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COMPARATIVE GROWTH ANALYSIS OF DIFFERENTIAL MONOMIALS AND DIFFERENTIAL POLYNOMIALS DEPENDING ON THEIR RELATIVE ${}_{p}L^{*}$- ORDERS

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire and meromorphic functions using relative ${}_{p}L^{*}$-order, relative ${}_{p}L^{*}$-lower order and differential monomials, differential polynomials generated by one of the factors.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers and f be a meromorphic function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [6, 8, 13, 15] and [12]. Henceforth, we do not explain those in details. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. Now we just recall the following properties of meromorphic functions which will be needed in the sequel.

Let $n_{0j}, n_{1j}, ..., n_{kj} (k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. For a non-constant meromorphic function f, we call $M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} ... \left(f^{(k)}\right)^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{Mj} = \sum\limits_{i=0}^{k} n_{ij}$ and

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$\Gamma_{Mj} = \sum_{i=0}^{k}(i+1)n_{ij}$ are called respectively the degree and weight of $M_j[f]$ ([5], [11]). The expression $P[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max_{1\leq j\leq s} \gamma_{Mj}$ and $\Gamma_P = \max_{1\leq j\leq s} \Gamma_{Mj}$ are called respectively the degree and weight of $P[f]$ (see [5, 11]). Also we call the numbers $\underline{\gamma_P} = \min_{1\leq j\leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_p} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper, we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f, i.e., for which $n_{0j} = 0$ for $j = 1, 2, ...s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

However, the Nevanlinna's Characteristic function of a meromorphic function f is define as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \overline{n}_f(0,a) \log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a) \log r\right),$$

in addition we represent by $n_f(r,a)$ $\left(\overline{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞-point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geq 0\ .$$

Also we may employ $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r)\,.$$

Moreover for any non-constant entire function f, $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (|T_f(0)|,\infty) \to (0,\infty)$ is exists where $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection we immediately remind the following definition which is relevant:

Definition 1.1. Let a be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of a with respect to a meromorphic function f are defined as

$$\delta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \underline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}$$

and

$$\Delta(a;f) = 1 - \underline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \overline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}\ .$$

Definition 1.2. The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows

$$\Theta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}_f(r,a)}{T_f(r)}.$$

Definition 1.3. [16] For $a \in \mathbb{C}\cup\{\infty\}$, we denote by $n_{f|=1}(r,a)$, the number of simple zeros of $f-a$ in $|z|\leq r$. $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way. We put

$$\delta_1(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{f|=1}(r,a)}{T_f(r)}\ ,$$

the deficiency of 'a' corresponding to the simple a-points of f, i.e., simple zeros of $f-a$.

Yang [14] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$.

Definition 1.4. [9] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_p(r, a; f)}{T_f(r)}.$$

Definition 1.5. [2] $P[f]$ is said to be admissible if
(i) $P[f]$ is homogeneous, or
(ii) $P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.

However in case of any two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called as the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions. Further the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders are normally defined in terms of their growth with respect to the exp function which are shown in the following definition:

Definition 1.6. The order ρ_f (the lower order λ_f) of a meromorphic function f is defined as

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

Somasundaram and Thamizharasi [10] introduced the notions of L-order and L-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. The more generalized concept of L-order and L-lower order of meromorphic functions are L^*-order and L^*-lower order respectively which are as follows:

Definition 1.7. [10] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]}.$$

Lahiri and Banerjee [7] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 1.8. [7] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned}\rho(f,g) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [7] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In order to make some progress in the study of relative order, now we introduce relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of a meromorphic function f with respect to an entire g which are as follows:

Definition 1.9. The relative ${}_pL^*$-order denoted as $\rho_p^{L^*}(f,g)$ and relative ${}_pL^*$- lower order denoted as $\lambda_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire g are defined as

$$\rho_p^{L^*}(f,g) = \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} \text{ and } \lambda_p^{L^*}(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]},$$

where p is any positive integers.

In the paper we establish some new results depending on the comparative growth properties of composite entire and meromorphic functions using relative ${}_pL^*$-order (resp. relative ${}_pL^*$- lower order) and differential monomials, differential polynomials generated by one of the factors.

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3] *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and g be an entire function with regular growth and non zero finite order. Also let $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Then for homogeneous $P_0[f]$ and $P_0[g]$,*

$$\lim_{r\to\infty} \frac{\log T^{-1}_{P_0[g]} T_{P_0[f]}(r)}{\log T^{-1}_g T_f(r)} = 1.$$

Lemma 2.2. [4] *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and g be a transcendental entire function with regular growth and non zero finite order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then*

$$\lim_{r\to\infty} \frac{\log T^{-1}_{M[g]} T_{M[f]}(r)}{\log T^{-1}_g T_f(r)} = 1 .$$

Lemma 2.3. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and g be an entire function with regular growth having non zero finite order and $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g for homogeneous $P_0[f]$ and $P_0[g]$.*

Proof. By Lemma 2.1 we obtain that

$$\begin{aligned}
\rho_p^{L^*}(P_0[f], P_0[g]) &= \overline{\lim_{r\to\infty}} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log\left[r \exp^{[p]} L(r)\right]} \\
&= \overline{\lim_{r\to\infty}} \left\{ \frac{\log T_g^{-1} T_f(r)}{\log\left[r \exp^{[p]} L(r)\right]} \cdot \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \right\} \\
&= \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log\left[r \exp^{[p]} L(r)\right]} \cdot \lim_{r\to\infty} \frac{\log T_{P_0[g]}^{-1} T_{P_0[f]}(r)}{\log T_g^{-1} T_f(r)} \\
&= \rho_p^{L^*}(f, g) \cdot 1 \\
&= \rho_p^{L^*}(f, g).
\end{aligned}$$

In a similar manner, $\lambda_p^{L^*}(P_0[f], P_0[g]) = \lambda_p^{L^*}(f, g)$. This proves the lemma. □

In the line of Lemma 2.3 and with the help of Lemma 2.2, we may state the following lemma without its proof :

Lemma 2.4. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and g be a transcendental entire function with regular growth and non zero finite order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $M[f]$ with respect to $M[g]$ are same as those of f with respect to g. i.e.,*

$$\rho_p^{L^*}(M[f], M[g]) = \rho_p^{L^*}(f, g) \text{ and } \lambda_p^{L^*}(M[f], M[g]) = \lambda_p^{L^*}(f, g) \ .$$

Lemma 2.5. [1] *Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r) \ge T_f(\exp(r)^\mu).$$

3. Theorems.

In this section we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ for meromorphic f will be needed as per the requirements of the theorems.

Theorem 3.1. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$. Also let h be a entire function with regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ and g be any entire function such that $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$ where p is any positive integer. Then for any $A > 0$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right) + K(r, A; L)} = \infty ,$$

where $0 < \mu < \rho_g$ *and* $K(r, A; L) = \begin{cases} 0 \text{ if } r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \text{ as } r \to \infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise .}\end{cases}$

Proof. Let $0 < \mu < \mu' < \rho_g$. Since $T_h^{-1}(r)$ is an increasing functions, from the definition of relative ${}_pL^*$-lower order we obtain in view of Lemma 2.5, for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right) \geq \log T_h^{-1} T_f\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right),$$

that is,

$$\begin{aligned}&\log T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)\\ &\geq \left(\lambda_p^{L^*}(f, h) - \varepsilon\right) \cdot \log\left\{\exp\left(\exp\left(r^A\right)\right)^{\mu'} \cdot \exp^{[p]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\}\end{aligned}$$

$$\begin{aligned}&\Rightarrow \log T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)\\ &\quad\geq \left(\lambda_p^{L^*}(f, h) - \varepsilon\right) \cdot \left\{\left(\exp\left(r^A\right)\right)^{\mu'} + \exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)\right\}\end{aligned}$$

$$\begin{aligned}&\Rightarrow \log T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)\\ &\quad\geq \left(\lambda_p^{L^*}(f, h) - \varepsilon\right) \cdot \left\{\left(\exp\left(r^A\right)\right)^{\mu'}\left(1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right)\right\}\end{aligned}$$

$$\Rightarrow \log^{[2]} T_h^{-1} T_{f \circ g}\left(\exp\left(r^A\right)\right)$$
$$\geq O(1) + \mu' \log \exp\left(r^A\right) + \log\left\{1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right\}$$
$$\Rightarrow \log^{[2]} T_h^{-1} T_{f \circ g}\left(\exp\left(r^A\right)\right)$$
$$\geq O(1) + \mu' r^A + \log\left\{1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(r^A\right)\right)^{\mu'}\right)}{\left(\exp\left(r^A\right)\right)^{\mu'}}\right\}$$
$$\Rightarrow \log^{[2]} T_h^{-1} T_{f \circ g}\left(\exp\left(r^A\right)\right)$$
$$\geq O(1) + \mu' r^A + \log\left[1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left(\mu' r^A\right)}\right]$$
$$\Rightarrow \log^{[2]} T_h^{-1} T_{f \circ g}\left(\exp\left(r^A\right)\right)$$
$$\geq O(1) + \mu' r^A + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) - \log\left[\exp\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}\right]$$
$$+ \log\left[1 + \frac{\exp^{[p-1]} L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left(\mu' r^A\right)}\right]$$
$$\Rightarrow \log^{[2]} T_h^{-1} T_{f \circ g}\left(\exp\left(r^A\right)\right)$$
$$\geq O(1) + \mu' r^A + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)$$
$$+ \log\left[\frac{\exp\left(\mu' r^A\right) + \exp^{[p-1]} L\left(\exp\left(\exp\left(\mu' r^A\right)\right)\right)}{\exp\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \cdot \exp\left(\mu' r^A\right)}\right]$$

(3.1)
$$\Rightarrow \log^{[2]} T_h^{-1} T_{f \circ g}\left(\exp\left(r^A\right)\right) \geq O(1) + \mu' r^{(A-\mu)} \cdot r^{\mu} + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right).$$

Again in view of Lemma 2.3, we have for all sufficiently large values of r that

$$\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)$$
$$\leq log\left(\rho_p^{L^*}\left(P_0[f], P_0[h]\right) + \varepsilon\right) \log\left\{\exp\left(r^{\mu}\right) \cdot \exp^{[p]} L\left(\exp\left(r^{\mu}\right)\right)\right\}$$
$$\Rightarrow \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)$$
$$\leq \left(\rho_p^{L^*}(f, h) + \varepsilon\right) \log\left\{\exp\left(r^{\mu}\right) \cdot \exp^{[p]} L\left(\exp\left(r^{\mu}\right)\right)\right\}$$
$$\Rightarrow \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)$$

$$\leq \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left\{\log\exp\left(r^{\mu}\right)+\exp^{[p-1]}L\left(\exp\left(r^{\mu}\right)\right)\right\}$$

$$\Rightarrow \log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)$$

$$\leq \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left\{r^{\mu}+\exp^{[p-1]}L\left(\exp\left(r^{\mu}\right)\right)\right\}$$

(3.2)

$$\Rightarrow \frac{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)-\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\exp^{[p-1]}L\left(\exp\left(r^{\mu}\right)\right)}{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)}\leq r^{\mu}.$$

Now from (3.1) and (3.2) it follows for a sequence of values of r tending to infinity that

(3.3)

$$\log^{[2]}T_h^{-1}T_{f\circ g}\left(\exp\left(r^{A}\right)\right)\geq O(1)+\left(\frac{\mu' r^{(A-\mu)}}{\rho_p^{L^*}(f,h)+\varepsilon}\right)\times$$
$$\left[\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)-\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\exp^{[p-1]}L\left(\exp\left(r^{\mu}\right)\right)\right]$$
$$+L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)$$

(3.4)

$$\Rightarrow\frac{\log^{[2]}T_h^{-1}T_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)}\geq\frac{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)+O(1)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)}$$
$$+\frac{\mu' r^{(A-\mu)}}{\rho_p^{L^*}(f,h)+\varepsilon}\left\{1-\frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\exp^{[p-1]}L\left(\exp\left(r^{\mu}\right)\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)}\right\}.$$

Again from (3.3) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[2]}T_h^{-1}T_{f\circ g}\left(\exp\left(r^{A}\right)\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)+L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$
$$\geq\frac{O(1)-\mu' r^{(A-\mu)}\cdot\exp^{[p-1]}L\left(\exp\left(r^{\mu}\right)\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)+L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$
$$+\frac{\left(\frac{\mu' r^{(A-\mu)}}{\rho_h^{L^*}(f)+\varepsilon}\right)\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)+L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$
$$+\frac{L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left(r^{\mu}\right)\right)+L\left(\exp\left(\exp\left(\mu r^{A}\right)\right)\right)}$$

(3.5)

$$\Rightarrow \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)}$$

$$\geq \frac{\frac{O(1)-\mu' r^{(A-\mu)}\cdot \exp^{[p-1]} L(\exp(r^\mu))}{L(\exp(\exp(\mu r^A)))}}{\frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp(r^\mu))}{L(\exp(\exp(\mu r^A)))} + 1} + \frac{\left(\frac{\mu' r^{(A-\mu)}}{\rho_h^{L^*}(f)+\varepsilon}\right) \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right)}{1 + \frac{L(\exp(\exp(\mu r^A)))}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp(r^\mu))}}$$

$$+ \frac{1}{1 + \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp(r^\mu))}{L(\exp(\exp(\mu r^A)))}}.$$

Case I. If $r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}$ then it follows from (3.4) that

$$\varlimsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right)} = \infty .$$

Case II. $r^\mu \neq o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\}$ then two sub cases may arise.

Sub case (a). If $L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) = o\left\{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right)\right\}$, then we get from (3.5) that

$$\varlimsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} = \infty .$$

Sub case (b). If $L\left(\exp\left(\exp\left(\mu r^{\rho_g^{L^*}}\right)\right)\right) \sim \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right)$ then

$$\lim_{r\to\infty} \frac{L\left\{\exp\left(\exp\left(\mu r^A\right)\right)\right\}}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right)} = 1$$

and we obtain from (3.5) that

$$\varlimsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right) + L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)} = \infty .$$

Combining Case I and Case II we may obtain that

$$\varlimsup_{r\to\infty} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp\left(r^\mu\right)\right) + K\left(r, A; L\right)} = \infty ,$$

where $K(r, A; L) = \begin{cases} 0 \text{ if } r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \text{ as } r \to \infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise.} \end{cases}$

This proves the theorem. □

Theorem 3.2. *Let g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty;g) = \sum_{a\neq\infty}\delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum_{a\neq\infty}\delta(a;g) = 1$. Also let h be a entire function of regular growth having non zero finite order with $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ and f be any meromorphic function such that $\lambda_p^{L^*}(f,h) > 0$ and $\rho_p^{L^*}(g,h) < \infty$ where p is any positive integer. Then for any $A > 0$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}\left(\exp\left(r^\mu\right)\right) + K(r,A;L)} = \infty ,$$

where $0<\mu<\rho_g$ *and* $K(r,A;L) = \begin{cases} 0 \text{ if } r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise.} \end{cases}$

The proof is omitted because it can be carried out in the line of Theorem 3.1.

In the line of Theorem 3.1 and Theorem 3.2 respectively and with the help of Lemma 2.4, one can easily proof the following two theorems and therefore their proofs are omitted:

Theorem 3.3. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h) = 4$ and g be any entire function such that $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$ where p is any positive integer. Then for any $A > 0$*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1} T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{M[h]}^{-1} T_{M[f]}\left(\exp\left(r^\mu\right)\right) + K(r,A;L)} = \infty ,$$

where $0<\mu<\rho_g$ *and* $K(r,A;L) = \begin{cases} 0 \text{ if } r^\mu = o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise .} \end{cases}$

Theorem 3.4. *Let g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g) = 4$. Also let h be a transcendental entire function of regular growth having non* *zero finite order with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and* f *be any meromorphic function such that* $\lambda_p^{L^*}(f,h)>0$ *and* $\rho_p^{L^*}(g,h)<\infty$ *where* p *is any positive integer. Then for any* $A>0$

$$\overline{\lim_{r\to\infty}}\frac{\log^{[2]}T_h^{-1}T_{f\circ g}\left(\exp\left(r^A\right)\right)}{\log T_{Mh]}^{-1}T_{M[g]}\left(\exp\left(r^{\mu}\right)\right)+K\left(r,A;L\right)}=\infty\ ,$$

where $0<\mu<\rho_g$ *and* $K(r,A;L)=\begin{cases}0 \text{ if } r^{\mu}=o\left\{L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right)\right\} \\ \qquad\qquad\qquad\qquad \text{as } r\to\infty \\ L\left(\exp\left(\exp\left(\mu r^A\right)\right)\right) \text{ otherwise .}\end{cases}$

THEOREM 3.5. *Let* f *be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f)=\sum\limits_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum\limits_{a\neq\infty}\delta(a;f)=1$ *. Also let* h *be a entire function with regular growth having non zero finite order and* $\Theta(\infty;h)=\sum\limits_{a\neq\infty}\delta_p(a;h)=1$ *or* $\delta(\infty;h)=\sum\limits_{a\neq\infty}\delta(a;h)=1$ *and* g *be any entire function such that* $0<\lambda_p^{L^*}(f\circ g,h)\leq\rho_p^{L^*}(f\circ g,h)<\infty$ *and* $0<\lambda_p^{L^*}(f,h)\leq\rho_p^{L^*}(f,h)<\infty$ *where* p *is any positive integer. If* $L\left(r^A\right)=o\left\{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)\right\}$ *as* $r\to\infty$ *then for any positive number* A*,*

$$\begin{aligned}\frac{\lambda_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(f,h)}&\leq\underline{\lim_{r\to\infty}}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)}\\&\leq\min\left\{\frac{\lambda_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(f,h)},\frac{\rho_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(f,h)}\right\}\\&\leq\max\left\{\frac{\lambda_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(f,h)},\frac{\rho_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(f,h)}\right\}\\&\leq\overline{\lim_{r\to\infty}}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)}\leq\frac{\rho_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(f,h)}\ .\end{aligned}$$

Proof. From the definition of relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of a meromorphic function with respect to an entire function and in view of Lemma 2.3, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right) \log\left[r \exp^{[p]} L(r)\right]$$
$$\Rightarrow \log T_h^{-1} T_{f\circ g}(r) \geq \left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right) \left[\log r + \exp^{[p-1]} L(r)\right]$$

(3.6)

$$\Rightarrow \log T_h^{-1} T_{f\circ g}(r) \geq \left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right) \left[\log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right] + \left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right) \left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$

and

$$\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \leq \left(\rho_p^{L^*}(P_0[f], P_0[h]) + \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r^A\right)\right]$$
$$\Rightarrow \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \leq \left(\rho_p^{L^*}(f, h) + \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r^A\right)\right]$$
$$\Rightarrow \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \leq \left(\rho_p^{L^*}(f, h) + \varepsilon\right) \left[A\log r + \exp^{[p-1]} L\left(r^A\right)\right]$$

(3.7)

$$\Rightarrow \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{A\left(\rho_p^{L^*}(f, h) + \varepsilon\right)} \leq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right).$$

Now from (3.6) and (3.7) it follows for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \frac{\left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right)}{A\left(\rho_p^{L^*}(f, h) + \varepsilon\right)} \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) + \left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right) \left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$
$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right)}{A\left(\rho_p^{L^*}(f, h) + \varepsilon\right)} \cdot \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} + \frac{\left(\lambda_p^{L^*}(f\circ g, h) - \varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$

(3.8)

$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$

$$\geq \frac{\frac{\lambda_p^{L^*}(f\circ g,h)-\varepsilon}{A\left(\rho_p^{L^*}(f,h)+\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}} + \frac{\left(\lambda_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^A)} - \frac{1}{A}\right]}{1+\frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}{L(r^A)}}.$$

Since $L(r^A) = o\left\{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)\right\}$ as $r \to \infty$, it follows from (3.8) that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\left(\lambda_p^{L^*}(f\circ g,h)-\varepsilon\right)}{A\left(\rho_p^{L^*}(f,h)+\varepsilon\right)}. \tag{3.9}$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (3.9) that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\lambda_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(f,h)}. \tag{3.10}$$

Again for a sequence of values of r tending to infinity,

$$\log T_h^{-1} T_{f\circ g}(r) \leq \left(\lambda_p^{L^*}(f\circ g,h)+\varepsilon\right) \log\left[r \exp^{[p]} L(r)\right]$$

(3.11)

$$\Rightarrow \log T_h^{-1} T_{f\circ g}(r) \leq \left(\lambda_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$
$$+ \left(\lambda_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$

and for all sufficiently large values of r,

$$\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \geq \left(\lambda_p^{L^*}(P_0[f],P_0[h])-\varepsilon\right)\log\left[r^A \exp^{[p]} L\left(r^A\right)\right]$$
$$\Rightarrow \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \geq \left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[A\log r + \exp^{[p-1]} L\left(r^A\right)\right]$$

(3.12)

$$\Rightarrow \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)} \geq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right).$$

Combining (3.11) and (3.12) we get for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\lambda_p^{L^*}(f\circ g, h)+\varepsilon\right)}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)} \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)$$
$$+\left(\lambda_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$

$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$
$$\leq \frac{\lambda_p^{L^*}(f\circ g,h)+\varepsilon}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)} \cdot \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$
$$+\frac{\left(\lambda_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$

(3.13)

$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$
$$\leq \frac{\frac{\lambda_p^{L^*}(f\circ g,h)+\varepsilon}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}} + \frac{\left(\lambda_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^A)} - \frac{1}{A}\right]}{1+\frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}{L(r^A)}}.$$

As $L\left(r^A\right) = o\left\{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)\right\}$ as $r \to \infty$ we get from (3.13) that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \leq \frac{\lambda_p^{L^*}(f\circ g,h)+\varepsilon}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)}. \tag{3.14}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (3.14) that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \leq \frac{\lambda_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(f,h)}. \tag{3.15}$$

Also for a sequence of values of r tending to infinity,

$$\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \leq \left(\lambda_p^{L^*}(P_0[f], P_0[h])+\varepsilon\right)\log\left[r^A \exp^{[p]} L\left(r^A\right)\right]$$
$$\Rightarrow \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \leq \left(\lambda_p^{L^*}(f,h)+\varepsilon\right)\left[A\log r + \exp^{[p-1]} L\left(r^A\right)\right]$$

$$(3.16) \qquad \Rightarrow \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{A\left(\lambda_p^{L^*}(f,h)+\varepsilon\right)} \leq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right) .$$

Now from (3.6) and (3.16) we obtain for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \frac{\left(\lambda_p^{L^*}(f\circ g,h)-\varepsilon\right)}{A\left(\lambda_p^{L^*}(f,h)+\varepsilon\right)} \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) + \left(\lambda_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$

$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\lambda_p^{L^*}(f\circ g,h)-\varepsilon}{A\left(\lambda_p^{L^*}(f,h)+\varepsilon\right)} \cdot \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} + \frac{\left(\lambda_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$

(3.17)

$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\frac{\lambda_p^{L^*}(f\circ g,h)-\varepsilon}{A\left(\lambda_p^{L^*}(f,h)+\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}} + \frac{\left(\lambda_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^A)} - \frac{1}{A}\right]}{1+\frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}{L(r^A)}} .$$

In view of the condition $L\left(r^A\right) = o\left\{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)\right\}$ as $r \to \infty$ we obtain from (3.17) that

$$(3.18) \qquad \overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\lambda_p^{L^*}(f\circ g,h)-\varepsilon}{A\left(\lambda_p^{L^*}(f,h)+\varepsilon\right)} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (3.18) that

$$(3.19) \qquad \overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\lambda_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(f,h)} .$$

Also for all sufficiently large values of r,

$$\log T_h^{-1} T_{f\circ g}(r) \leq \left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]$$
$$\Rightarrow\ \log T_h^{-1} T_{f\circ g}(r) \leq \left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\log r+\exp^{[p-1]}L(r)\right]$$

(3.20)

$$\Rightarrow\ \log T_h^{-1} T_{f\circ g}(r) \leq \left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\log r+\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right] + \left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\exp^{[p-1]}L(r)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right].$$

So from (3.12) and (3.20) it follows for all sufficiently large values of r that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)}\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right) + \left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\exp^{[p-1]}L(r)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right]$$

$$\Rightarrow\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_p^{L^*}(f\circ g,h)+\varepsilon}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)}\cdot\frac{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)} + \frac{\left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\exp^{[p-1]}L(r)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right]}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)}$$

(3.21)

$$\Rightarrow\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)} \leq \frac{\frac{\rho_p^{L^*}(f\circ g,h)+\varepsilon}{A\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}(r^A)}} + \frac{\left(\rho_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[\frac{\exp^{[p-1]}L(r)}{\exp^{[p-1]}L(r^A)}-\frac{1}{A}\right]}{1+\frac{\log T_{P_0[h]}^{-1}T_{P_0[f]}(r^A)}{L(r^A)}}.$$

Using $L\left(r^A\right)=o\left\{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)\right\}$ as $r\to\infty$ we obtain from (3.21) that

$$\varlimsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \leq \frac{\rho_p^{L^*}(f\circ g, h) + \varepsilon}{A\left(\lambda_p^{L^*}(f,h) - \varepsilon\right)} \ . \tag{3.22}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (3.22) that

$$\varlimsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \leq \frac{\rho_p^{L^*}(f\circ g, h)}{A\lambda_p^{L^*}(f,h)} \ . \tag{3.23}$$

From the definition of $\rho_p^{L^*}(P_0[f], P_0[h])$ and in view of Lemma 2.3, we get for a sequence of values of r tending to infinity that

$$\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \geq \left(\rho_p^{L^*}(P_0[f], P_0[h]) - \varepsilon\right) \log\left[r^A \exp^{[p]} L\left(r^A\right)\right]$$

$$\Rightarrow\ \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) \geq \left(\rho_p^{L^*}(f,h) - \varepsilon\right)\left[A\log r + \exp^{[p-1]} L\left(r^A\right)\right]$$

$$\Rightarrow\ \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{A\left(\rho_p^{L^*}(f,h) - \varepsilon\right)} \geq \log r + \frac{1}{A}\exp^{[p-1]} L\left(r^A\right). \tag{3.24}$$

Now from (3.20) and (3.24) it follows for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\rho_p^{L^*}(f\circ g, h) + \varepsilon\right)}{A\left(\rho_p^{L^*}(f,h) - \varepsilon\right)} \log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right) + \left(\rho_p^{L^*}(f\circ g, h) + \varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]$$

$$\Rightarrow\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \leq \frac{\rho_p^{L^*}(f\circ g, h) + \varepsilon}{A\left(\rho_p^{L^*}(f,h) - \varepsilon\right)} \cdot \frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} + \frac{\left(\rho_p^{L^*}(f\circ g, h) + \varepsilon\right)\left[\exp^{[p-1]} L(r) - \frac{1}{A}\exp^{[p-1]} L\left(r^A\right)\right]}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$

$$\Rightarrow\ \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \tag{3.25}$$

$$\leq \frac{\frac{\rho_p^{L^*}(f\circ g,h)+\varepsilon}{A\left(\rho_p^{L^*}(f,h)-\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}(r^A)}} + \frac{\left(\rho_p^{L^*}\left(f\circ g,h\right)+\varepsilon\right)\left[\frac{\exp^{[p-1]}L(r)}{\exp^{[p-1]}L(r^A)}-\frac{1}{A}\right]}{1+\frac{\log T_{P_0[h]}^{-1}T_{P_0[f]}(r^A)}{L(r^A)}} .$$

Using $L\left(r^A\right) = o\left\{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)\right\}$ as $r\to\infty$ we obtain from (3.25) that

$$\underset{r\to\infty}{\underline{\lim}}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_p^{L^*}\left(f\circ g,h\right)+\varepsilon}{A\left(\rho_p^{L^*}\left(f,h\right)-\varepsilon\right)} . \tag{3.26}$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (3.26) that

$$\underset{r\to\infty}{\underline{\lim}}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)} \leq \frac{\rho_p^{L^*}\left(f\circ g,h\right)}{A\rho_p^{L^*}\left(f,h\right)} . \tag{3.27}$$

Again for a sequence of values of r tending to infinity,

$$\log T_h^{-1}T_{f\circ g}\left(r\right) \geq \left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)\log\left[r\exp^{[p]}L\left(r\right)\right]$$

$$\Rightarrow\ \log T_h^{-1}T_{f\circ g}\left(r\right) \geq \left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)\left[\log r+\exp^{[p-1]}L\left(r\right)\right]$$

(3.28)

$$\Rightarrow\ \log T_h^{-1}T_{f\circ g}\left(r\right) \geq \left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)\left[\log r+\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right]$$
$$+\left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)\left[\exp^{[p-1]}L\left(r\right)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right].$$

So combining (3.7) and (3.28) we get for a sequence of values of r tending to infinity that

$$\log T_h^{-1}T_{f\circ g}\left(r\right) \geq \frac{\left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)}{A\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)}\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)$$
$$+\left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)\left[\exp^{[p-1]}L\left(r\right)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right]$$

$$\Rightarrow\ \frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)}$$

$$\geq \frac{\left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)}{A\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)}\cdot\frac{\log T_h^{-1}T_f\left(r^A\right)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)}$$
$$+\frac{\left(\rho_p^{L^*}\left(f\circ g,h\right)-\varepsilon\right)\left[\exp^{[p-1]}L\left(r\right)-\frac{1}{A}\exp^{[p-1]}L\left(r^A\right)\right]}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r^A\right)+L\left(r^A\right)}$$

(3.29)

$$\Rightarrow \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)}$$

$$\geq \frac{\frac{\rho_p^{L^*}(f\circ g,h)-\varepsilon}{A\left(\rho_p^{L^*}(f,h)+\varepsilon\right)}}{1+\frac{L(r^A)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}} + \frac{\left(\rho_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[\frac{\exp^{[p-1]} L(r)}{\exp^{[p-1]} L(r^A)} - \frac{1}{A}\right]}{1+\frac{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A)}{L(r^A)}}.$$

Since $L\left(r^A\right) = o\left\{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(r^A\right)\right\}$ as $r \to \infty$, it follows from (3.29) that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\rho_p^{L^*}(f\circ g,h)-\varepsilon}{A\left(\rho_p^{L^*}(f,h)+\varepsilon\right)} . \tag{3.30}$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (3.30) that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(r^A) + L(r^A)} \geq \frac{\rho_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(f,h)} . \tag{3.31}$$

Thus the theorem follows from $(3.10), (3.15), (3.19), (3.23), (3.27)$ and (3.31). □

Similarly in view of Theorem 3.5, we may state the following theorem without proof for the right factor g of the composite function $f\circ g$:

Theorem 3.6. *Let g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty;g) = \sum_{a\neq\infty}\delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum_{a\neq\infty}\delta(a;g) = 1$. Also let h be a entire function of regular growth having non zero finite order with $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$ and f be any meromorphic function such that $0 < \lambda_p^{L^*}(f\circ g,h) \leq \rho_p^{L^*}(f\circ g,h) < \infty$, and $0 < \lambda_p^{L^*}(g,h) \leq \rho_p^{L^*}(g,h) < \infty$ where p is any positive integer. If $L\left(r^A\right) = o\left\{\log T_{P_0[h]}^{-1} T_{P_0[g]}\left(r^A\right)\right\}$ as*

$r \to \infty$ then for any positive number A,

$$\frac{\lambda_p^{L^*}(f \circ g, h)}{A\rho_p^{L^*}(g, h)} \leq \varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(r^A) + L(r^A)}$$
$$\leq \min\left\{\frac{\lambda_p^{L^*}(f \circ g, h)}{A\lambda_p^{L^*}(g, h)}, \frac{\rho_p^{L^*}(f \circ g, h)}{A\rho_p^{L^*}(g, h)}\right\}$$
$$\leq \max\left\{\frac{\lambda_p^{L^*}(f \circ g, h)}{A\lambda_p^{L^*}(g, h)}, \frac{\rho_p^{L^*}(f \circ g, h)}{A\rho_p^{L^*}(g, h)}\right\}$$
$$\leq \varlimsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(r^A) + L(r^A)} \leq \frac{\rho_p^{L^*}(f \circ g, h)}{A\lambda_p^{L^*}(g, h)}.$$

In the line of Theorem 3.5 and Theorem 3.6 respectively and with the help of Lemma 2.4, one can easily proof the following two theorems and therefore their proofs are omitted:

THEOREM 3.7. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; h) = 4$ and g be any entire function such that $0 < \lambda_p^{L^*}(f \circ g, h) \leq \rho_p^{L^*}(f \circ g, h) < \infty$ and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$ where p is any positive integer. If $L(r^A) = o\left\{\log T_{M[h]}^{-1} T_{M[f]}(r^A)\right\}$ as $r \to \infty$ then for any positive number A,*

$$\frac{\lambda_p^{L^*}(f \circ g, h)}{A\rho_p^{L^*}(f, h)} \leq \varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(r^A) + L(r^A)}$$
$$\leq \min\left\{\frac{\lambda_p^{L^*}(f \circ g, h)}{A\lambda_p^{L^*}(f, h)}, \frac{\rho_p^{L^*}(f \circ g, h)}{A\rho_p^{L^*}(f, h)}\right\}$$
$$\leq \max\left\{\frac{\lambda_p^{L^*}(f \circ g, h)}{A\lambda_p^{L^*}(f, h)}, \frac{\rho_p^{L^*}(f \circ g, h)}{A\rho_p^{L^*}(f, h)}\right\}$$
$$\leq \varlimsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(r^A) + L(r^A)} \leq \frac{\rho_p^{L^*}(f \circ g, h)}{A\lambda_p^{L^*}(f, h)}.$$

THEOREM 3.8. *Let g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Also* *let h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ *and f be any meromorphic function such that* $0<\lambda_p^{L^*}(f\circ g,h)\leq\rho_p^{L^*}(f\circ g,h)<\infty$, *and* $0<\lambda_p^{L^*}(g,h)\leq\rho_p^{L^*}(g,h)<\infty$ *where p is any positive integer. If* $L\left(r^A\right)=o\left\{\log T_{M[h]}^{-1}T_{M[g]}\left(r^A\right)\right\}$ *as $r\to\infty$ then for any positive number A,*

$$\begin{aligned}\frac{\lambda_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(g,h)} &\leq \varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[g]}(r^A)+L(r^A)}\\ &\leq\min\left\{\frac{\lambda_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(g,h)},\frac{\rho_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(g,h)}\right\}\\ &\leq\max\left\{\frac{\lambda_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(g,h)},\frac{\rho_p^{L^*}(f\circ g,h)}{A\rho_p^{L^*}(g,h)}\right\}\\ &\leq\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{M[h]}^{-1}T_{M[g]}(r^A)+L(r^A)}\leq\frac{\rho_p^{L^*}(f\circ g,h)}{A\lambda_p^{L^*}(g,h)}\ .\end{aligned}$$

THEOREM 3.9. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ *or* $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$. *Also let h be a entire function of regular growth having non zero finite order with* $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ *or* $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$ *and g be any entire function such that* $\rho_p^{L^*}(f,h)<\infty$ *and* $\lambda_p^{L^*}(f\circ g,h)=\infty$ *where p is any positive integer. Then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}(r)}=\infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta>0$ such that for a sequence of values of r tending to infinity

$$\log T_h^{-1}T_{f\circ g}(r)\leq\beta\cdot\log T_{P_0[h]}^{-1}T_{P_0[f]}(r)\ . \tag{3.32}$$

Again from the definition of $\rho_p^{L^*}(P_0[f],P_0[h])$ and in view of Lemma 2.3, it follows for all sufficiently large values of r that

$$\log T_{P_0[h]}^{-1}T_{P_0[f]}(r)\leq\left(\rho_p^{L^*}(P_0[f],P_0[h])+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]\ .$$

$$(3.33)\quad i.e.,\ \log T^{-1}_{P_0[h]}T_{P_0[f]}(r) \leq \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right] .$$

Thus from (3.32) and (3.33) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1}T_{f\circ g}(r) &\leq \beta\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right] \\ i.e.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[r\exp^{[p]}L(r)\right]} &\leq \frac{\beta\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]}{\log\left[r\exp^{[p]}L(r)\right]} \\ i.e.,\ \varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left[r\exp^{[p]}L(r)\right]} &= \lambda_p^{L^*}(f\circ g,h)<\infty . \end{aligned}$$

This is a contradiction. Hence the theorem follows. □

In the line of Theorem 3.9, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 3.10. *Let g be an entire function with finite order or of non-zero lower order such that $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. Also let h be a entire function of regular growth having non zero finite order with $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$ and f be any meromorphic function such that $\rho_p^{L^*}(g,h)<\infty$ and $\lambda_p^{L^*}(f\circ g,h)=\infty$ where p is any positive integer. Then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T^{-1}_{P_0[h]}T_{P_0[g]}(r)}=\infty .$$

Remark 3.11. Theorem 3.9 is also valid with “limit superior” instead of “limit” if $\lambda_p^{L^*}(f\circ g,h)=\infty$ is replaced by $\rho_p^{L^*}(f\circ g,h)=\infty$ and the other conditions remain the same.

Remark 3.12. Theorem 3.10 is also valid with “limit superior” instead of “limit” if $\lambda_p^{L^*}(f\circ g,h)=\infty$ is replaced by $\rho_p^{L^*}(f\circ g,h)=\infty$ and the other conditions remain the same.

Corollary 3.13. *Under the assumptions of Theorem 3.9 and Remark 3.11,*

$$\lim_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}=\infty \text{ and } \varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}=\infty$$

respectively.

Proof. By Theorem 3.9 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &> K \log T_{P_0[h]}^{-1} T_{P_0[f]}(r) \\ i.e.,\ T_h^{-1} T_{f \circ g}(r) &> \left\{ T_{P_0[h]}^{-1} T_{P_0[f]}(r) \right\}^K, \end{aligned}$$

from which the first part of the corollary follows.

Similarly using Remark 3.11, we obtain the second part of the corollary. □

COROLLARY 3.14. *Under the assumptions of Theorem 3.10 and Remark 3.12,*

$$\lim_{r\to\infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} = \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[g]}(r)} = \infty$$

respectively.

In the line of Corollary 3.13, one can easily verify Corollary 3.14 with the help of Theorem 3.10 and Remark 3.12 respectively and therefore its proof is omitted.

In the line of Theorem 3.9 and Theorem 3.10 respectively and with the help of Lemma 2.4, one can easily proof the following two theorems and therefore their proofs are omitted:

THEOREM 3.15. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order such that $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$ and g be any entire function such that $\rho_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(f \circ g, h) = \infty$ where p is any positive integer. Then*

$$\lim_{r\to\infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(r)} = \infty .$$

THEOREM 3.16. *Let g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$ and f be any meromorphic* *function such that* $\rho_p^{L^*}(g,h) < \infty$ *and* $\lambda_p^{L^*}(f \circ g, h) = \infty$ *where* p *is any positive integer. Then*

$$\lim_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(r)} = \infty .$$

Remark 3.17. Theorem 3.15 is also valid with "limit superior" instead of "limit" if $\lambda_p^{L^*}(f \circ g, h) = \infty$ is replaced by $\rho_p^{L^*}(f \circ g, h) = \infty$ and the other conditions remain the same.

Remark 3.18. Theorem 3.16 is also valid with "limit superior" instead of "limit" if $\lambda_p^{L^*}(f \circ g, h) = \infty$ is replaced by $\rho_p^{L^*}(f \circ g, h) = \infty$ and the other conditions remain the same.

Corollary 3.19. *Under the assumptions of Theorem 3.15 and Remark 3.17,*

$$\lim_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} = \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r)} = \infty$$

respectively.

Corollary 3.20. *Under the assumptions of Theorem 3.16 and Remark 3.18,*

$$\lim_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} = \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[g]}(r)} = \infty$$

respectively.

In the line of Corollary 3.13, one can easily verify the above two corollaries with the help of Theorem 3.15; Remark 3.17 and Theorem 3.16, Remark 3.18 respectively and therefore their proofs are omitted.

From the definitions of relative ${}_pL^*$-order and relative ${}_pL^*$- lower order and with help of Lemma 2.3, one can easily verify the following theorem.

Theorem 3.21. *Let* f *and* g *be any two meromorphic functions both either of finite order or of non-zero lower order such that* $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ *and* $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ *or* $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$ *respectively. Also let* h *and* k *be any two entire functions both of regular growth having non zero finite order with* $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$ *and* $\Theta(\infty; k) = \sum_{a\neq\infty} \delta_p(a; k) = 1$ *or* $\delta(\infty; k) = \sum_{a\neq\infty} \delta(a; k) = 1$. *If*

$0 < \lambda_p^{L^*}(f,k) \le \rho_p^{L^*}(f,k) < \infty$ *and* $0 < \lambda_p^{L^*}(g,h) \le \rho_p^{L^*}(g,h) < \infty$ *where* p *is any positive integer, then*

$$\frac{\lambda_p^{L^*}(f,k)}{\rho_p^{L^*}(g,h)} \le \underline{\lim}_{r\to\infty} \frac{\log T^{-1}_{P_0[k]} T_{P_0 f]}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[g]}(r)} \le \min\left\{\frac{\lambda_p^{L^*}(f,k)}{\lambda_p^{L^*}(g,h)}, \frac{\rho_p^{L^*}(f,k)}{\rho_p^{L^*}(g,h)}\right\}$$

$$\le \max\left\{\frac{\lambda_p^{L^*}(f,k)}{\lambda_p^{L^*}(g,h)}, \frac{\rho_p^{L^*}(f,k)}{\rho_p^{L^*}(g,h)}\right\} \le \overline{\lim}_{r\to\infty} \frac{\log T^{-1}_{P_0[k]} T_{P_0 f]}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[g]}(r)} \le \frac{\rho_p^{L^*}(f,k)}{\lambda_p^{L^*}(g,h)}.$$

The proof of the above theorem is omitted.

THEOREM 3.22. *Let* f *and* g *be any two transcendental meromorphic functions both of finite order or of non-zero lower order with*

$$\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4 \quad \textit{and} \quad \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$$

respectively. Also let h *and* k *be any two transcendental entire functions both of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$ *and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$. *If* $0 < \lambda_p^{L^*}(f,k) \le \rho_p^{L^*}(f,k) < \infty$ *and* $0 < \lambda_p^{L^*}(g,h) \le \rho_p^{L^*}(g,h) < \infty$ *where* p *is any positive integer, then*

$$\frac{\lambda_p^{L^*}(f,k)}{\rho_p^{L^*}(g,h)} \le \underline{\lim}_{r\to\infty} \frac{\log T^{-1}_{M[k]} T_{M f]}(r)}{\log T^{-1}_{M[h]} T_{M[g]}(r)} \le \min\left\{\frac{\lambda_p^{L^*}(f,k)}{\lambda_p^{L^*}(g,h)}, \frac{\rho_p^{L^*}(f,k)}{\rho_p^{L^*}(g,h)}\right\}$$

$$\le \max\left\{\frac{\lambda_p^{L^*}(f,k)}{\lambda_p^{L^*}(g,h)}, \frac{\rho_p^{L^*}(f,k)}{\rho_p^{L^*}(g,h)}\right\} \le \overline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T^{-1}_{P[h]} T_{P[g]}(r)} \le \frac{\rho_p^{L^*}(f,k)}{\lambda_p^{L^*}(g,h)}.$$

In the line of Theorem 3.21 and with the help of Lemma 2.4 one may easily establish the conclusion of the above theorem and therefore its proof is omitted.

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