JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 1, February 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.75

EXISTENCE OF NON-CONSTANT POSITIVE SOLUTIONS FOR A RATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH DISEASE IN THE PREY

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Abstract. In this paper, we consider ratio-dependent predator-prey models with disease in the prey under Neumann boundary condition. We investigate sufficient conditions for the existence and non-existence of non-constant positive steady-state solutions by the effects of the induced diffusion rates.

1. Introduction

In this paper, we investigate the existence of non-constant positive steady-states of the following ratio-dependent predator-prey system with disease in the prey:

(1.1)
$$\begin{cases} u_t - d\Delta u = u(a - au - av - v) & \\ v_t - d\Delta v = v(u - b_2 - \frac{lw}{mw+v}) & \\ w_t - d_3\Delta w = w(-b_1 + \frac{klv}{mw+v}) & \text{in } (0,\infty)\times\Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x),\ \ v(0,x) = v_0(x),\ \ w(0,x) = w_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$; the given coefficients a, m, l, b_i, d and d_3 are positive constants; ν is the outward directional derivative normal to $\partial\Omega$; and the nonnegative initial functions $u_0(x)$, $v_0(x)$ and $w_0(x)$ are not identically zero in Ω. Here u, v

Received January 05, 2018; Accepted January 19, 2018.
2010 Mathematics Subject Classification: 35J60, 35Q80.
Key words and phrases: positive steady-states, ratio-dependent, disease-free, non-constant positive solution.
This work was supported by the research grant of Cheongju University (2016.03. 01.∼2018.02.28.).

and w represent the population densities of susceptible prey, the infected prey and the predator, respectively.

The ratio-dependent predator-prey models have been proposed first by R. Arditi and L. R. Ginzburg in [2]. The actual evidence and justification of the ratio-dependent predator-prey models can be found in [3, 4, 6, 7], and the related models have been widely studied for spatially homogeneous case [9, 10, 11, 12] and for spatially inhomogeneous case [5, 16]. For the dynamics of diffusive ratio-dependent three species predator-prey interaction systems have been partially studied [13]. In [1], the authors investigate the asymptotic behavior of positive constant solutions and the non-negative equilibria to the system (1.1).

The main concern of this paper is to study the existence and non-existence of positive steady-states of (1.1), that is, we investigate the existence and non-existence of non-constant positive solutions to the following elliptic system

$$\begin{cases} -d\Delta u = u[a - au - av - v] \\ -d\Delta v = v[u - b_2 - \frac{lw}{mw+v}] \\ -d_3\Delta w = w[-b_1 + \frac{klv}{mw+v}] & \text{in } \Omega, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Note that (1.1) have the following four non-negative equilibria:

(i) $\mathbf{e_0} = (0,0,0)$,
(ii) $\mathbf{e_1} = (1,0,0)$,
(iii) $\mathbf{e_2} = \left(b_2, \frac{a(1-b_2)}{a+1}, 0\right)$ when $b_2 < 1$,
(iv) $\mathbf{u}_* = (u_*, v_*, w_*)$, where $u_* = b_2 + \frac{kl-b_1}{km}$, $v_* = \frac{a}{1+a}(1-u_*)$ and $w_* = \frac{kl-b_1}{b_1 m} v_*$ when $kl > b_1$ and $b_2 + \frac{kl-b_1}{km} < 1$.

In this paper, we define $\frac{vw}{mw+v} = 0$ at $(v,w) = (0,0)$ to avoid the singularity at $(0,0)$. Note that $\lim_{(v,w)\to(0,0)} \frac{vw}{mw+v} = 0$.

This paper is organized as follow. In Section 2, we state some useful known results of (1.1) obtained in [1]. Finally, in Section 3, we study the existence and non-existence of non-constant positive solutions of (1.2).

2. Preliminaries

In this section, we state some known results of the system (1.1) in [1], which is useful in the later section.

First, the following theorem shows that the solution of (1.1) is uniformly bounded, and thus no blow up occurs.

THEOREM 2.1. *Assume that $kl > b_1$. Then the non-negative solution (u, v, w) of (1.1) satisfies*

$$0 \le u(t,x) \le B_1,\ 0 \le v(t,x) \le B_2,\ 0 \le w(t,x) \le B_3$$

on $[0,\infty) \times \overline{\Omega}$, *where*

$$\begin{aligned} B_1 &:= \max\{1,\ ||u_0||_\infty\}, \\ B_2 &:= \max\left\{\frac{a+b_2}{(1+a)b_2}B_1,\ \frac{1}{1+a}||u_0||_\infty + ||v_0||_\infty\right\}, \\ B_3 &:= \max\left\{||w_0||_\infty,\ \frac{kl-b_1}{b_1 m}B_2\right\}. \end{aligned}$$

Proof. See Theorem 2.1 in [1]. □

Now we introduce the following notations, similarly as in [13] and [15].

NOTATION. (i) μ_i denotes the eigenvalue of $-\Delta$ on Ω under Neumann boundary condition.
(ii) $E(\mu_i)$ is the eigenspace corresponding to μ_i.
(iii) $\{\varphi_{ij} : j = 1, ..., \dim E(\mu_i)\}$ is an orthonormal basis of $E(\mu_i)$.
(iv) $\mathbf{X_{ij}} = \{\mathbf{c} \cdot \varphi_{ij} | \mathbf{c} \in \mathbb{R}^3\}$
(v) $\mathbf{X} = \left\{\mathbf{u} = (u, v, w) \in [C^1(\overline{\Omega})]^3 | \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \ \ on \ \ \partial\Omega\right\}.$

We point out that $\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X_i}$, where $\mathbf{X_i} = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X_{ij}}$ (for more details, see [13, 15]).

THEOREM 2.2. (*Asymptotic stability at* $\mathbf{u}_*$) *Assume that one of the followings holds:*

(i) $(1-b_2)m \ge l,\ k > b_1/l,$
(ii) $l > (1-b_2)m,\ \frac{b_1}{\sqrt{l(l-(1-b_2)m)}} \ge k > b_1/l.$

Further, if

$$a > \frac{b_1}{kl}\frac{kl-b_1}{b_2 km + kl - b_1}\max\left\{1,\ \frac{k^2 l(m(1-b_2)-l) + b_1^2 + (kl-b_1)b_1}{k^2 l(m(1-b_2)-l) + b_1^2 + (kl-b_1)b_1 km}\right\},$$

then the positive equilibrium point $\mathbf{u}_*$ *of (1.1) is locally asymptotically stable.*

Proof. See Theorem 2.4 in [1]. □

3. Positive coexistence of (1.2)

In this section, we show the existence of a non-constant positive solution of elliptic system (1.2) by using the degree theory. To do this, it is necessary to estimate an *a-priori* bound of solutions for (1.2).

3.1. An *a priori* bound

First, we give an *a-priori* bound for (1.2).

THEOREM 3.1. *Assume $kl > b_1$. Then the non-negative solution (u, v, w) of (1.1) satisfies*

$$\limsup_{t\to\infty} u \leq 1,\ \limsup_{t\to\infty} v \leq \frac{a+b_2}{b_2(a+1)},$$
$$\limsup_{t\to\infty} w \leq \left(\frac{kl-b_1}{b_1 m}\right)\frac{a+b_2}{b_2(a+1)} \quad on\ \overline{\Omega}.$$

Proof. See Theorem 2.2 in [1]. □

Next we estimate a positive lower bound of classical positive solutions for (1.2).

THEOREM 3.2. *Assume that $1 - b_2 - \frac{kl-b_1}{km} > 0$ and $kl > b_1$. Let $d \in [d^*, \infty)$ and $d_3 \in [d^*, d_3^*]$ for a fixed positive d^* and d_3^*. Then there exists a positive constant $C_\sharp(N, \Omega, d^*, d_3^*, \Gamma)$ such that a positive solution (u, v, w) of (1.2) satisfies*

$$\min_{\overline{\Omega}} u(x),\ \ \min_{\overline{\Omega}} v(x),\ \ \min_{\overline{\Omega}} w(x) > C_\sharp, \tag{3.1}$$

if

$$1 - b_2 - \frac{l}{m} - \frac{kl-b_1}{m} > -2\sqrt{b_1 m}\sqrt{1-b_2}. \tag{3.2}$$

Proof. It is easy to see that $\frac{f_1}{d}$, $\frac{f_2}{d}$, $\frac{f_3}{d_3} \in C(\overline{\Omega})$ for d, $d_3 \geq d^*$. By using Harnack inequality, there exists a positive constant $C_*(N, \Omega, d^*, \Gamma)$ such that

$$\max_{\overline{\Omega}} u \leq C_* \min_{\overline{\Omega}} u,\ \ \max_{\overline{\Omega}} v \leq C_* \min_{\overline{\Omega}} v,\ \ \max_{\overline{\Omega}} w \leq C_* \min_{\overline{\Omega}} w. \tag{3.3}$$

Suppose by contradiction that (3.1) does not hold. Then there are sequences $\{d_n\}$, $\{d_{3,n}\}$; and the corresponding positive solution (u_n, v_n, w_n) of (1.2) such that $d_n \geq d^*$, $d_{3,n} \in [d^*, d_3^*]$ for $n \in \mathbb{N}$, and $\max_{\overline{\Omega}} u_n \to 0$ or $\max_{\overline{\Omega}} v_n \to 0$ or $\max_{\overline{\Omega}} w_n \to 0$ as $n \to \infty$.

By Theorem 2.1, it is easy to see that $||u_n||_\infty$, $||v_n||_\infty$ and $||w_n||_\infty < \infty$ for all $n \geq 1$. By Agmon, Douglis, and Nirenberg inequality,

$$||u_n||_{W^{2,p}} \leq C(||u_n||_{L^p} + ||u_n f_1(u_n, v_n)||_{L^p}) < \infty$$

for all $n \geq 1$, $p \geq 2$, and some positive constant C. Then it follows from Sobolev imbedding theorem that $\{u_n\}$ is also bounded in $C^{1,\alpha}$-norm. Moreover, since $||u_n||_{C^{2,\alpha}} \leq C(||u_n||_{C^{0,\alpha}} + ||u_n f_1(u_n, v_n)||_{C^{0,\alpha}})$ for some constant C depending on α, we see that $\{u_n\}$ is also bounded in $C^{2,\alpha}$-norm by the Schauder estimate. By using similar arguments, one can show that $\{v_n\}$ and $\{w_n\}$ is bounded in $C^{2,\alpha}$-norm. Thus Arzela-Ascoli Theorem shows that there exists a subsequence of $\{(u_n, v_n, w_n)\}$, which is denoted by itself again for a convenience, and nonnegative functions $\widetilde{u}, \widetilde{v}, \widetilde{w} \in C^2(\overline{\Omega})$, such that $(u_n, v_n, w_n) \to (\widetilde{u}, \widetilde{v}, \widetilde{w})$ as $n \to \infty$. Since $\max_{\overline{\Omega}} u_n \to 0$ or $\max_{\overline{\Omega}} v_n \to 0$ or $\max_{\overline{\Omega}} w \to 0$ as $n \to \infty$, $\widetilde{u} \equiv 0$ or $\widetilde{v} \equiv 0$ or $\widetilde{w} \equiv 0$. We have the following five cases:

i) $\widetilde{u} \equiv 0,\ \ \widetilde{v} \not\equiv 0,\ \ \widetilde{w} \not\equiv 0$ or $\widetilde{u} \equiv 0,\ \ \widetilde{v} \not\equiv 0,\ \ \widetilde{w} \equiv 0$,
ii) $\widetilde{u} \not\equiv 0,\ \ \widetilde{v} \equiv 0,\ \ \widetilde{w} \not\equiv 0$ or $\widetilde{u} \equiv 0,\ \ \widetilde{v} \equiv 0,\ \ \widetilde{w} \not\equiv 0$,
iii) $\widetilde{u} \not\equiv 0,\ \ \widetilde{v} \not\equiv 0,\ \ \widetilde{w} \equiv 0$,
iv) $\widetilde{u} \not\equiv 0,\ \ \widetilde{v} \equiv 0,\ \ \widetilde{w} \equiv 0$,
v) $\widetilde{u} \equiv 0,\ \ \widetilde{v} \equiv 0,\ \ \widetilde{w} \equiv 0$.

Case i) Note that v_n, $w_n > 0$ and $\widetilde{u}$, $\widetilde{v}$, $\widetilde{w}$ satisfy the inequality (3.3). Thus $\widetilde{v} > 0$ since $\widetilde{v} \not\equiv 0$. Also since $v_n \to \widetilde{v} > 0$ and $u_n \to 0$ as $n \to \infty$, $v_n f_2(u_n, v_n, w_n) < 0$ for a sufficient large n. By applying Green's first identity to the second equation in (1.2), we have $\int_\Omega v_n f_2(u_n, v_n, w_n) = 0$. This derives a contradiction.

Case ii) since $w_n > 0$ and $\widetilde{v} = 0$, one can similarly show that $f_3(v_n, w_n) < 0$ for a sufficient large n. This is a contradiction to the fact that $\int_\Omega w_n f_3(v_n, w_n) = 0$ for all n.

Case iii) It is obvious that u_n, $v_n > 0$ on $\overline{\Omega}$ for a sufficient large n. First, note that $\widetilde{v} > 0$ as in the case i). Thus $\frac{v_n}{mw_n + v_n} \to 1$ uniformly on $\overline{\Omega}$ since $w_n \to 0$ uniformly on $\overline{\Omega}$, as $n \to \infty$. Since $kl > b_1$, this derives also a contradiction.

Case iv) By applying Green's first identity to the first equation in (1.2) and using the fact that $v_n \to \widetilde{v} \equiv 0$ as $n \to \infty$, we have $0 = \int_\Omega u_n f_1(u_n, v_n) \to \int_\Omega \widetilde{u}(a - a\widetilde{u})$ as $n \to \infty$. Thus shows that $\widetilde{u} \equiv 1$ since $0 < \widetilde{u} \leq 1$.

Consider the following elliptic system under Neumann boundary condition:

$$\begin{cases} -d_n \Delta V_n = V_n f_2(\widetilde{u}, V_n, W_n) \\ -d_{3,n} \Delta W_n = W_n f_3(V_n, W_n) \quad \text{ in } \Omega, \end{cases} \tag{3.4}$$

where $V_n = \frac{v_n}{||v_n||_\infty + ||w_n||_\infty}$ and $W_n = \frac{w_n}{||v_n||_\infty + ||w_n||_\infty}$. By applying Green's first identity, one can get the following integral equations

$$\int_\Omega V_n f_2(\widetilde{u}, V_n, W_n) = 0, \quad \int_\Omega W_n f_3(V_n, W_n) = 0 \;\; for \;\; n \geq 1. \tag{3.5}$$

Similarly as in the case i), there exists a subsequence (V_n, W_n), which is denoted by itself, such that $\lim_{n\to\infty} V_n = \widetilde{V}$ and $\lim_{n\to\infty} W_n = \widetilde{W}$ in $C^2(\overline{\Omega})$. Since $||V_n||_\infty + ||W_n||_\infty = 1$, $||\widetilde{V}||_\infty + ||\widetilde{W}||_\infty = 1$ and $\widetilde{V} + \widetilde{W} > 0$ on $\overline{\Omega}$. And these nonnegative pairs satisfy the Harnack inequality.

Now We assume that $d_n \to D^* \in [d^*, \infty]$ and $d_{3,n} \to D_3^* \in [d^*, d_3^*]$, by taking a subsequence if necessary.

First, consider the case of $D^* < \infty$. If $\widetilde{W} \equiv 0$, then $\widetilde{V} > 0$ on $\overline{\Omega}$ and $\int_\Omega \widetilde{V}(\widetilde{u} - b_2) = 0$. But since $b_2 < 1 \equiv \widetilde{u}$ from the given assumption, this is a contradiction, and thus $\widetilde{W} \not\equiv 0$. If $\widetilde{V} \equiv 0$, then $\widetilde{W} > 0$ and so we have $\int_\Omega -b_1\widetilde{W} = 0$ which is also impossible. Hence $\widetilde{V}$, $\widetilde{W} > 0$ on $\overline{\Omega}$ by Harnack inequality. After taking the limit in (3.5), by subtracting $\int_\Omega \widetilde{V} f_2(1, \widetilde{V}, \widetilde{W}) = 0$ from the equation $\int_\Omega \widetilde{W} f_3(\widetilde{V}, \widetilde{W}) = 0$, we have

$$\int_\Omega \left[\frac{b_1 m\widetilde{W}^2 + (1-b_2)\widetilde{V}^2 + (b_1 + m(1-b_2) - l(k+1))\widetilde{V}\widetilde{W}}{m\widetilde{W} + \widetilde{V}} \right] = 0.$$

On the other hand, by using (3.2), one can easily show that the above integral is positive. This derives a contradiction.

Next, consider the case of $D^* = \infty$, $\widetilde{V}$. As in the case of $D^* < \infty$, one can show that $\widetilde{W} = A$ for some positive constant A. Thus $\widetilde{W} \equiv \frac{kl - b_1}{b_1 m} A$ since $\widetilde{W}$ satisfies

$$\begin{cases} -D_3^* \Delta \widetilde{W} = \widetilde{W} f_3(A, \widetilde{W}) & \text{in } \Omega, \\ \frac{\partial \widetilde{W}}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $B = 1 - b_2 - \frac{l\widetilde{W}}{m\widetilde{W} + A} = 0$, then $B = 0$ since the first integral equation in (3.5) holds as $n \to \infty$, and thus we have $1 - b_2 - \frac{kl - b_1}{km} = 0$, which derives a contradiction.

Case v) Consider the system (3.4) with $\widetilde{u} \equiv 0$. Then one can get $\widetilde{V} + \widetilde{W} > 0$ on $\overline{\Omega}$ and (3.5) with $\widetilde{u} = 0$.

If $D^* = \infty$, then $\widetilde{V} \equiv A > 0$ and $\widetilde{W} \equiv \frac{kl - b_1}{b_1 m} A$ for some positive constant A. But, since $\int_\Omega A\Big(-b_2 - \frac{l\widetilde{W}}{m\widetilde{W} + A}\Big) = 0$, A must be zero. This is a contradiction.

Now assume that $D^* < \infty$. If $\widetilde{W} \equiv 0$ or $\widetilde{V} \equiv 0$, then $\int_\Omega \widetilde{V}(-b_2) = 0$ or $\int_\Omega \widetilde{W}(-b_1) = 0$, and thus $\widetilde{V}$ and $\widetilde{W} > 0$ on $\overline{\Omega}$. By the way, $-D^*\Delta\widetilde{V} = \widetilde{V}\Big[-b_2 - \frac{l\widetilde{W}}{m\widetilde{W}+\widetilde{V}}\Big] < 0$ in Ω and $\frac{\partial\widetilde{V}}{\partial\eta} = 0$ on $\partial\Omega$. Moreover, by the strong maximum principle and Hopf Boundary Lemma, we see that $\frac{\partial\widetilde{V}}{\partial\eta}(p) > 0$ at some point $p \in \partial\Omega$.(If not, then $\widetilde{V}$ is a constant in Ω.) Hence $\widetilde{V}$ is a nonnegative constant since $\frac{\partial\widetilde{V}}{\partial\eta} = 0$ on $\partial\Omega$. As in the case of $D^* = \infty$, this derives a contradiction. □

3.2. Nonexistence of non-constant positive solution

In this subsection, we investigate the nonexistence of non-constant positive solution of (1.2).

THEOREM 3.3. *Assume that $d_3\mu_2 > kl - b_1 > 0$. If there exists a positive constant $\widetilde{D}(N,\Omega,d_3,\Gamma)$ such that $d > \widetilde{D}$, then (1.2) has no non-constant positive solution, where μ_2 is a eigenvalue defined in Notation.*

Proof. Define $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u$, $\overline{v} = \frac{1}{|\Omega|}\int_\Omega v$ and $\overline{w} = \frac{1}{|\Omega|}\int_\Omega w$. By multiplying $(u-\overline{u})$, $(v-\overline{v})$ and $(w-\overline{w})$ to the first, second and third equation in (1.2), respectively, we have

$$\left\{\begin{array}{ll} -d(u-\overline{u})\Delta u & = u(u-\overline{u})f_1(u,v), \\ -d(v-\overline{v})\Delta v & = v(v-\overline{v})f_2(u,v,w), \\ -d_3(w-\overline{w})\Delta w & = w(w-\overline{w})f_3(v,w). \end{array}\right. \tag{3.6}$$

Therefore by Green's first identity and Cauchy inequality, we have

(3.7)

$$\begin{aligned}
&\int_\Omega (d|\nabla u|^2 + d|\nabla v|^2 + d_3|\nabla w|^2) \\
&= \int_\Omega \Big[(u-\overline{u})\Big(uf_1(u,v) - \overline{u}f_1(\overline{u},\overline{v})\Big) + (v-\overline{v})\Big(vf_2(u,v,w) \\
&\qquad -\overline{v}f_2(\overline{u},\overline{v},\overline{w})\Big) + (w-\overline{w})\Big(wf_3(v,w) - \overline{w}f_3(\overline{v},\overline{w})\Big)\Big] \\
&= \int_\Omega \Big[a(u-\overline{u})^2 - a(u-\overline{u})^2(u+\overline{u}) - (a+1)\overline{v}(u-\overline{u})^2 \\
&\qquad -(a+1)u(v-\overline{v})(u-\overline{u}) + v(u-\overline{u})(v-\overline{v}) + \overline{u}(v-\overline{v})^2 \\
&\qquad -b_2(v-\overline{v})^2 - \frac{lmw\overline{w}(v-\overline{v})^2}{(mw+v)(m\overline{w}+\overline{v})} - \frac{lv\overline{v}(v-\overline{v})(w-\overline{w})}{(mw+v)(m\overline{w}+\overline{v})} - b_1(w-\overline{w})^2 \\
&\qquad +\frac{klmw\overline{w}(v-\overline{v})(w-\overline{w})}{(mw+v)(m\overline{w}+\overline{v})} + \frac{klv\overline{v}(w-\overline{w})^2}{(mw+v)(m\overline{w}+\overline{v})}\Big]
\end{aligned}$$

$$\leq \int_\Omega \Big[a(u-\overline{u})^2 + (a+1)|v-\overline{v}|\,|u-\overline{u}| + \tfrac{a+b_2}{b_2(1+a)}|u-\overline{u}|\,|v-\overline{v}| \\ +(v-\overline{v})^2 + l|w-\overline{w}|\,|v-\overline{v}| + \tfrac{kl}{m}|w-\overline{w}|\,|v-\overline{v}| \\ +(kl-b_1)(w-\overline{w})^2\Big]$$

$$\leq \int_\Omega \Big[a(u-\overline{u})^2 + \tfrac{a+1}{2}(u-\overline{u})^2 + \tfrac{a+1}{2}(v-\overline{v})^2 + \tfrac{a+b_2}{2b_2(a+1)}(u-\overline{u})^2 \\ +\tfrac{a+b_2}{2b_2(a+1)}(v-\overline{v})^2 + (v-\overline{v})^2 + \tfrac{l+kl/m}{2\varepsilon}(v-\overline{v})^2 \\ +\tfrac{(l+kl/m)\varepsilon}{2}(w-\overline{w})^2 + (kl-b_1)(w-\overline{w})^2\Big],$$

where ε is an arbitrary positive constant. Synthetically, we have (3.8)

$$\int_\Omega (d|\nabla u|^2 + d|\nabla v|^2 + d_3|\nabla w|^2) \\ \leq \int_\Omega \Big\{\Big(a+\tfrac{a+1}{2}+\tfrac{a+b_2}{2b_2(1+a)}\Big)(u-\overline{u})^2 + \Big(\tfrac{a+1}{2}+\tfrac{a+b_2}{2b_2(1+a)}+1 \\ +\tfrac{l+kl/m}{2\varepsilon}\Big)(v-\overline{v})^2 + \Big(\tfrac{l+kl/m}{2}\varepsilon + kl - b_1\Big)(w-\overline{w})^2\Big\}.$$

It follows from Poincaré inequality that

$$\int_\Omega (d|\nabla u|^2 + d|\nabla v|^2 + d_3|\nabla w|^2) \\ \geq \int_\Omega d\mu_2(u-\overline{u})^2 + d\mu_2(v-\overline{v})^2 + d_3\mu_2(w-\overline{w})^2.$$

Since $d_3\mu_2 > kl - b_1$, there is a sufficient small ε_0 such that $d_3\mu_2 > \frac{l+kl/m}{2}\varepsilon_0 + kl - b_1$. Let $\widetilde{D} = \frac{1}{\mu_2}\max\Big\{a+\frac{a+1}{2}+\frac{a+b_2}{2b_2(1+a)},\ \frac{a+1}{2}+\frac{a+b_2}{2b_2(1+a)}+1+\frac{l+kl/m}{2\varepsilon_0}\Big\}$, then we conclude that $u=\overline{u}$, $v=\overline{v}$ and $w=\overline{w}$. This completes the proof. □

3.3. Existence of non-constant positive solution

In this subsection, we study the existence of non-constant positive solution using Leray-Schauder Theorem. For the sake of convenience, define $\mathbf{u} = (u(x), v(x), w(x))^T$ and

$$\mathcal{F}(\mathbf{u}) = \begin{pmatrix} (I-d\Delta)^{-1}[u(f_1(u,v)+1)] \\ (I-d\Delta)^{-1}[v(f_2(u,v,w)+1)] \\ (I-d_3\Delta)^{-1}[w(f_3(v,w)+1)] \end{pmatrix}.$$

Then (1.2) becomes $(\mathbf{I}-\mathcal{F})\mathbf{u} = 0$. Notice that $\mathcal{F} : \mathbf{X} \to \mathbf{X}$ is a compact operator and the operator $(I-\rho\Delta)^{-1} : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is compact for some positive constant ρ.

To apply the index theory, we must investigate the eigenvalue of the following problem

$$-(\mathbf{I}-\mathcal{F}_{\mathbf{u}}(\mathbf{u}_*))\Psi=\lambda\Psi,\ \ \Psi\neq\mathbf{0}, \tag{3.9}$$

where $\Psi=(\psi_1,\psi_2,\psi_3)$ and $\mathbf{u}_*$ is the unique positive equilibrium point of (1.2). Then by Leray-Schauder Theorem(Theorem 2.8.1 in [14]),

$$index(I-\mathcal{F},\mathbf{u}_*)=(-1)^{\gamma},\ \ \gamma=\sum_{\lambda>0}n_{\lambda},$$

where n_λ is the multiplicity of all the positive eigenvalues λ of (3.9). After some computations, one can have the following elliptic system which is equivalent to (3.9)

(3.10)
$$\begin{cases} -d(\lambda+1)\Delta\psi_1+(\lambda+au_*)\psi_1+(1+a)u_*\psi_2=0 \\ -d(\lambda+1)\Delta\psi_2+(-v_*)\psi_1+\left(\lambda-\frac{lv_*w_*}{(mw_*+v_*)^2}\right)\psi_2+\frac{lv_*^2}{(mw_*+v_*)^2}\psi_3=0 \\ -d_3(\lambda+1)\Delta\psi_3+\left(-\frac{klmw_*^2}{(mw_*+v_*)^2}\right)\psi_2+\left(\lambda+\frac{klmv_*w_*}{(mw_*+v_*)^2}\right)\psi_3=0 & \text{in } \Omega, \\ \frac{\partial\psi_1}{\partial\eta}=\frac{\partial\psi_2}{\partial\eta}=\frac{\partial\psi_3}{\partial\eta}=0 & \text{on } \partial\Omega, \\ \psi_1\neq 0,\psi_2\neq 0,\psi_3\neq 0. \end{cases}$$

Hence we see that investigating the eigenvalue of (3.9) is equivalent to find positive roots of the characteristic equation $B_k(\lambda)=0$, where

$$B_k(\lambda)=\det\begin{pmatrix} \lambda+\frac{d\mu_k+au_*}{1+d\mu_k} & \frac{(1+a)u_*}{1+d\mu_k} & 0 \\ -\frac{v_*}{1+d\mu_k} & \lambda+\frac{d\mu_k-\frac{lv_*w_*}{(mw_*+v_*)^2}}{1+d\mu_k} & \frac{1}{1+d\mu_k}\frac{lv_*^2}{(mw_*+v_*)^2} \\ 0 & -\frac{1}{1+d_3\mu_k}\frac{klmw_*^2}{(mw_*+v_*)^2} & \lambda+\frac{d_3\mu_k+\frac{klmv_*w_*}{(mw_*+v_*)^2}}{1+d_3\mu_k} \end{pmatrix}$$

for $k\geq 1$. Therefore it follows from Leray-Schauder Theorem that

$$index(I-\mathcal{F},\mathbf{u}_*)=(-1)^{\gamma},\ \ \gamma=\sum_{k\geq 1}\sum_{\lambda_k>0}n_{\lambda_k},$$

where $n_{\lambda_k}=m_{\lambda_k}\dim E(\mu_k)$ and m_{λ_k} is the multiplicity of λ_k as a positive root of $B_k(\lambda)=0$.

In view of Theorem 3.3, we see that there is no nonconstant positive solution of (1.2) if $d>\widetilde{D}$ for a sufficient large $\widetilde{D}$ when $d_3>\frac{kl-b_1}{\mu_2}$. Thus it is necessary to investigate the index value at $\mathbf{u}_*$ when d is a sufficient large.

LEMMA 3.4. *Assume that* $k > \max\{\frac{b_1}{l}, \quad \frac{1}{m}\}$ *and* $\min\{a - \frac{b_1}{kl} + \frac{b_1}{l}m, \quad 1 - b_2 - \frac{kl-b_1}{km}\} > 0$. *If there exists a positive constants* $\widehat{D} = \widehat{D}(N, \Omega, \Gamma, d_3)$ *such that* $d \geq \widehat{D}$, *then*

$$index(I - \mathcal{F}, \mathbf{u}_*) = 1.$$

Proof. It is easy to see that the unique positive constant solution $\mathbf{u}_*$ exists.

Since $\mu_1 = 0$, we have

$$\begin{aligned} B_1(\lambda) = \; & \lambda^3 + \Big(\tfrac{klmv_*w_*}{(mw_*+v_*)^2} + au_* - \tfrac{lv_*w_*}{(mw_*+v_*)^2}\Big)\lambda^2 + \Big((1+a)u_*v_* \\ & + a\tfrac{klmu_*v_*w_*}{(mw_*+v_*)^2} - a\tfrac{lu_*v_*w_*}{(mw_*+v_*)^2}\Big)\lambda + (a+1)\tfrac{klmu_*v_*^2w_*}{(mw_*+v_*)^2}. \end{aligned}$$

Since $km > 1$ and $a - \frac{b_1}{kl} + \frac{b_1}{l}m > 0$, $\lambda > 0$ and $\lambda^2 > 0$, and thus $B_1(\lambda) > 0$ for all $\lambda \geq 0$.

Now assume that $k \geq 2$. Note that $\mu_k > 0$. Then

$$B_k(\lambda) = (\lambda+1)^2\left(\lambda + \frac{d_3\mu_k + \frac{klmv_*w_*}{(mw_*+v_*)^2}}{1 + d_3\mu_k}\right) + \mathcal{O}\Big(\frac{1}{d}\Big).$$

Thus there exists a large positive constant $\widehat{D}$ depending on Γ, N, Ω and d_3 such that $B_k(\lambda) > 0$ for all $d \geq \widehat{D}$ and $\lambda \geq 0$.

Therefore one can conclude that $B_k(\lambda) > 0$ for all $\lambda \geq 0$, $k \geq 1$ and $d \geq \widehat{D}$, and so $\gamma = \sum_{k\geq 1}\sum_{\lambda_k > 0} n_{\lambda_k} = 0$. This implies the desired result. □

LEMMA 3.5. *Assume that* $\widetilde{\mu} \in (\mu_{k_0}, \mu_{k_0+1})$ *for some* $k_0 \geq 2$ *and*

$$\begin{aligned} & 1 < km, \\ & \tfrac{kl-b_1}{km} < 1 - b_2 < \Big(\tfrac{b_1}{kl} + 1\Big)\tfrac{kl-b_1}{km}, \\ & ab_2 < \Big(\tfrac{b_1}{kl} - a\Big)\tfrac{kl-b_1}{km} < b_1\tfrac{m}{l}\tfrac{kl-b_1}{km}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \widetilde{\mu} \; = \; & \tfrac{1}{2d}\Big\{\tfrac{lv_*w_*}{(mw_*+v_*)^2} - au_* \\ & + \sqrt{\Big(au_* - \tfrac{lv_*w_*}{(mw_*+v_*)^2}\Big)^2 - 4\Big((1+a)u_*v_* - \tfrac{alu_*v_*w_*}{(mw_*+v_*)^2}\Big)}\Big\}. \end{aligned} \tag{3.12}$$

Then there exist a positive constant $\widehat{D}_3(N, \Omega, \Gamma, d)$ *such that the polynomial* $B_k(\lambda) = 0$ *has one simple positive root for* $2 \leq k \leq k_0$, *provided that* $d_3 \geq \widehat{D}_3$.

Proof. If $k=1$, the similarly as in Lemma 3.4, we have $k>\max\{\frac{b_1}{l},\ \frac{1}{m}\}$ and $a-\frac{b_1}{kl}+\frac{b_1}{l}m>0$ and thus $B_1(\lambda)>0$ for all $\lambda\geq 0$.

If $k\geq 2$, then

$$B_k(\lambda)=(\lambda+1)\left(\lambda^2+p(\mu_k)\lambda+q(\mu_k)\right)+\mathcal{O}\Big(\frac{1}{d_3}\Big),$$

where

$$p(\mu_k)=\frac{2d\mu_k+au_*-\frac{lv_*w_*}{(mw_*+v_*)^2}}{1+d\mu_k}$$

and

$$q(\mu_k)=\frac{(d\mu_k+au_*)\Big(d\mu_k-\frac{lv_*w_*}{(mw_*+v_*)^2}\Big)+(1+a)u_*v_*}{(1+d\mu_k)^2}.$$

Note that $au_*-\frac{lv_*w_*}{(mw_*+v_*)^2}<0$ and $(1+a)u_*v_*-\frac{alu_*v_*w_*}{(mw_*+v_*)^2}<0$ since $ab_2<\Big(\frac{b_1}{kl}-a\Big)\frac{kl-b_1}{km}$ and $1-b_2<\Big(\frac{b_1}{kl}+1\Big)\frac{kl-b_1}{km}$ in (3.11).

Now we investigate roots of $r_k(\lambda)=\lambda^2+p(\mu_k)\lambda+q(\mu_k)=0$. First, if $p(\mu_k)^2-4q(\mu_k)>0$, then $r_k(\lambda)=0$ has two real roots. In fact, $p^2(\mu_k)-4q(\mu_k)=\frac{1}{(1+d\mu_k)^2}[(G+H)^2-4(G\cdot H+(1+a)u_*v_*)]$, where $G=d\mu_k+au_*$ and $H=d\mu_k-\frac{lv_*w_*}{(mw_*+v_*)^2}$. We know that $(G-H)^2-4(1+a)u_*v_*=\Big(au_*+\frac{lv_*w_*}{(mw_*+v_*)^2}\Big)^2-4(1+a)u_*v_*=\Big(au_*-\frac{lv_*w_*}{(mw_*+v_*)^2}\Big)^2-4\Big[(1+a)u_*v_*-\frac{alu_*v_*w_*}{(mw_*+v_*)^2}\Big]>0$, and hence $r_k(\lambda)=0$ has two real roots.

Next,we investigate the sign of $q(\mu_k)=\frac{1}{(1+d\mu_k)^2}\widetilde{q}(\mu_k)$, where $\widetilde{q}(\mu_k)=d^2\mu_k^2+\Big(au_*-\frac{lv_*w_*}{(mw_*+v_*)^2}\Big)d\mu_k+(1+a)u_*v_*-\frac{alu_*v_*w_*}{(mw_*+v_*)^2}$. It is easy to check that the equation $\widetilde{q}(\mu_k)=0$ has two real roots. Moreover, these two roots has a different sign. Since $\mu_k>0$ for $k\geq 2$, we just handle the only positive one $\widetilde{\mu}$, defined in (3.12).By the given assumption $\widetilde{\mu}\in(\mu_{k_0},\mu_{k_0+1})$, it is concluded that $q(\mu_k)<0$ for $2\leq k\leq k_0$. Consequently, we see that $r_k(\lambda)=0$ has two roots which have a different sign for $2\leq k\leq k_0$.

If $k\geq k_0+1$, then $q(\mu_k)>0$ and $p(\mu_k)>0$ since $\widetilde{\mu}\in(\mu_{k_0},\mu_{k_0+1})$. Thus $r_k(\lambda)=0$ has two negative real roots.

Therefore the coefficients of $B_k(\lambda)$ converge to the coefficients of $(\lambda+1)(\lambda^2+p(\mu_k)\lambda+q(\mu_k))$ as $d_3\to\infty$. If $d_3\geq\widehat{D}_3$ for some positive constant $\widehat{D}_3$, then we have the desired result. □

Remark 3.6. The condition (3.11) guarantees the existence of positive constant $\mathbf{u}_*$. Moreover, this violates the inequality $\frac{b_1}{kl}\frac{kl-b_1}{km}\leq$

$\min\left\{a\Big(b_2+\frac{kl-b_1}{km}\Big),\quad 1-b_2-\frac{kl-b_1}{km}\right\}$ in Theorem 2.2. Hence we may expect the non-constant positive solutions.

Finally, we show the existence of non-constant positive solutions of (1.2) by using Theorem 3.1-3.3 and Lemma 3.4, 3.5.

THEOREM 3.7. *Assume that (3.2), (3.11) and* $\widetilde{\mu}\in(\mu_{k_0},\mu_{k_0+1})$ *for some* $k_0\geq 2$. *If* $\sum_{k=2}^{k_0}\dim E(\mu_k)$ *is odd, then there exist a positive constant* $\widehat{D}_3=\widehat{D}_3(\Gamma,N,\Omega,d)$ *such that (1.2) has at least one non-constant positive solution when* $d_3\geq\widehat{D}_3$.

Proof. For $\theta\in[0,1]$, define

$$\mathcal{F}_\theta(u)=\begin{pmatrix}(I-\theta d\Delta-(1-\theta)\widehat{D}\Delta)^{-1}[u(f_1(u,v)+1)]\\(I-\theta d\Delta-(1-\theta)\widehat{D}\Delta)^{-1}[v(f_2(u,v,w)+1)]\\(I-\theta d_3\Delta-(1-\theta)(\frac{kl-b_1}{\mu_2}+1)\Delta)^{-1}[w(f_3(v,w)+1)]\end{pmatrix},$$

where $\widehat{D}$ is a constant defined in Lemma 3.4 with $\widehat{D}\geq\widetilde{D}$. Here $\widetilde{D}$ is a constant defined in Theorem 3.3. By Theorem 3.1 and 3.2, there exist positive constants $C_\sharp(\Gamma,\widetilde{D},\widehat{D}_3,N,\Omega)$ and $C^\sharp(\Gamma)$ such that the positive solutions of problem $\mathcal{F}_\theta(\mathbf{u})=0$ is contained in $\Lambda=\{\mathbf{u}\in\mathbf{X}|C_\sharp<u,v,w<C^\sharp\}$ for all $\theta\in[0,1]$. Then for all $\mathbf{u}\in\partial\Lambda$, $\mathcal{F}_\theta(\mathbf{u})\neq 0$. Thus the degree $deg(I-\mathcal{F}_\theta(\mathbf{u}),\Lambda,0)$ is well-defined since $\mathcal{F}_\theta(\mathbf{u}):\Lambda\times[0,1]\to X$ is compact. Moreover, by applying the homotopy invariance of the Leray-Schauder degree theory, we have

$$deg(I-\mathcal{F}_0(\mathbf{u}),\Lambda,0)=deg(I-\mathcal{F}_1(\mathbf{u}),\Lambda,0).$$

If $\theta=0$, then $\mathcal{F}_0(\mathbf{u})=0$ has no non-constant positive solutions by Theorem 3.3 since $\widehat{D}\geq\widetilde{D}$. Hence $deg(I-\mathcal{F}_0(\mathbf{u}),\Lambda,0)=index(I-\mathcal{F}_0,\mathbf{u}_*)$. Furthermore, Lemma 3.4 gives

$$index(I-\mathcal{F}_0,\mathbf{u}_*)=1.$$

On the other hand, we have

$$deg(I-\mathcal{F}_1(\mathbf{u}),\Lambda,0)=index(I-\mathcal{F}_1,\mathbf{u}_*)=(-1)^{\sum_{k=2}^{k_0}\dim E(\mu_k)}=-1$$

by Leray-Schauder Theorem. This contradiction implies the existence of non-constant positive solutions of (1.2), the desired result. □

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