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THE EFFECTS OF TAXATION ON OPTIMAL CONSUMPTION AND INVESTMENT

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Abstract. We investigate the optimal consumption and investment problem of working agent who faces tax system on consumption, labor income, savings and investment. By applying martingale method, we obtain the closed-form solutions so it is possible to verify the effect of tax system analytically.

1. Introduction

How taxes affect the economic agent's behavior? Nowadays we always face tax system where there is a cash flow. A typical consumption tax is a VAT(value-added tax) which is the tax on good and service. Labor income tax is also the tax which every wage earners should pay. In financial market, there are taxes on savings and investment. Each tax may affect the corresponding economic behavior but it is uncertain whether consumption tax affects the investment behavior or vice versa.

There are large strands of literature on the effect of tax on household finance or the tax incentives on savings, insurance, or borrowing. Poterba [8] and Bruhn [2] examine the effect of taxation on household decisions and verify the impact of tax. Jappelli and Pistaferri [4] investigates the tax incentives on saving and borrowing. Tax incentive on insurance, pension, and mortgage are well-addressed in Amromin *et al.* [1]. Many countries adopt capital gains tax for investment. Seifried [9] and Fischer *et al.* [3] provide the effect of the deferred capital gains tax on investment.

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In this paper, we consider the working agent's optimal portfolio choice problem, where there are taxes on consumption, labor, savings, and investment. Our model is based on Merton's model [6, 7] where there are a risk free asset and a risky asset. We consider a stochastic income process and symmetric tax system on investment, which implies the same tax rates are offered on capital losses. By applying martingale approach we obtain the analytic solutions. We can show that the consumption tax has no impact on the gross consumption rate whereas the other taxations including labor income tax and taxes on savings and investment affect both consumption and investment.

This paper is organized as follows. Section 2 introduces the financial market and tax system. The objective function and martingale approach are explained in Section 3. Section 4 provides the closed form solutions including the optimal consumption and investment.

2. The Financial Market

2.1. Financial assets

We consider the complete market in the continuous-time framework. There two kinds of assets which are a risk-free and a risky asset. The risk free asset has constant interest rate $\bar{r}$ and it has its dynamics as

$$dS_t^0 = \bar{r} S_t^0 dt.$$

The risky asset is supposed to follow a geometric Brownian motion with constant coefficients, which evolves

$$\frac{dS_t}{S_t} = \bar{\mu} dt + \bar{\sigma} dB_t,$$

where B_t is the standard Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2. Labor income

In the model, the agent receives wage income from labor and it is also a stochastic process with same uncertainty as risky asset. More specifically, the labor income process denoted by $\bar{I}_t$ follows a geometric Brownian motion,

$$\frac{d\bar{I}_t}{\bar{I}_t} = \mu_I dt + \sigma_I dB_t,$$

where μ_I and σ_I are constant coefficients. Due to the fact that there is only an uncertain source, our financial market is complete in the sense that all the risky sources can be hedged by the existing risky assets.

2.3. Consumption and investment

Let's denote a feasible gross consumption rate and the gross investment ratio by $\bar{c}_t$ and $\bar{\pi}_t$ respectively. We assume that they are $\mathcal{F}_t$-progressively measurable and satisfy

$$\int_0^\infty \bar{c}_t dt < \infty, \ a.s., \qquad \int_0^\infty \bar{\pi}_t^2 dt < \infty, \ a.s..$$

2.4. Tax scheme

Now we introduce the taxes on consumption, investment and labor income. The agent faces the taxes on economic behavior which leads to income or consumption. For simplicity, we suppose a taxation of investments as symmetric non-progressive mark-to-market. Note that in most countries, there is no immediate taxation on capital losses but this assumption makes the problem tractable. Let us denote the consumption tax and labor income tax by τ_c and τ_L respectively. Moreover, the taxes on saving and investment are also defined by τ_1 and τ_2. Obviously, all the taxes have the values in $[0, 1]$. When $\tau_i = 0$ for $i = c, L, 1, 2$, no tax is offered.

2.5. Wealth dynamics

Let us denote any gross value before taxation by $\bar{v}$. Then the net value v is defined by $v = \bar{v}(1 - \tau)$, where v is the income from labor or investment. In the similar manner, for gross consumption $\bar{c}_t$ the net consumption is defined by $\bar{c}_t = c_t/(1 - \tau_c)$.

Then the wealth dynamics X_t is unfolded by

$$\begin{aligned} dX_t &= (1 - \bar{\pi}_t)(1 - \tau_1)\frac{dS_t^0}{S_t^0} + \bar{\pi}_t(1 - \tau_2)\frac{dS_t}{S_t} - \bar{c}_t dt + \bar{I}_t(1 - \tau_L)dt \\ &= rX_t dt + \pi_t(\mu - r)dt + \sigma\pi_t dB_t - c_t/(1 - \tau_c)dt + I_t dt, \quad X_0 = x, \end{aligned}$$

where the second equality holds for the real values.

2.6. Static budget

We apply the martingale approach to solve the problem. To do that it is necessary to transform the dynamic budget into the static form. Let us introduce a Radon-Nykodym derivative defined by

$$Z_t = e^{-\frac{1}{2}\theta^2 t - \theta B_t},$$

where $\theta = (\mu - r)/\sigma$ is the market price of risk. Then from the Girsanov's theorem, there exists an equivalent martingale measure $\mathbb{Q}$ under which $\tilde{B}_t = B_t + \theta t$ is the standard Brownian motion. Thus, under the new measure, the wealth process is governed by

$$dX_t = [rX_t - c_t/(1-\tau_c) + I_t]dt + \pi_t \sigma d\tilde{B}_t.$$

By multiplying e^{-rt} on both sides and taking integration, we can obtain the following static budget by applying Bayes' rule,

$$\mathbb{E}\left[\int_0^\infty H_t(c_t/(1-\tau_c) - I_t)dt\right] \leq x. \tag{2.1}$$

3. Optimization Problem

3.1. Preference

The agent is assumed to have a CRRA(constant relative risk aversion) utility function, which is given by

$$u(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma},$$

where $\gamma (> 0)$ is the coefficient of risk tolerance. When $\gamma = 1$, a CRRA utility function becomes log, which is defined by $u(c_t) = \log(c_t)$. Then our main goal is to find the maximization value of the expected utility by choosing consumption and investment in risky asset.

3.2. Value function

With the agent's subjective discount factor $\beta > 0$, the value function of infinitely-lived agent is formulated by

$$V(x, I_0) = \max_{c,\pi} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t) dt\right], \tag{3.1}$$

subject to the static budget constraint (2.1). Notice that in this primal problem we consider the net values only. We can convert the net values which are the solutions to the primal problem into the gross values using the given tax rates.

3.3. Martingale approach

We apply the martingale approach, which introduces the Lagrangian with Lagrange multiplier $\lambda > 0$. Let us define the dual value function by

(3.2)

$$J(\lambda, I_0) = \max_{c,\pi} \mathbb{E}\left[\int_0^\infty e^{-\beta t} u(c_t)dt\right] - \lambda \mathbb{E}\left[\int_0^\infty H_t(c_t/(1-\tau_c) - I_t)dt\right].$$

Note that since the labor income is a stochastic process, which has to be taken as another state variable, the dual value is a function of two variables λ and I_0.

Let us define the convex dual of $u(\cdot)$ by

$$\widetilde{u}(y) = \sup_{x\in\mathbb{R}}\{u(x) - xy\}, \qquad y \in \mathbb{R},$$

then we have

$$\widetilde{u}(y) = u(I(y)) - yI(y), \qquad y > 0,$$

where $I(y) = y^{-\frac{1}{\gamma}}$ is the inverse function of $u'(\cdot)$. The existence of $I(y)$ is guaranteed because the utility function $u(\cdot)$ is concave and non-decreasing and its derivative is strictly decreasing.

From the first order condition in (3.2), the optimal consumption rate is given by

$$c_t^* = \left(\frac{y_t}{1-\tau_c}\right)^{-\frac{1}{\gamma}},$$

where $y_t = \lambda e^{\beta t} H_t$. Thus, we can rewrite the dual value function as

$$J(\lambda, I_0) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(\frac{1}{1-\gamma}\left(\frac{y_t}{1-\tau_c}\right)^{1-\frac{1}{\gamma}} + y_t I_t\right)dt\right].$$

It can be shown that the value function defined in (3.1) is the concave conjugate of $J(\cdot, I_0)$. In other words, if we consider the dual value as the function of λ for fixed initial I_0, the primal value function is obtained from the conventional convex duality theory. The following lemma summarizes the relation between primal value and dual value.

Lemma 3.1. *When the dual value function is given by the equation in (3.2), the value function of the primal problem is determined by*

$$V(x, I_0) = \inf_{\lambda>0}\{J(\lambda, I_0) + \lambda x\}.$$

Proof. See Karatzas and Shereve [5]. □

4. The Solutions

Now we will find the dual value as the solution to the one-dimensional problem. Let us rewrite the dual value function as

$$J(\lambda, I_0) = \mathbb{E}\left[\int_0^\infty e^{-\beta t} I_t^{1-\gamma}\left(\frac{1}{1-\gamma}\left(\frac{y_t I_t^\gamma}{1-\tau_c}\right)^{1-\frac{1}{\gamma}} + y_t I_t^\gamma\right) dt\right]$$
$$= I_0^{1-\gamma}\mathbb{E}\left[\int_0^\infty e^{-\hat{\beta} t}\tilde{Z}_t\left(\frac{1}{1-\gamma}\left(\frac{y_t I_t^\gamma}{1-\tau_c}\right)^{1-\frac{1}{\gamma}} + y_t I_t^\gamma\right) dt\right],$$

where the adjusted discounted factor $\hat{\beta}$ is determined by

$$\hat{\beta} = \beta - (1-\gamma)\mu_I + \frac{\gamma(1-\gamma)}{2}\sigma_I^2,$$

and the exponential martingale $\tilde{Z}_t$ is also given by

$$\tilde{Z}_t = e^{-\frac{1}{2}(1-\gamma)^2\sigma_I^2 t - (1-\gamma)\sigma_I B_t}.$$

Now let us introduce an equivalent martingale measure defined by a Radon-Nikodym derivative $\tilde{Z}_t$. Under the new measure, $\bar{B}_t = B_t - (1-\gamma)\sigma_I t$ is the standard Brownian motion and the variable $z_t \equiv y_t I_t^\gamma$ has its dynamics as

$$\frac{dz_t}{z_t} = \mu_z dt + \sigma_z d\bar{B}_t,$$

where the coefficients are given by

$$\mu_z = \beta - r + \gamma\mu_I + \frac{\gamma(1-\gamma)}{2}\sigma_I^2 - \theta\sigma_I, \qquad \sigma_z = \gamma\sigma_I - \theta.$$

In sum, the dual value function is restated as

$$J(\lambda, I_0) = I_0^{1-\gamma}\bar{\mathbb{E}}\left[\int_0^\infty e^{-\hat{\beta} t}\left(\frac{1}{1-\gamma}\left(\frac{z_t}{1-\tau_c}\right)^{1-\frac{1}{\gamma}} + z_t\right) dt\right],$$

where $\bar{\mathbb{E}}[\cdot]$ is the expectation under the new measure. Note that the dual value function is now represented by the expected value which can be obtained by one dimensional ordinary differential equation (ODE).

Let us define the function $\varphi(z_0)$ by

$$\varphi(z_0) \equiv \bar{\mathbb{E}}\left[\int_0^\infty e^{-\hat{\beta} t}\left(\frac{1}{1-\gamma}\left(\frac{z_t}{1-\tau_c}\right)^{1-\frac{1}{\gamma}} + z_t\right) dt\right],$$

then by Feynman-Kac's formula leads to the following ODE.

$$\frac{1}{2}\sigma_z^2 z^2\varphi''(z)+\left(\mu_z-\frac{1}{2}\sigma_z^2\right)z\varphi'(z)-\hat{\beta}\varphi(z)+\frac{1}{1-\gamma}\left(\frac{z}{1-\tau_c}\right)^{1-\frac{1}{\gamma}}+z=0.$$

Due to the growth condition, there is no general solution to the ODE and the particular solution is derived by

$$\varphi(z)=\frac{\gamma}{K(1-\gamma)}\left(\frac{z}{1-\tau_c}\right)^{1-\frac{1}{\gamma}}+\frac{1}{r_I}z, \tag{4.1}$$

where $K=r+\frac{\beta-r}{\gamma}+\frac{\gamma-1}{2\gamma^2}\theta^2$ and $r_I=r-\mu_I+\sigma_I\theta$. We summarize the result in the next proposition.

Proposition 4.1. *The value function* (3.1) *is determined by*

$$V(x,I_0)=\inf_{\lambda>0}\left\{I_0^{1-\gamma}\varphi(\lambda I_0^{\gamma})+\lambda x\right\},$$

where the function $\varphi(\cdot)$ *is given in* (4.1).

Notice that the optimal wealth is determined from the first order condition for λ in Proposition 4.1, i.e., the optimal wealth is given by

$$X_t^*=-\varphi'(z_t^*)I_t,$$

where z_0^* satisfies the algebraic equation $x=-\varphi'(z_0^*)I_0$. Since the function $\varphi'(\cdot)$ is strictly increasing and convex, it can be confirmed that there exists one-to-one correspondence between wealth to income ratio and the variable z. Now we provide our main results in the next theorem.

Theorem 4.2. *The infinitely-lived agent who should pay taxes on consumption, investment, and labor income has its value function as*

$$V(x,I_t)=\frac{1}{K^{\gamma}(1-\gamma)}\left((1-\tau_c)\left(x+\frac{I_0}{r_I}\right)\right)^{1-\gamma}.$$

Moreover, the optimal controls are given by

$$\frac{c_t^*}{I_t}=K(1-\tau_c)\left(\frac{X_t}{I_t}+\frac{1}{r_I}\right),$$

$$\frac{\pi_t^*}{I_t}=\frac{\theta}{\gamma\sigma}\left(\frac{X_t}{I_t}+\frac{1}{r_I}\right)-\frac{\sigma_I}{\sigma}\frac{1}{r_I}.$$

Proof. The direct computation leads to the value function. More specifically, the first order condition for λ implies

$$\lambda^*=K^{-\gamma}(1-\tau_c)^{1-\gamma}\left(x+\frac{I_0}{r_I}\right)^{-\gamma}.$$

By substituting this value into the value function in Proposition 4.1, we obtain the result.

On the other hand, the optimal portfolio can be determined from the fact that there exists one-to-one correspondence between wealth X_t^* and z_t^*. By applying Itô's formula to the optimal wealth X_t^*, we have

$$\begin{aligned} dX_t^* &= -d\left(\varphi'(z_t^*)\cdot I_t\right) = -d(\varphi'(z_t^*))\cdot I_t - \varphi'(z_t^*)\cdot dI_t - d(\varphi'(z_t^*))\cdot dI_t \\ &= -\varphi''(z_t^*)I_t(\hat{\beta}-r_I)z_t^* dt - \varphi'(z_t^*)\mu_I I_t dt - \varphi''(z_t^*)z_t^* I_t\sigma_z\sigma_I dt \\ &\qquad - \varphi''(z_t^*)I_t\sigma_z z_t^* dB_t - \varphi'(z_t^*)I_t\sigma_I dB_t. \end{aligned}$$

Then the diffusion term is exactly same as the portfolio amount so we can obtain the optimal investment in terms of z_t^*. The conversion to the notation with wealth process is another direct computation using $\varphi(z_t)$ in (4.1). □

Notice that the optimal controls are the net values which are the values after taxation. We can show that the consumption tax is the only one which influences the value function or consumption rate. Especially, the consumption tax reduces consumption proportionally.

Now, let us convert the net optimal values into the gross values. The next proposition provides the results.

PROPOSITION 4.3. *The gross optimal consumption and investment of the net values in Theorem 4.2 are given by*

$$\begin{aligned} \bar{c}_t^* &= \frac{c_t^*}{1-\tau_c} = \bar{K}\left(X_t + (1-\tau_L)\frac{\bar{I}_t}{\bar{r}_I}\right), \\ \bar{\pi}_t^* &= \frac{\bar{\mu}(1-\tau_2)-\bar{r}(1-\tau_1)}{\gamma\bar{\sigma}^2(1-\tau_2)^2}\left(X_t + (1-\tau_L)\frac{\bar{I}_t}{\bar{r}_I}\right) - \frac{\sigma_I}{(1-\tau_2)\bar{\sigma}}\frac{(1-\tau_L)\bar{I}_t}{\bar{r}_I}, \end{aligned}$$

where $\bar{r}_I = (1-\tau_1)r - \mu_I + \sigma_I\frac{(1-\tau_2)\bar{\mu}-(1-\tau_1)\bar{r}}{(1-\tau_2)\bar{\sigma}}$ *and*

$$\begin{aligned} \bar{K} &= \bar{r}(1-\tau_1) + (\beta - r(1-\tau_1))/\gamma \\ &\quad + (\gamma-1)/2\gamma^2\left((\bar{\mu}(1-\tau_2)-\bar{r}(1-\tau_1))/\bar{\sigma}(1-\tau_2)\right)^2. \end{aligned}$$

Proof. This is the direct computation by substituting the gross values instead of net values. □

It is worth to note that the consumption tax has no impact on gross consumption but the other taxes affect the consumption and investment.

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