_X$. Then

$$\begin{aligned} & p(T_n(t)R(\lambda_0, A)x - T(t)R(\lambda_0, A)x) \\ & \quad \leq \; p(T_n(t)(R(\lambda_0, A)x - R(\lambda_0, A_n)x)) \\ & \qquad + p(R(\lambda_0, A_n)T_n(t)x - R(\lambda_0, A_n)T(t)x) \\ & \qquad + p(R(\lambda_0, A_n)T(t)x - R(\lambda_0, A)T(t)x). \end{aligned}$$

By the uniform C_0-sequential continuity of $\{T_n(t) : t \geq 0\}$, the first term converges to zero uniformly on $[0,\ T]$. Since $T(t)x$ is continuous in $t \geq 0$, $\{T(t)x : 0 \leq t \leq T\}$ is compact in X. Thus the third term converges to zero uniformly on $[0,\ T]$. Next, we prove the second term converges to zero uniformly.

By the differentiability of $T(t)x$ for $x \in D(A)$, we have

$$\begin{aligned} & \frac{d}{ds}(T_n(t-s)R(\lambda_0, A_n)T(s)R(\lambda_0, A)x) \\ & = \; T_n(t-s)(-A_n)R(\lambda_0, A_n)T(s)R(\lambda_0, A)x \\ & \quad + T_n(t-s)R(\lambda_0, A_n)T(s)AR(\lambda_0, A)x \\ & = \; T_n(t-s)(R(\lambda_0, A) - R(\lambda_0, A_n))T(s)x. \end{aligned}$$

By integrating both sides from 0 to t, we have

$$\begin{aligned} & R(\lambda_0, A_n)T(t)R(\lambda_0, A)x - R(\lambda_0, A_n)T_n(t)R(\lambda_0, A)x \\ & = \; \int_0^t T_n(t-s)\left(R(\lambda_0, A) - R(\lambda_0, A_n)\right)T(s)x\,ds, \end{aligned}$$

which converges to zero uniformly on $[0,\ T]$, since $\{T(t)x : 0 \leq t \leq T\}$ is compact and $\{T_n(t) : t \geq 0\}$ is uniformly C_0-sequentially equicontinuous. Hence the second term converges to zero uniformly on $D(A^2)$. Since $D(A^2)$ is sequentially dense in X, the result follows. So we have the convergence of semigroups on $D(A)$. By the uniform C_0-sequential equicontinity and the sequential denseness of $D(A)$, we have $\lim_{n\to\infty} T_n(t)x = T(t)x$ for all $x \in X$, uniformly on $[0,\ T]$. □

The following theorem presents the equivalent condition about the convergence of sequentially continuous inverses of generators and it may be useful to apply Theorem 2.5.

THEOREM 2.6. *Let $\{T_n(t) : t \geq 0\}$ and $\{T(t) : t \geq 0\}$ be uniformly C_0-sequentially equicontinuous semigroup with generator A_n and A, respectively. The following statements are equivalent.*

(i) *For $x \in D(A)$ there exist $x_n \in D(A_n)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} A_n x_n = Ax$.*
(ii) *$\lim_{n\to\infty} R(\lambda_0, A_n)x = R(\lambda_0, A)x$ for all $x \in X$ and some $\lambda_0 > 0$.*

(iii) $\lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x$ *for all* $x \in X$ *and* $\lambda > 0$.

Proof. Let $y = (\lambda_0 I - A)x$ and $y_n = (\lambda_0 I - A)x_n$. Then $y_n \to y$. For $p \in P_X$,

$$\begin{aligned} & p\left(R(\lambda_0, A_n)y - R(\lambda_0, A)y\right) \\ & \quad \leq \quad p(R(\lambda_0, A_n)(y - y_n)) + p(R(\lambda_0, A_n)y_n - R(\lambda_0, A)y) \\ & \quad \leq \quad \frac{1}{\lambda_0} \sup_{t\geq 0} p(T_n(t)(y - y_n)) + p(x_n - x). \end{aligned}$$

By the sequential equicontinuity of $\{T_n(t) : t \geq 0\}$, $\lim_{n\to\infty} R(\lambda_0, A_n)y = R(\lambda_0, A)y$. Since $\lambda_0 I - A$ is onto, (i) implies (ii).

Since $p\left((\lambda R(\lambda, A_n))^{m+1}x\right) \leq \sup_{t\geq 0} p(T_n(t)x)$,

$$R(\lambda, A_n)x = \sum_{m=0}^{\infty} (\lambda - \lambda_0)^m R(\lambda_0, A_n)^{m+1} x$$

exists for $|\lambda - \lambda_0|/\lambda_0 < 1$ and exists uniformly for $|\lambda - \lambda_0|/\lambda_0 \leq r < 1$.

For $\varepsilon > 0$ there exists m_0 such that

$$\begin{aligned} & p\left(R(\lambda, A_n)x - R(\lambda, A)x\right) \\ & \quad \leq \sum_{m=0}^{m_0} |\lambda - \lambda_0|^m p\left(R(\lambda_0, A_n)^{m+1}x - R(\lambda_0, A)^{m+1}x\right) \\ & \qquad + \varepsilon \left(\sup_{t\geq 0, n\in N} p(T_n(t)x) + \sup_{t\geq 0} p(T(t)x) \right). \end{aligned}$$

So we have $\lim_{n\to\infty} R(\lambda, A_n)x = R(\lambda, A)x$. So (ii) implies (iii).

Let $y \in X$. Take $x = R(\lambda, A)y \in D(A)$ and $x_n = R(\lambda, A_n)y \in D(A_n)$. Then $x_n \to x$ and $A_n x_n = A_n R(\lambda, A_n)y = \lambda R(\lambda, A_n)y - y \to \lambda R(\lambda, A)y - y = AR(\lambda, A)y = Ax$. Hence (iii) implies (i). □

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Department of Mathematics
Seoul Women's University
Seoul 01797, Republic of Korea
E-mail: younglee@swu.ac.kr