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A NOTE ON GENERALIZED SINGULAR GRONWALL INEQUALITIES

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Abstract. This paper deals an impulsive fractional integral inequality with singular kernel which can be used in getting the explicit estimate of solutions of impulsive fractional differential equations.

1. Introduction

Integral inequalities of various Gronwall types play important roles in the study of the qualitative properties of solutions of differential and integral equations. The classic Gronwall-Bellman inequality provided explicit bounds on solutions of a class of linear integral inequalities. On the basis of various motivations, this inequality has been extended and used in various contexts [1, 4, 7, 8, 12, 13, 15]. L. Wang *et al.* [14] developed some generalized singular Gronwall inequalities to study existence, uniqueness and data dependent results of solutions to impulsive fractional differential equations.

In this paper we obtain an impulsive integral inequality with singular kernel by using the Caputo fractional integral inequality of Gronwall type. This result improves some generalized singular Gronwall inequalities given in [14].

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2. Main results

In this section we present an integral inequality which can be used in getting the explicit estimate of solutions of impulsive fractional differential equations. This proof is based on the fractional integral inequalities.

We also obtain the integral inequality with singular kernel which obtained from the similar argument to the proof of Corollary 2.2.1 in [11].

Throughout this paper, let $0 < q < 1$, $t_0 \in \mathbb{R}_+ = [0, \infty)$, and $J = [t_0, T]$ be a subset of $\mathbb{R}$ (some $T \le +\infty$). Suppose that (t_k) is a finite sequence in $\mathbb{R}$ satisfying $0 \le t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. For a function $u : J \to \mathbb{R}$, $u(t_k^+) = \lim_{\varepsilon \to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon \to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$. Denote by $C(J, \mathbb{R})$ the set of all continuous functions from J into $\mathbb{R}$. Also, let $PC(J, \mathbb{R})$ be the set of all functions from J into $\mathbb{R}$ as follows:

$$\begin{aligned} PC(J,\mathbb{R}) = \{ & u : J \to \mathbb{R} | u \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \cdots, m, \text{ and} \\ & \text{there exist } u(t_k^-) \text{ and } u(t_k^+), k = 1, \cdots, m, \text{ with } u(t_k^-) = u(t_k) \}. \end{aligned}$$

For a detailed discussion of impulsive integral inequalities and some basic concepts concerning impulsive differential equations, we refer the reader to [2, 10].

Let us recall the Caputo fractional integral inequality of Gronwall type which can be found in [3, Theorem 3.6] and [15, Corollary 2].

LEMMA 2.1. [11] *Let $m \in C(J, \mathbb{R})$ and suppose that*

$$m(t) \le m_0 + \frac{L}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} m(s) ds, \ t \in J. \tag{2.1}$$

Then we have

$$m(t) \le m_0 E_q(L(t - t_0)^q), \ t \in J,$$

where m_0 and L are nonnegative constants, and E_q is the Mittag-Leffler function [9] *given by*

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}, z \in \mathbb{C}.$$

We can extend the result of Theorem 2.6 in [6] to case of infinite sequence $(t_k)_{k=1}^{\infty}$ with $T = \infty$. The proof of the following result is adapted from the proof of Theorem 2.6 in [6].

THEOREM 2.2. *Let $u \in PC(\mathbb{R}_+, \mathbb{R})$ satisfies the following inequality*

$$u(t) \le c + \lambda \int_{t_0}^{t} (t-s)^{q-1} u(s) ds + \sum_{t_0 < t_k < t} \beta_k u(t_k^-), \quad k \in \mathbb{N}, \tag{2.2}$$

where c, λ, and $\beta_k (k \in \mathbb{N})$ are nonnegative constants. Then

$$u(t) \le c E_q(\Gamma(q)\lambda(t-t_0)^q), \ t \in [t_0, t_1] \tag{2.3}$$

and

$$u(t) \le c \prod_{i=1}^{k} (1 + \beta_i E_q(\Gamma(q)\lambda(t_i - t_0)^q)) E_q(\Gamma(q)\lambda(t-t_0)^q), \tag{2.4}$$

where $t \in (t_k, t_{k+1}], k \in \mathbb{N}$.

Proof. Let $t \in [t_0, \infty)$. From Lemma 2.1, we have

$$u(t) \le c E_q(\Gamma(q)\lambda(t-t_0)^q), \ t \in [t_0, t_1].$$

Note that (2.4) holds for $k = 0$. Suppose that (2.4) holds for $k = j \in \mathbb{N}$.

For $t \in (t_{j+1}, t_{j+2}]$, we derive

$$\begin{aligned}
u(t) &\le [c + \sum_{i=1}^{j+1} \beta_i u(t_i^-)] E_q(\Gamma(q)\lambda(t-t_0)^q) \\
&= [c + \sum_{i=1}^{j} \beta_i u(t_i^-) + \beta_{j+1} u(t_{j+1}^-)] E_q(\Gamma(q)\lambda(t-t_0)^q) \\
&\le [c(\prod_{i=1}^{j} (1 + \beta_i E_q(\Gamma(q)\lambda(t_i - t_0)^q))) + \beta_{j+1} u(t_{j+1}^-)] E_q(\Gamma(q)\lambda(t-t_0)^q) \\
&\le [c(\prod_{i=1}^{j} (1 + \beta_i E_q(\Gamma(q)\lambda(t_i - t_0)^q))) + \beta_{j+1} c(\prod_{i=1}^{j} (1+ \\
&\quad \beta_i E_q(\Gamma(q)\lambda(t_i - t_0)^q))) E_q(\Gamma(q)\lambda(t_{j+1} - t_0)^q)] E_q(\Gamma(q)\lambda(t-t_0)^q) \\
&= c \prod_{i=1}^{j} (1 + \beta_i E_q(\Gamma(q)\lambda(t_i - t_0)^q)) \\
&\qquad (1 + \beta_{j+1} E_q(\Gamma(q)\lambda(t_{j+1} - t_0)^q)) E_q(\Gamma(q)\lambda(t-t_0)^q) \\
&= c \prod_{i=1}^{j+1} (1 + \beta_i E_q(\Gamma(q)\lambda(t_i - t_0)^q)) E_q(\Gamma(q)\lambda(t-t_0)^q), t \in (t_{j+1}, t_{j+2}].
\end{aligned}$$

Hence, it follows from induction that (2.4) holds for every $k \in \mathbb{N}$. This completes the proof. $\square$

We obtain the following result using similar argument as in the proof of Corollary 2.7 in [6].

Theorem 2.3. [6, Corollary 2.7] *Suppose that $u \in PC(\mathbb{R}_+, \mathbb{R})$ satisfies the following inequality*

$$|u(t)| \leq c_1(t) + c_2 \int_{t_0}^{t} (t-s)^{q-1} |u(s)| ds + \sum_{t_0 < t_k < t} \theta_k |u(t_k^-)|, \quad k \in \mathbb{N},$$

where $c_1(t)$ is positive continuous and nondecreasing on $\mathbb{R}_+$, and $c_2, \theta_k (k \in \mathbb{N})$ are nonnegative constants. Then

$$|u(t)| \leq c_1(t) \prod_{i=1}^{k} (1 + \theta_i E_q(c_2 \Gamma(q)(t_i - t_0)^q)) E_q(c_2 \Gamma(q)(t - t_0)^q),$$

wheret $\in (t_k, t_{k+1}]$, and $k \in \mathbb{N}$.

We obtain the following result in [14, Lemma 2.8] as a corollary of Theorem 2.3.

Corollary 2.4. *Let $c_1(t)$ be nonnegative continuous and nondecreasing on $\mathbb{R}_+$. Suppose that $u \in PC(\mathbb{R}_+, \mathbb{R})$ satisfies the following inequality*

$$|u(t)| \leq c_1(t) + c_2 \int_{0}^{t} (t-s)^{q-1} |u(s)| ds + \sum_{0 < t_k < t} \theta_k |u(t_k^-)|, \quad k \in \mathbb{N},$$

where c_2 and $\theta_k (k \in \mathbb{N})$ are nonnegative constants. Then

$$\begin{aligned} |u(t)| &\leq c_1(t) \prod_{i=1}^{k} (1 + \theta_i E_q(c_2 \Gamma(q) t^q)) E_q(c_2 \Gamma(q) t^q) \\ &\leq c_1(t)(1 + \theta E_q(c_2 \Gamma(q) t^q))^k E_q(c_2 \Gamma(q) t^q), t \in (t_k, t_{k+1}], \quad k \in \mathbb{N}, \end{aligned}$$

where $\theta = \sup\{\theta_k : k \in \mathbb{N}\}$ exists.

Proof. From Theorem 2.3 and the nondecreasing property of $E_q(t^q)$ [5, Lemma 2.4], we have

$$
\begin{aligned}
|u(t)| &\leq c_1(t)\prod_{i=1}^{k}(1+\theta_i E_q(c_2\Gamma(q)t_i^q))E_q(c_2\Gamma(q)t^q) \\
&\leq c_1(t)[\prod_{i=1}^{k}(1+\theta E_q(c_2\Gamma(q)t^q))]E_q(c_2\Gamma(q)t^q) \\
&= c_1(t)(1+\theta E_q(c_2\Gamma(q)t^q))^k E_q(c_2\Gamma(q)t^q), t\in(t_k,t_{k+1}],
\end{aligned}
$$

where $k \in \mathbb{N}$ and $\theta = \sup\{\theta_k : k \in \mathbb{N}\}$. This completes the proof. □

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