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ON AP-HENSTOCK-STIELTJES INTEGRAL FOR FUZZY-NUMBER-VALUED FUNCTIONS

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Abstract. In this paper we introduce the AP-Henstock-Stieltjes integral for fuzzy-number-valued functions which is an extension of the Henstock-Stieltjes integral and investigate some properties.

1. Introduction and Preliminaries

It is well-known that the Henstock integral for real valued function was first defined by Henstock [3, 4] in 1963. The Henstock integral is more powerful and simpler than the Lebesgue, Feynman integrals.

In 2012, Z. Gong and L. Wang introduced the concept of the Henstock-Stieltjes integrals of fuzzy-number-valued functions and obtained some properties [2].

In this paper we introduce the concept of the AP-Henstock-Stieltjes integral of fuzzy-number-valued functions and investigate some properties.

A Henstock partition of $[a, b]$ is a finite collection $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ such that $\{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $\xi_i \in [x_{i-1}, x_i]$ for each $1 \leq i \leq n$. A gauge on $[a, b]$ is a function $\delta : [a, b] \to (0, \infty)$. A Henstock partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is said to be δ -fine on $[a, b]$ if $[x_{i-i}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for each $1 \leq i \leq n$.

Let α be an increasing function on $[a, b]$. A function $f : [a, b] \to R$ is said to be Henstock-Stieltjes integrable to $L \in R$ with respect to α on $[a, b]$ if for every $\epsilon > 0$ there exists a positive function δ on $[a, b]$ such that $|\sum_{i=1}^{n} f(\xi_i)(\alpha(v_i) - \alpha(u_i)) - L| < \epsilon$ whenever $P = \{([u_i, v_i], \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a, b]$. We denote this fact as $(HS)\int_a^b f(x)d\alpha = L$ and $f \in HS[a, b]$. The function f is

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Henstock-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if f_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a,b]$, where χ_E denotes the characteristic function of E.

Fuzzy set $u : R \to [0,1]$ is called a fuzzy number if u is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in R | u(x) > 0\}}$ is compact. Here $\overline{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [4].

Let $u, v \in E^1, k \in R$, the addition and scalar multiplication are defined by

$$[u+v]_\lambda = [u]_\lambda + [v]_\lambda,\ [ku]_\lambda = k[u]_\lambda,$$

where $[u]_\lambda = \{x : u(x) \geq \lambda\} = [u_\lambda^-, u_\lambda^+]$ for any $\lambda \in [0,1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, \infty)$ as follows

$$D(u,v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u_\lambda^- - v_\lambda^-|, |u_\lambda^+ - v_\lambda^+|\},$$

where d is the Hausdorff metric. $D(u,v)$ is called the distance between u and v.

Lemma 1.1. [2] *If $u \in E^1$, then*

(1) $[u]_\lambda$ *is non-empty bounded closed interval for all* $\lambda \in [0,1]$.
(2) $[u]_{\lambda_1} \supset [u]_{\lambda_2}$ *for any* $0 \leq \lambda_1 \leq \lambda_2 \leq 1$.
(3) *for any* $\{\lambda_n\}$ *converging increasingly to* $\lambda \in (0,1]$,

$$\bigcap_{n=1}^{\infty} [u]_{\lambda_n} = [u]_\lambda.$$

Conversely, if for all $\lambda \in [0,1]$, *there exists* $A_\lambda \subset R$ *satisfying* $(1) \sim (3)$, *then there exists a unique* $u \in E^1$ *such that* $[u]_\lambda = A_\lambda$, $\lambda \in (0,1]$, *and* $[u]_0 = \overline{\cup_{\lambda \in (0,1]} [u]_\lambda} \subset A_0$.

Definition 1.2. [2] Let α be an increasing function on $[a,b]$. A fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on $[a,b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a positive function $\delta(x)$ such that

$$D\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ K\right) < \epsilon$$

whenever $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is a δ -fine Henstock partition of $[a,b]$. We denote this fact as $(FHS)\int_a^b F(x)d\alpha = K$ and $(F,\alpha) \in FHS[a,b]$.

The fuzzy number-valued function F is Henstock-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if F_{χ_E} is Henstock-Stieltjes integrable with respect to α on $[a, b]$, where χ_E denotes the characteristic function of E.

2. The fuzzy number-valued AP-Henstock-Stieltjes Integral

In this section, we will define the fuzzy number-valued AP-Henstock-Stieltjes integral, which is an extension of the fuzzy number-valued Henstock-Stieltjes integral [4] and investigate some of their properties.

Let E be a mesurable set and let x be a real number. The density of E at x is defined by

$$d_x E = \lim_{h \to 0+} \frac{\mu(E \cap (x-h, x+h))}{2h},$$

provided the limit exists. The point x is called a point of density of E if $d_x E = 1$. The E^d represents the set of all $x \in E$ such that x is a point of density of E.

An approximate neighborhood(or ad-nbd) of $x \in [a, b]$ is a measurable set $S_x \subset [a, b]$ containing x as a point of density. For every $x \in E \subset [a, b]$, choose an ad-nbd $S_x \subset [a, b]$ of x. then we say that $S = \{S_x : x \in E\}$ is a choice on E. A tagged interval $([u, v], x)$ is said to fine to the choice $S = \{S_x\}$ if $u, v \in S_x$. Let $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ is fine to a choice S for each i, then we say that P is S-fine. Let $E \subset [a, b]$. If P is S-fine and each $\xi_i \in E$, then P is called S-fine on E. If P is S-fine and $[a, b] = \cup_{i=1}^n [u_i, v_i]$, then we say that P is S-fine partition of $[a, b]$.

Definition 2.1. [6] A function $f : [a, b] \to R$ is AP-Henstock integrable if there exists a real number $A \in R$ such that for each $\epsilon > 0$ there is a choice S on $[a, b]$ such that

$$\left| \sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - A \right| < \epsilon$$

for each S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ of $[a, b]$. In this case, A is called AP-Henstock integral of f on $[a, b]$, and we write $A = (APH) \int_a^b f$.

Definition 2.2. Let α be an increasing function on $[a,b]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ if there exists a fuzzy number $K \in E^1$ such that for every $\epsilon > 0$ there exists a choice S on $[a,b]$ such that

$$D\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i)-\alpha(x_{i-1})),\, K\right) < \epsilon$$

whenever $P = \{([x_{i-1},x_i],\xi_i) : 1 \le i \le n\}$ is a S-fine Henstock partition of $[a,b]$. We write $(APFHS)\int_a^b F(x)d\alpha = K$ and $(F,\alpha) \in APFHS[a,b]$.

The fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if F_{χ_E} is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$, where χ_E denotes the characteristic function of E.

Remark 2.3. If $F \in APFHS[a,b]$, then the integral value is unique.

From the definition of the fuzzy number-valued AP-Henstock-Stieltjes integral and the fact that (E^1, D) is a complete metric space, we can easily obtain the following theorem.

Theorem 2.4. *Let α be an increasing function on $[a,b]$. A fuzzy number-valued function F is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a choice S on $[a,b]$ such that for any S-fine partitions $P = \{([u,v],\xi)\}$ and $P' = \{([u',v'],\xi')\}$, we have*

$$D\left(\sum_{P} F(\xi)(\alpha(v)-\alpha(u)),\, \sum_{P'} F(\xi')(\alpha(v')-\alpha(u'))\right) < \epsilon.$$

Theorem 2.5. *Let α be an increasing function on $[a,b]$ and let F be a fuzzy number-valued function on $[a,b]$. Then the following statements are equivalent:*

(1) *$(F,\alpha) \in APFHS[a,b]$ and $(APFHS)\int_a^b F(x)d\alpha = K$.*
(2) *for any $\lambda \in [0,1]$, F_λ^- and F_λ^+ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a,b]$ for and $\lambda \in [0,1]$ uniformly (Choice S is independent of $\lambda \in [0,1]$) and*

$$\Big[(APFHS)\int_a^b F(x)d\alpha\Big]_\lambda = \Big[(APHS)\int_a^b F_\lambda^-(x)d\alpha,\, (APHS)\int_a^b F_\lambda^+(x)d\alpha\Big].$$

Proof. (1) implies (2) : Since $(APFHS)\int_a^b F(x)d\alpha = K$, then for any $\epsilon > 0$, there exists a choice S on $[a,b]$ such that for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$, we have

$$D\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\, K\right) < \epsilon.$$

By definition of D, for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a,b]$

$$\sup_{\lambda\in[0,1]} \max\Big\{\Big|\big[\textstyle\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))\big]_\lambda^- - K_\lambda^-\Big|, \Big|\Big[\displaystyle\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1}))\Big]_\lambda^+ - K_\lambda^+\Big|\Big\} < \epsilon.$$

Hence, we have

$$\Big|\sum_{i=1}^{n} F_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^-\Big| < \epsilon,$$

$$\Big|\sum_{i=1}^{n} F_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^+\Big| < \epsilon.$$

Therefore, $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are AP-Henstock-Stieltjes with respect to α on $[a,b]$ for any $\lambda \in [0,1]$, and

$$(APHS)\int_a^b F_\lambda^-(x)d\alpha = K_\lambda^-, \quad (APHS)\int_a^b F_\lambda^+(x)d\alpha = K_\lambda^+.$$

(2) implies (1) : Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are AP-Henstock-Stieltjes integrable functions with respect to α on $[a,b]$ for any $\lambda \in [0,1]$, For $\epsilon > 0$, there exists a choice S on $[a,b]$ such that for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a,b]$ and for any $\lambda \in [0,1]$, we have

$$\Big|\sum_{i=1}^{n} F_\lambda^-(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^-\Big| < \epsilon,$$

$$\Big|\sum_{i=1}^{n} F_\lambda^+(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) - K_\lambda^+\Big| < \epsilon.$$

Then we can prove that $\{[K_\lambda^-, K_\lambda^+] : \lambda \in [0,1]\}$ satisfies the conditions of Lemma 1.1. By the Lemma 1.1, $\{[K_\lambda^-, K_\lambda^+] : \lambda \in [0,1]\}$ determines a fuzzy number K. By the definition of D, for any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \le i \le n\}$ on $[a, b]$, we obtain

$$D\left(\sum_{i=1}^{n} F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})),\ K\right) < 2\epsilon$$

Thus, $(F, \alpha) \in APFHS[a, b]$ and $(APFHS)\int_a^b F(x)d\alpha = K$. $\square$

From theorem 2.5 and the properties of AP-Henstock-Stieltjes integral, the followings theorems are obvious.

THEOREM 2.6. *Let $F, G \in APFHS[a, b]$ and $\beta, \gamma \in R$. Then*

(1) *$\beta F + \gamma G \in\in APFHS[a, b]$ and*

$$(APFHS)\int_a^b (\beta F + \gamma G)d\alpha$$
$$= \beta(APFHS)\int_a^b Fd\alpha + \gamma(APFHS)\int_a^b Gd\alpha.$$

(2) *If $F(x) \le G(x)$ in $[a, b]$, then*

$$(APFHS)\int_a^b Fd\alpha \le (APFHS)\int_a^b Gd\alpha.$$

THEOREM 2.7. *Let $F \in APFHS[a, c]$ and $F \in APFHS[c, b]$. Then $F \in APFHS[a, b]$ and $(APFHS)\int_a^b Fd\alpha = (APFHS)\int_a^c Fd\alpha + (APFHS)\int_c^b Fd\alpha$.*

THEOREM 2.8. *Let $F, G \in APFHS[a, b]$ and $D(F, G)$ is Lebesgue-Stieltjes integrable on $[a, b]$. Then*

$$D\left((APFHS)\int_a^b F\ d\alpha, (APFHS)\int_a^b G\ d\alpha\right) \le (L)\int_a^b D(F, G)\ d\alpha.$$

Proof. By definition of distance,

$$D\Big((APFHS)\int_a^b Fd\alpha,\ (APFHS)\int_a^b Gd\alpha\Big)$$
$$= \sup_{\lambda\in[0,1]} \max\Big(\Big|[((APHS)\int_a^b Fd\alpha)]_\lambda^- - [((APHS)\int_a^b Gd\alpha)]_\lambda^-\Big|,$$
$$\Big|[((APHS)\int_a^b Fd\alpha)]_\lambda^+ - [((APHS)\int_a^b Gd\alpha)]_\lambda^+\Big|\Big)$$
$$= \sup_{\lambda\in[0,1]} \max\Big(\Big|(APHS)\int_a^b ([F]_\lambda^- - [G]_\lambda^-)d\alpha\Big|,$$

$$\Big|(APHS)\int_a^b ([F]_\lambda^+ - [G]_\lambda^+)d\alpha\Big|\Big)$$

$$\leq \sup_{\lambda\in[0,1]} \max\Big((L)\int_a^b \Big|[F]_\lambda^- - [G]_\lambda^-\Big|d\alpha,\ (L)\int_a^b \Big|[F]_\lambda^+ - [G]_\lambda^+\Big|d\alpha\Big)$$

$$\leq (LS)\int_a^b D(F,G)d\alpha.$$

This completes the proof. □

THEOREM 2.9. *Let α be an increasing function on $[a,b]$ with $\alpha \in C^1[a,b]$ and let F be a bounded fuzzy number-valued on $[a,b]$. Then F is AP-Henstock-Stieltjes integrable with respect to α on $[a,b]$ if and only if $f\alpha'$ is AP-Henstock integrable on $[a,b]$. In this case, we have*

$$(APFHS)\int_a^b F(x)d\alpha = (APFH)\int_a^b F(x)\alpha'(x)dx,$$

where $(APFH)$integral denotes the fuzzy number-valued Henstock AP-integral.

Proof. Since F is bounded, $\sup_{x\in[a,b]} D(F(x),0)$ exists. Also since α' is uniformly continuous on $[a,b]$, for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$|\alpha'(x) - \alpha'(y)| < \frac{\epsilon}{3\sup_{x\in[a,b]} D(F(x),0)(b-a)}$$

for any $x, y \in [a,b]$ satisfying $|x-y| < \eta$. Define a choice $S_1 = \{S_x^1\}$ on $[a,b]$ with Lebesgue measure $|S_x^1| < \eta$ for all $x \in [a,b]$.

Let $P = \{([x_{i-1},x_i],\xi_i) : 1 \leq i \leq n\}$ be a S_1-fine partition on $[a,b]$, then by Lagrange mean value theorem, there exists $t_i \in (x_{i-1},x_i)$ such that $\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})$ $(1 \leq i \leq n)$. Since $|t_i - x_i| \leq |S_{\xi_i}| < \eta$ for $1 \leq i \leq n$, we have

$$|\alpha'(t_i) - \alpha'(x_i)| < \frac{\epsilon}{3\sup_{x\in[a,b]} D(F(x),0)(b-a)}$$

for $1 \leq i \leq n$. Hence, we have

$$D\Big(\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})), \sum_{i=1}^n F(\xi_i)\alpha'(\xi_i)(x_i - x_{i-1})\Big)$$

$$= D\Big(\sum_{i=1}^n F(\xi_i)\alpha'(t_i)(x_i - x_{i-1}), \sum_{i=1}^n F(\xi_i)\alpha'(\xi_i)(x_i - x_{i-1})\Big)$$

$$\leq \sum_{i=1}^{n} D\Big(F(\xi_i)\alpha'(t_i)(x_i - x_{i-1}), F(\xi_i)\alpha'(\xi_i)(x_i - x_{i-1})\Big)$$

$$= \sum_{i=1}^{n} \sup_{\lambda\in[0,1]} \max\Big\{ \mid F_\lambda^-(\xi_i)(\alpha'(t_i) - \alpha'(\xi_i)) \mid, \qquad \mid F_\lambda^+(\xi_i)(\alpha'(t_i) - \alpha'(\xi_i)) \mid \Big\}(x_i - x_{i-1})$$

$$\leq \sum_{i=1}^{n} \mid \alpha'(t_i) - \alpha'(\xi_i) \mid (x_i - x_{i-1}) \sup_{\lambda\in[0,1]} \max\{\mid F_\lambda^-(\xi_i) \mid, \mid f_\lambda^+(\xi_i) \mid\}$$

$$\leq (b-a)\frac{\epsilon}{3\sup_{x\in[a,b]} D(F(x),0)(b-a)} \sup_{x\in[a,b]} D(F(x),0) \leq \frac{\epsilon}{3}.$$

On the other hand, since F is fuzzy number-valued Henstock-stieltjes integrable with respect to α on $[a,b]$, by Theorem 2.4 there is a choice S_2 on $[a,b]$ such that for any S_2-fine partitions $P = \{[u,v],\xi\}$ and $P' = \{[u',v'],\xi'\}$, we have

$$D\Big(\sum_P f(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} f(\xi')(\alpha(v') - \alpha(u'))\Big) < \frac{\epsilon}{3}.$$

Define a choice $S_3 = \{S_x^1 \cap S_x^2 : S_x^1 \in S_1, S_x^2 \in S_2\}$. Let $P = \{[u,v],\xi\}$ and $P' = \{[u',v'],\xi'\}$ be S_3-fine partitions on $[a,b]$. we have

$$\begin{aligned} &D\Big(\sum_P F(\xi)\alpha'(\xi)(v-u), \sum_{P'} F(\xi')\alpha'(\xi')(v'-u')\Big) \\ &\quad \leq D\Big(\sum_P F(\xi)\alpha'(\xi)(v-u), \sum_P F(\xi)(\alpha(v) - \alpha(u))\Big) \\ &\quad\quad + D\Big(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} F(\xi')\alpha(v') - \alpha(u'))\Big) \\ &\quad\quad + D\Big(\sum_{P'} F(\xi')\alpha(v') - \alpha(u'), \sum_{P'} F(\xi')\alpha'(\xi')(v'-u')\Big) < \epsilon. \end{aligned}$$

Hence, $F\alpha'$ is AP-Henstock integrable on $[a,b]$.

Conversely, if $F\alpha'$ is AP-Henstock integrable on $[a,b]$, then for any $\epsilon > 0$, there is a choice $S_3 = \{S_x^3\}$ on $[a,b]$ such that for any S_3-fine partitions $P = \{[u,v],\xi\}$ and $P' = \{[u',v'],\xi'\}$, we have

$$D\Big(\sum_P F(\xi)\alpha'(\xi)(v-u), \sum_{P'} F(\xi')\alpha'(\xi')(v'-u')\Big) < \frac{\epsilon}{3}.$$

For a choice $S_4 = \{S_x^1 \cap S_x^3 : S_x^1 \in S_1, S_x^3 \in S_3\}$, let $P = \{[u, v], \xi\}$ and $P' = \{[u', v'], \xi'\}$ be S_4-fine partitions on $[a, b]$. then we have

$$\begin{aligned} D\Big(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_{P'} F(\xi')(\alpha(v') - \alpha(u'))\Big) \\ \leq D\Big(\sum_P F(\xi)(\alpha(v) - \alpha(u)), \sum_P F(\xi)\alpha'(\xi)(v - u)\Big) \\ + D\Big(\sum_P F(\xi)\alpha'(\xi)(v - u), \sum_{P'} F(\xi')\alpha'(\xi')(v' - u')\Big) \\ + D\Big(\sum_{P'} F(\xi')\alpha'(\xi')(v' - u'), \sum_{P'} F(\xi')(\alpha(v') - \alpha(u'))\Big) < \epsilon. \end{aligned}$$

Hence, F is fuzzy number-valued Henstock-Stieltjes integrable with respect to α on $[a, b]$.

Next, we will show that

$$(APFHS) \int_a^b F(x)d\alpha = (APFH) \int_a^b F(x)\alpha'(x)dx.$$

For any S-fine partition $P = \{([x_{i-1}, x_i], \xi_i) : 1 \leq i \leq n\}$ on $[a, b]$, by the Lagrange mean value theorem there exists $t_i \in (x_i - x_{i-1})$ such that

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})(1 \leq i \leq n).$$

Hence, we have

$$\sum_{i=1}^n F(\xi_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^n F(\xi)\alpha'(t_i)(x_i - x_{i-1}).$$

Since F and $F\alpha'$ are bounded functions on $[a, b]$, we have

$$(APFHS) \int_a^b F(x)d\alpha = (APFH) \int_a^b F(x)\alpha'(x)dx.$$

This completes the proof. □

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