# A SHORT NOTE ON THE HYERS-ULAM STABILITY IN MULTI-VALUED DYNAMICS 

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#### Abstract

In this paper, we consider the Hyers-Ulam stability on multi-valued dynamics. For a generalized n-dimensional quadratic set-valued functional equation, we prove the Hyers-Ulam stability for the functional equation in multi-valued dynamics.


## 1. Introduction

The aim of this article is to establish the Hyers-Ulam stability of the generalized quadratic set-valued functional equation. The original stability problem of functional equation concerning group homomorphisms had been first raised by S. M. Ulam [25]. D. H. Hyers [12] gave a first affirmative partial answer to the question of S. M. Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mapping. Th. M. Rassias [22] proved the stability of the linear mapping by being a Cauchy difference of $\varepsilon\left(\|x\|^{p}+\|y\|^{p}\right)$ for some $\varepsilon \geq 0$ and $0 \leq p<1$. J. M. Rassias [21] investigated the same problem with $\varepsilon\left(\|x\|^{p} \cdot\|y\|^{p}\right)$. Thereafter, P. Găvruta [11] provided a generalization of Th. M. Rassias' theorem in which replaced the bound $\varepsilon\left(\|x\|^{p}+\|y\|^{p}\right)$ by a general control function $\phi(x, y)$ for the existence of a unique linear mapping. The functional equation $f(x+y)+f(x-y)=2 f(x)+2 f(y)$ is called the quadratic functional equation and every solution of the quadratic functional equation is called a quadratic function.

The Hyers-Ulam stability of quadratic functional equation was proved by F. Skof [24] for function $f: E_{1} \rightarrow E_{2}$ where $E_{1}$ is normed space and

[^0]$E_{2}$ is a Banach space. P. W. Cholewa [5] extended Skof's theorem by replacing $X$ by an abelian group. Skof's result was generalized by S. Czerwik [10]. He proved the generalized Hyers-Ulam stability of quadratic functional equation in the spirit of Rassias approach. Chu et al. [6] extended the quadratic functional equation to the following generalized form
${ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right)+{ }_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right)=\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)$,
where $n \geq 3$ and $2 \leq m \leq n-1$. In $[6,7]$, they investigated the HyersUlam stability for the generalized quadratic functional equation. Lu and Park [15] defined the additive set-valued functional equations and proved the Hyers-Ulam stability of the set-valued functional equations. Park et al. [17] futher investigated stability problems of the Jensen additive, quadratic, cubic and quartic set-valued functional equation. Kenary et al. [14] proved the stability for various types of the set-valued functional equation using the fixed point alternative.

In [2], Brzdek investigated some earlier classical results concerning the stability of the additive Cauchy equation. He also disprove a conjecture of Th. M. Rassias and present a new method for proving stability results for functional equations in [3]. In [18, 19], Piszczek obtained some results of stability of functional equation in some classes of multi-valued functions.

Recently, Chu and Yoo [9] investigated the Hyers-Ulam stability of the $n$-dimensional additive set-valued functional equation. In [8], they also investigated the Hyers-Ulam stability of the $n$-dimensional cubic set-valued functional equation.

Now we briefly introduce some definitions and notations which are needed to prove main theorems. Let $C B(Y)$ be the set of all closed bounded subsets of $Y$ and $C C(Y)$ the set of all closed convex subsets of $Y$. Let $C B C(Y)$ be the set of all closed bounded convex subsets of $Y$. For elements $A, B$ of $C C(Y)$ and $\alpha, \beta \in \mathbb{R}^{+}$, we denote $A \oplus B:=\overline{A+B}$. If $A$ is convex, then we obtain that $(\alpha+\beta) A=\alpha A+\beta A$ for all $\alpha, \beta \in \mathbb{R}^{+}$. And ${ }_{n} \mathrm{C}_{m}$ is defined by ${ }_{n} \mathrm{C}_{m}=\frac{n!}{(n-m)!m!}$. Let $f: X \rightarrow C B C(Y)$ be a mapping. The quadratic set-valued functional equation is defined by

$$
\begin{equation*}
f(x+y) \oplus f(x-y)=2 f(x) \oplus 2 f(y) \tag{1.1}
\end{equation*}
$$

for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is said to be a quadratic set-valued mapping. In the present paper, we define the generalized $n$-dimensional quadratic set-valued functional equation and investigate the Hyers-Ulam-Rassias stability of the functional equation as follows

$$
\begin{align*}
{ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) & \oplus{ }_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right)  \tag{1.2}\\
& =\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)
\end{align*}
$$

where $n \geq 3$ and $2 \leq m \leq n-1$. Every solution of the generalized n-dimensional quadratic set-valued functional equation is called a $n$ dimensional quadratic set-valued mapping.

In the next section, to obtain the Hyers-Ulam-Rassias stability of a generalized n-dimensional quadratic functional equation, we use the most popular method induced from the completeness of the phase spaces and another method to gain the stability which is called the fixed point method.

Before we deal with the method, we need a terminology. For a set $X$, we say a function $d: X \times X \rightarrow[0, \infty)$ a generalized metric on $X$ if $d$ satisfies the following properties:
(1) $d(x, y)=0$ if and only if $x=y$;
(2) $d(x, y)=d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y)+d(y, z)$ for all $x, y, z \in X$.

## 2. Stability of the quadratic set-valued functional equation

In this section, we first give basic definitions to prove main theorems and prove the Hyers-Ulam-Rassias stability.

For $A, B \in C B(Y)$, the Hausdorff distance $d_{H}(A, B)$ is defined by

$$
d_{H}(A, B):=\inf \left\{\alpha \geq 0 \mid A \subseteq B+\alpha B_{Y}, B \subseteq A+\alpha B_{Y}\right\}
$$

where $B_{Y}$ is the closed unit ball in $Y$.
In [4], it was proved that a Hausdorff metric space $\left(C B C(Y), \oplus, d_{H}\right)$ is a complete metric semigroup. Rådström [20] proved that $(C B C(Y), \oplus$, $\left.d_{H}\right)$ is isometrically embedded in a Banach space. To prove main theorems, we need the next remark which states fundamental properties for the Hausdorff distance.

Remark 2.1. Let $A, A^{\prime}, B, B^{\prime}, C \in C B C(Y)$ and $\alpha>0$. Then we have that
(1) $d_{H}\left(A \oplus A^{\prime}, B \oplus B^{\prime}\right) \leq d_{H}(A, B)+d_{H}\left(A^{\prime}, B^{\prime}\right)$;
(2) $d_{H}(\alpha A, \alpha B)=\alpha d_{H}(A, B)$;
(3) $d_{H}(A, B)=d_{H}(A \oplus C, B \oplus C)$.

Now, we prove the Hyers-Ulam stability of the $n$-dimensional quadratic set-valued functional equation.

Theorem 2.2. Let $n \geq 3$ be an integer and let $\phi: X^{n} \rightarrow[0, \infty)$ be a function such that

$$
\begin{equation*}
\sum_{i=0}^{\infty} \frac{1}{4^{i}} \phi\left(2^{i} x_{1}, \ldots, 2^{i} x_{n}\right)<\infty \tag{2.1}
\end{equation*}
$$

for all $x_{1}, \ldots, x_{n} \in X$. Suppose that $f: X \longrightarrow\left(C B C(Y), d_{H}\right)$ is an even set-valued mapping with $f(0)=\{0\}$ and

$$
\begin{align*}
& d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right),\right.  \tag{2.2}\\
&\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \phi\left(x_{1}, \ldots, x_{n}\right)
\end{align*}
$$

for all $x_{1}, \ldots, x_{n} \in X$. Then for any $m \in\{2,3, \ldots, n-1\}$, there exists a unique n-dimensional quadratic set-valued mapping $Q: X \rightarrow$ $\left(C B C(Y), d_{H}\right)$ such that

$$
\begin{equation*}
d_{H}(f(x), Q(x)) \leq \frac{1}{4_{n-3} C_{m-2}} \sum_{i=0}^{\infty} \frac{1}{4^{i}} \phi\left(2^{i} x,-2^{i} x, 2^{i} x, 0, \ldots, 0\right) \tag{2.3}
\end{equation*}
$$

for all $x \in X$.
Proof. Put $x_{1}=x, x_{2}=-x, x_{3}=x$ and $x_{4}=x_{5}=\cdots=x_{n}=0$ in (2.2). We have

$$
\begin{align*}
d_{H}\left({ }_{n-2} C_{m-2} f(x)\right. & \oplus 3_{n-2} C_{m-1} f(x), 3_{n-3} C_{m-1} f(x) \\
& \left.\oplus{ }_{n-3} C_{m-3} f(x) \oplus{ }_{n-3} C_{m-2} f(2 x)\right)  \tag{2.4}\\
& \leq \phi(x,-x, x, 0, \ldots, 0)
\end{align*}
$$

for all $x \in X$. From the condition of remark 2.1, we obtain

$$
\begin{equation*}
d_{H}\left(f(x), \frac{1}{4} f(2 x)\right) \leq \frac{1}{4_{n-3} C_{m-2}} \phi(x,-x, x, 0, \ldots, 0) \tag{2.5}
\end{equation*}
$$

for all $x \in X$. Replace $x$ by $2 x$ and divide by 4 in (2.5). Then we get

$$
\begin{align*}
d_{H}\left(\frac{1}{4} f(2 x) \quad,\right. & \left.\frac{1}{4^{2}} f(4 x)\right) \leq  \tag{2.6}\\
& \frac{1}{4^{2}{ }_{n-3} C_{m-2}} \phi(2 x,-2 x, 2 x, 0, \ldots, 0)
\end{align*}
$$

for all $x \in X$. From (2.5) and (2.6), we obtain

$$
\begin{align*}
d_{H}\left(f(x), \frac{1}{4^{2}} f(4 x)\right) \leq & \frac{1}{4_{n-3} C_{m-2}} \phi(x,-x, x, 0, \ldots, 0)  \tag{2.7}\\
& +\frac{1}{4^{2}{ }_{n-3} C_{m-2}} \phi(2 x,-2 x, 2 x, 0, \ldots, 0)
\end{align*}
$$

for all $x \in X$. Using the induction on $i$, we get that

$$
\begin{align*}
d_{H}\left(f(x), \frac{1}{4^{s}} f\left(2^{s} x\right)\right) & \leq  \tag{2.8}\\
& \frac{1}{4_{n-3} C_{m-2}} \sum_{i=0}^{s-1} \frac{1}{4^{i}} \phi\left(2^{i} x,-2^{i} x, 2^{i} x, 0, \ldots, 0\right)
\end{align*}
$$

for any positive integer $s$ and for all $x \in X$.
For all integer $r$ and $l(r>l>0)$, we have

$$
\begin{align*}
d_{H}\left(\frac{1}{4^{r}} f\left(2^{r} x\right), \frac{1}{4^{l}} f\left(2^{l} x\right)\right) & \leq  \tag{2.9}\\
& \frac{1}{4_{n-3} C_{m-2}} \sum_{k=l}^{r-1} \frac{1}{4^{k}} \phi\left(2^{k} x,-2^{k} x, 2^{k} x, 0, \ldots, 0\right)
\end{align*}
$$

for all $x \in X$. Since the right-hand side of the inequality (2.9) tends to zero as $k$ tends to infinity, the sequence $\left\{\frac{f\left(2^{s} x\right.}{4^{s}}\right\}$ is a Cauchy sequence in $\left(C B C(Y), d_{H}\right)$. Therefore, we can define a mapping $Q: X \rightarrow$ $\left(C B C(Y), d_{H}\right)$ as $Q(x):=\lim _{s \rightarrow \infty} \frac{1}{4^{s}} f\left(2^{s} x\right)$ for all $x \in X$. Now, we show that $Q: X \rightarrow\left(C B C(Y), d_{H}\right)$ is a quadratic set-valued mapping. By taking $x_{1}=\cdots=x_{n}=0$ in (1.2), we have

$$
{ }_{n-2} C_{m-2} Q(0) \oplus n_{n-2} C_{m-1} Q(0)={ }_{n} C_{m} Q(0) .
$$

Then we obtain

$$
\frac{(m-1)(n-1)!}{m!(n-m-1)!} Q(0)=\{0\} .
$$

Since $n \geq 3, Q(0)=\{0\}$. By putting $x_{1}=x, x_{2}=-y, x_{3}=y$ and $x_{4}=\cdots=x_{n}=0$ in (1.2), we have

$$
\begin{aligned}
{ }_{n-2} C_{m-2} Q(x) \oplus & { }_{n-2} C_{m-1} Q(x) \oplus 2_{n-2} C_{m-1} Q(y) \\
= & { }_{n-3} C_{m-2}(Q(x+y) \oplus Q(x-y)) \oplus{ }_{n-3} C_{m-3} Q(x) \\
& \oplus{ }_{n-3} C_{m-1} Q(x) \oplus 2_{n-3} C_{m-1} Q(y) .
\end{aligned}
$$

Hence we may have ${ }_{n-3} C_{m-2}(Q(x+y) \oplus Q(x-y))=2_{n-3} C_{m-2}(Q(x) \oplus$ $Q(y)$ ), that is, $Q$ is a quadratic. Also, a mapping $Q$ satisfies

$$
\begin{aligned}
&{ }_{n-2} \mathrm{C}_{m-2} Q\left(\sum_{j=1}^{n} x_{j}\right) \oplus_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} Q\left(x_{i}\right) \\
&=\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} Q\left(x_{i_{1}}+\cdots+x_{i_{m}}\right),
\end{aligned}
$$

where $n \geq 3$ is an integer and $2 \leq m \leq n-1$.
Now, letting $l=0$ and taking the limit $r \rightarrow \infty$ in (2.9), we obtain the inequality (2.3).

To prove the uniqueness of the n-dimensional quadratic set-valued mapping, we assume that $Q^{\prime}: X \rightarrow\left(C B C(Y), d_{H}\right)$ be another ndimensional quadratic set-valued mapping satisfying (2.3). Then

$$
\begin{align*}
d_{H}\left(Q(x), Q^{\prime}(x)\right) & \leq d_{H}(Q(x), f(x)) \oplus d_{H}\left(f(x), Q^{\prime}(x)\right)  \tag{2.10}\\
& \leq \frac{2^{1-2 r}}{4_{n-3} C_{m-2}} \sum_{i=0}^{r-1} \frac{1}{4^{i}} \phi\left(2^{i} x,-2^{i} x, 2^{i} x, 0, \ldots, 0\right)
\end{align*}
$$

for all $x \in X$. Taking the limit as $r \rightarrow \infty$ in (2.10), we have $Q(x)=Q^{\prime}(x)$ for all $x \in X$. This completes the proof.

REMARK 2.3. Let $n \geq 3$ be an integer. Consider a change of control function $\phi$ in the theorem 2.2. Let $\phi: X^{n} \rightarrow[0, \infty)$ be a function such that

$$
\begin{equation*}
\sum_{i=0}^{\infty} 4^{i} \phi\left(\frac{x_{1}}{2^{i}}, \ldots, \frac{x_{n}}{2^{i}}\right)<\infty \tag{2.11}
\end{equation*}
$$

for all $x_{1}, \ldots, x_{n} \in X$. Suppose that $f: X \longrightarrow\left(C B C(Y), d_{H}\right)$ is an even set-valued mapping with $f(0)=\{0\}$ and

$$
\begin{align*}
& d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right),\right.  \tag{2.12}\\
&\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \phi\left(x_{1}, \ldots, x_{n}\right)
\end{align*}
$$

for all $x_{1}, \ldots, x_{n} \in X$. Then for any $m \in\{2,3, \ldots, n-1\}$, there exists a unique n-dimensional quadratic set-valued mapping $Q: X \rightarrow$ $\left(C B C(Y), d_{H}\right)$ such that

$$
\begin{align*}
& d_{H}(f(x), Q(x))  \tag{2.13}\\
& \leq \frac{1}{{ }_{n-3} C_{m-2}} \sum_{i=0}^{\infty} 4^{i} \phi\left(\frac{1}{2^{i}} x,-\frac{1}{2^{i}} x, \frac{1}{2^{i}} x, 0, \ldots, 0\right)
\end{align*}
$$

for all $x \in X$.
Corollary 2.4. Let $n \geq 3$ be an integer, $0<p<2$ and $\theta \geq 0$ be real numbers. Suppose that $f: X \rightarrow\left(C B C(Y), d_{H}\right)$ is an even mapping satisfying

$$
\begin{aligned}
& d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right),\right. \\
&\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}
\end{aligned}
$$

for all $x_{1}, \ldots, x_{n} \in X$ and $m \in\{2,3, \ldots, n-1\}$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q: X \rightarrow$ $\left(C B C(Y), d_{H}\right)$ such that

$$
d_{H}(f(x), Q(x)) \leq \frac{1}{2_{n-3} C_{m-2}} \frac{\theta}{4-2^{p}}\|x\|^{p}
$$

for all $x \in X$.
Proof. The result follows theorem 2.2 by setting $\phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=$ $\theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}$ for all $x_{1}, \ldots, x_{n} \in X$.

Remark 2.5. By setting $\phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}$ for all $x_{1}, \ldots, x_{n} \in X$ in the remark 2.3, we obtain the following statement. Let $n \geq 3$ be an integer, $p>2$ and $\theta \geq 0$ be real numbers. Suppose that $f: X \rightarrow\left(C B C(Y), d_{H}\right)$ is an even mapping satisfying

$$
\begin{aligned}
& d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus{ }_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right),\right. \\
&\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}
\end{aligned}
$$

for all $x_{1}, \ldots, x_{n} \in X$ and $m \in\{2,3, \ldots, n-1\}$. Then there exists a unique n -dimensional quadratic set-valued mapping $Q: X \rightarrow$
$\left(C B C(Y), d_{H}\right)$ such that

$$
d_{H}(f(x), Q(x)) \leq \frac{1}{2_{n-3} C_{m-2}} \frac{\theta}{2^{p}-4}\|x\|^{p}
$$

for all $x \in X$.
Next we use another method closely related to a fixed point theory to prove the Hyers-Ulam stability of the generalized quadratic set-valued functional equation. We first introduce a useful theorem to prove our results. In [16], the following lemma is due to Margolis and Diaz.

Lemma 2.6. Let $(X, d)$ be a complete generalized metric space and let $J: X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each element $x \in X$, either

$$
d\left(J^{n} x, J^{n+1} x\right)=\infty
$$

for all nonnegative integers $n$ or there exists a positive integer $n_{0}$ such that
(1) $d\left(J^{n} x, J^{n+1} x\right)<\infty, \quad \forall n \geq n_{0}$;
(2) the sequence $\left\{J^{n} x\right\}$ converges to a fixed point $y^{*}$ of $J$;
(3) $y^{*}$ is the unique fixed point of $J$ in the set $Y=\left\{y \in X \mid d\left(J^{n_{0}} x, y\right)<\right.$ $\infty$;
(4) $d\left(y, y^{*}\right) \leq \frac{1}{1-L} d(y, J y)$ for all $y \in Y$.

Generally, the fixed point method is so popular technique to prove the Hyers-Ulam stability. In the set-valued version, we also use this useful method to prove the Hyers-Ulam stability.

Theorem 2.7. Let $2 \leq m \leq n-1$ be an integer. Suppose that an even mapping $f: X \longrightarrow\left(C B C(Y), d_{H}\right)$ with $f(0)=\{0\}$ satisfies the inequality

$$
\begin{align*}
d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus{ }_{n-2} \mathrm{C}_{m-1}\right. & \sum_{i=1}^{n} f\left(x_{i}\right),  \tag{2.14}\\
& \left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \phi\left(x_{1}, \ldots, x_{n}\right)
\end{align*}
$$

for all $x_{1}, \ldots, x_{n} \in X$ and there exists a constant $L$ with $0<L<1$ for which the function $\phi: X^{n} \rightarrow[0, \infty)$ satisfies

$$
\begin{equation*}
\phi(2 x,-2 x, 2 x, 0, \ldots, 0) \leq 4 L \phi(x,-x, x, 0, \ldots, 0) \tag{2.15}
\end{equation*}
$$

for all $x \in X$. Then there exists a unique $n$-dimensional quadratic setvalued mapping $Q: X \rightarrow\left(C B C(Y), d_{H}\right)$ such that

$$
\begin{equation*}
d_{H}(f(x), Q(x)) \leq \frac{1}{4(1-L)} \phi(x,-x, x, 0, \ldots, 0) \tag{2.16}
\end{equation*}
$$

for all $x \in X$.
Proof. Set $x_{1}=x, x_{2}=-x, x_{3}=x$ and $x_{4}=\cdots=x_{n}=0$ in (2.14). Since $f$ is even and the range of $f$ is convex, we have that

$$
\begin{equation*}
d_{H}\left(f(x), \frac{1}{4} f(2 x)\right) \leq \frac{1}{4_{n-3} C_{m-2}} \phi(x,-x, x, 0, \ldots, 0) \tag{2.17}
\end{equation*}
$$

for all $x \in X$.
Let $S:=\{g \mid g: X \rightarrow C B C(Y), g(0)=\{0\}\}$. We define a generalized metric on $S$ defined by

$$
\begin{aligned}
d\left(g_{1}, g_{2}\right):=\inf \left\{\mu \in(0, \infty) \mid d_{H}( \right. & \left.g_{1}(x), g_{2}(x)\right) \\
& \leq \mu \phi(x,-x, x, 0, \ldots, 0), x \in X\}
\end{aligned}
$$

where, as usual, in $f \emptyset:=\infty$.
Now, we define the mapping $J:(S, d) \rightarrow(S, d)$ given by $J g(x)=$ $\frac{1}{4} g(2 x)$ for all $x \in X$. For $g_{1}, g_{2} \in S$, let $d\left(g_{1}, g_{2}\right)<\mu$. Then

$$
d_{H}\left(g_{1}(x), g_{2}(x)\right) \leq \mu \phi(x,-x, x, 0, \ldots, 0)
$$

for all $x \in X$. By (2.15), we have

$$
\begin{aligned}
d_{H}\left(J g_{1}(x), J g_{2}(x)\right) & =\frac{1}{4} d_{H}\left(g_{1}(2 x), g_{2}(2 x)\right) \\
& \leq \frac{1}{4} \mu \phi(2 x,-2 x, 2 x, 0, \ldots, 0) \\
& \leq L \mu \phi(x,-x, x, 0, \ldots, 0)
\end{aligned}
$$

for all $x \in X$.
Therefore, we have that $d\left(J g_{1}, J g_{2}\right) \leq L d\left(g_{1}, g_{2}\right)$ for all $g_{1}, g_{2} \in S$. Hence $J$ is a strictly contractive mapping with the Lipschitz constant $L$. From (2.17), we can obtain that $d(f, J f) \leq \frac{1}{4}$. By theorem 2.6, there exists a unique fixed point $Q: X \rightarrow\left(C B C(Y), d_{H}\right)$ of $J$ such that $\left\{J^{r} f\right\} \rightarrow 0$ as $r \rightarrow \infty$. Then we have

$$
\begin{equation*}
Q(x)=\lim _{r \rightarrow \infty} \frac{1}{4^{r}} f\left(2^{r} x\right) \tag{2.18}
\end{equation*}
$$

for all $x \in X$. Also, from the fixed point alternative, we get $d(f, Q) \leq$ $\frac{1}{1-L} d(J f, f) \leq \frac{1}{4(1-L)}$, which implies the inequality (2.16) holds.

From (2.14) and (2.18), it follows that

$$
\begin{aligned}
d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} Q\left(\sum_{j=1}^{n} x_{j}\right) \oplus{ }_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} Q\left(x_{i}\right)\right. \\
\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} Q\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \\
\leq \lim _{r \rightarrow \infty} \frac{1}{4^{r}} \phi\left(2^{r} x_{1}, \ldots, 2^{r} x_{n}\right)=0
\end{aligned}
$$

for all $x_{1}, \ldots, x_{n} \in X$.
Therefore, $Q$ is a unique n-dimensional quadratic set-valued mapping as desired.

REMARK 2.8. Let $2 \leq m \leq n-1$ be an integer. Suppose that an even mapping $f: X \longrightarrow\left(C B C(Y), d_{H}\right)$ with $f(0)=\{0\}$ satisfies the inequality

$$
\begin{aligned}
d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus{ }_{n-2} \mathrm{C}_{m-1}\right. & \sum_{i=1}^{n} f\left(x_{i}\right), \\
& \left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \phi\left(x_{1}, \ldots, x_{n}\right)
\end{aligned}
$$

for all $x_{1}, \ldots, x_{n} \in X$ and there exists a constant $L$ with $0<L<1$ for which the function $\phi: X^{n} \rightarrow[0, \infty)$ satisfies

$$
\phi\left(\frac{x}{2},-\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right) \leq \frac{L}{4} \phi(x,-x, x, 0, \ldots, 0)
$$

for all $x \in X$. Then there exists a unique n -dimensional quadratic setvalued mapping $Q: X \rightarrow\left(C B C(Y), d_{H}\right)$ such that

$$
d_{H}(f(x), Q(x)) \leq \frac{L}{4-4 L} \phi(x,-x, x, 0, \ldots, 0)
$$

for all $x \in X$.
Corollary 2.9. Let $0<p<2$ and $\theta \geq 0$ be real number. Suppose that $f: X \longrightarrow\left(C B C(Y), d_{H}\right)$ is an even mapping satisfying

$$
\begin{aligned}
& d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right)\right. \\
&\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}
\end{aligned}
$$

for all $x_{1}, \ldots, x_{n} \in X$. Then there exists a unique $n$-dimensional quadratic set-valued mapping $Q: X \rightarrow\left(C B C(Y), d_{H}\right)$ such that

$$
d_{H}(f(x), Q(x)) \leq \frac{3 \theta}{2^{2}-2^{p}}\|x\|^{p}
$$

for all $x \in X$.
Proof. The proof follows from theorem 2.7 by setting $\phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=$ $\theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}$ for every $x_{1}, \ldots, x_{n} \in X$. Then we can choose $L=2^{p-2}$ and we get the desired results.

REMARK 2.10. In remark 2.8, we set $\phi\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}$ for every $x_{1}, \ldots, x_{n} \in X$. Then we obtain the following statement. Let $p>2$ and $\theta \geq 0$ be real number. Suppose that $f: X \longrightarrow\left(C B C(Y), d_{H}\right)$ is an even mapping satisfying

$$
\begin{aligned}
& d_{H}\left({ }_{n-2} \mathrm{C}_{m-2} f\left(\sum_{j=1}^{n} x_{j}\right) \oplus{ }_{n-2} \mathrm{C}_{m-1} \sum_{i=1}^{n} f\left(x_{i}\right)\right. \\
&\left.\sum_{1 \leq i_{1}<\cdots<i_{m} \leq n} f\left(x_{i_{1}}+\cdots+x_{i_{m}}\right)\right) \leq \theta \sum_{i=1}^{n}\left\|x_{i}\right\|^{p}
\end{aligned}
$$

for all $x_{1}, \ldots, x_{n} \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $Q: X \rightarrow\left(C B C(Y), d_{H}\right)$ such that

$$
d_{H}(f(x), Q(x)) \leq \frac{\theta}{2^{p}-2^{2}}\|x\|^{p}
$$

for all $x \in X$.
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