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A NATURAL MAP ON AN ORE EXTENSION

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Abstract. Let δ be a derivation in a noetherian integral domain A. It is shown that a natural map induces a homeomorphism between the spectrum of $A[z;\delta]$ and the Poisson spectrum of $A[z;\delta]_p$ such that its restriction to the primitive spectrum of $A[z;\delta]$ is also a homeomorphism onto the Poisson primitive spectrum of $A[z;\delta]_p$.

Let R be a $\mathbf{k}$-algebra and let h be a nonzero, nonunit, non-zero-divisor and central element of R such that R/hR is commutative. Then R/hR becomes a Poisson algebra with Poisson bracket

$$\{\overline{a}, \overline{b}\} = \overline{h^{-1}(ab - ba)} \tag{1}$$

for $\overline{a}, \overline{b} \in R/hR$, which is called a semiclassical limit of R and R is called a quantization of its semiclassical limit. One estimates that a class D of nontrivial algebras $R/(h-\lambda)R$, $\lambda \in \mathbf{k}$, shares its algebraic structure with Poisson algebraic structure of R/hR since the multiplication of $R/(h-\lambda)R$ and the Poisson bracket (1) of R/hR are induced by that of R. In fact, there are many positive evidences, for instance, see [8], [4] and [1], [5], [10], [9]. In [9] and [5], the second author constructed a natural map from a quantized algebra onto its semiclassical limit which can explain relationships between algebraic structures of quantized algebra and Poisson structures of its semiclassical limit.

Let δ be a derivation in a noetherian integral domain A. Then, in [3], Jordan proved that the spectrum of $A[z;\delta]$ is homeomorphic to the Poisson spectrum of $A[z;\delta]_p$ such that its restriction to the primitive spectrum of $A[z;\delta]$ is also a homeomorphism onto the Poisson primitive spectrum of $A[z;\delta]_p$. In usual, it is difficult for a map to be a homeomorphism between two spaces. In this paper, it is established that the natural map in [9] and [5] induces a homeomorphism between the spectrum of $A[z;\delta]$ and the Poisson spectrum of $A[z;\delta]_p$ such that its

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restriction to the primitive spectrum of $A[z;\delta]$ is also a homeomorphism onto the Poisson primitive spectrum of $A[z;\delta]_p$.

Assume throughout the paper that $\mathbf{k}$ denotes a base field of characteristic zero and that all algebras considered have unities.

Let A be a finitely generated commutative $\mathbf{k}$-algebra and domain with a non-zero derivation δ. Then there exists the skew polynomial algebra $A[z;\delta]$. Refer to [2, Chapter 2] for details of skew polynomial algebra which is frequently called Ore extension. On the other hand, there exists the Poisson polynomial algebra $A[z;\delta]_p$ which is the Poisson algebra $A[z]$ with Poisson bracket

$$\{A, A\} = 0, \quad \{z, a\} = \delta(a)$$

for all $a \in A$. Refer to [6, 1.1] for details of Poisson polynomial algebra.

The derivation δ on A is extended to a $\mathbf{k}[t]$-derivation on $A[t]$, still denoted by δ, by setting $\delta(t) = 0$. Hence $(t-1)\delta$ is a derivation on $A[t]$ and thus there exists the skew polynomial $\mathbf{k}[t]$-algebra

$$B := A[t][z;(t-1)\delta].$$

Note that B is a domain and thus the central element $t-1 \in B$ is a nonzero, nonunit and non-zero-divisor such that

$$B_1 := B/(t-1)B$$

is commutative. Hence B_1 becomes a Poisson algebra with Poisson bracket

$$\{\overline{a}, \overline{b}\} = \overline{(t-1)^{-1}(ab-ba)}$$

for $\overline{a}, \overline{b} \in B_1$.

Lemma 1. *$B_1 \cong A[z;\delta]_p$ as Poisson algebras.*

Proof. It is easy to see that $B_1 \cong A[z]$ as commutative algebras since $A[z] \cap (t-1)B = \{0\}$. For all $a, b \in A$, $\{\overline{a}, \overline{b}\} = 0$ and $\{\overline{z}, \overline{b}\} = \overline{\delta(b)}$ in B_1. Hence the result follows. □

Set $\mathbf{K} = \mathbf{k} \setminus \{a \in \mathbf{k} | a^n = 1 \text{ for some positive integer } n\}$ and

$$B_\lambda := B/(t-\lambda)B$$

for all $\lambda \in \mathbf{K}$. Note that $1 \notin \mathbf{K}$ and that B_λ is a nontrivial $\mathbf{k}$-algebra since $t-\lambda$ is a nonzero and nonunit for all $\lambda \in \mathbf{K}$.

Lemma 2. *For each $\lambda \in \mathbf{K}$, $B_\lambda \cong A[z;(\lambda-1)\delta] \cong A[z;\delta]$ as $\mathbf{k}$-algebras.*

Proof. The map from B into $A[z;(\lambda-1)\delta]$ defined by $z \mapsto z$ and $a(t) \mapsto a(\lambda)$ for all $a(t) \in A[t]$ is an epimorphism with kernel $(t-\lambda)B$. Hence $B_\lambda \cong A[z;(\lambda-1)\delta]$.

The map from $A[z;(\lambda-1)\delta]$ into $A[z;\delta]$ defined by $z \mapsto (\lambda-1)z$ and $a \mapsto a$ for all $a \in A$ is an isomorphism since $\lambda \neq 1$ for all $\lambda \in \mathbf{K}$. It completes the proof. □

Lemma 3. *The map*

$$\gamma : B \to \prod_{\lambda\in\mathbf{K}} B_\lambda, \quad \gamma(b) = (\gamma_\lambda(b))_{\lambda\in\mathbf{K}}$$

is a monomorphism, where γ_λ is the canonical projection from B onto $B_\lambda = B/(t-\lambda)B$.

Proof. Since B is a skew polynomial algebra $A[t][z;(t-1)\delta]$, every element $b \in B$ is expressed uniquely by $b = \sum_i a_i(t)z^i$ for some $a_i(t) \in A[t]$ and each $a_i(t)$ is expressed uniquely by $a_i(t) = \sum_j c_{ij}t^j$ for some $c_{ij} \in A$. If $\gamma(b) = 0$ then $\sum_j c_{ij}\lambda^j = 0$ for all $\lambda \in \mathbf{K}$ and thus $c_{ij} = 0$ for all i, j. It follows that $b = 0$ and thus γ is a monomorphism. □

By Lemma 3, there exists the composition of γ^{-1} and γ_1

$$\Gamma : \gamma(B) \xrightarrow{\gamma^{-1}} B \xrightarrow{\gamma_1} B_1, \quad \Gamma(x) = \gamma_1\gamma^{-1}(x)$$

which is a $\mathbf{k}$-algebra epimorphism, where $\gamma_1 : B \to B_1 = B/(t-1)B$ is the canonical projection.

As in [5, Remark 3.2], let $\widehat{q}$ be a parameter taking values in $\mathbf{K}$ and let $B_{\widehat{q}}$ be the $\mathbf{k}$-algebra obtained by replacing λ in B_λ by $\widehat{q}$. That is, $B_{\widehat{q}}$ is the $\mathbf{k}$-algebra defined by $B/(t-\widehat{q})B$, which is isomorphic to $A[z;(\widehat{q}-1)\delta]$. Let

$$\widehat{\ } : B_{\widehat{q}} \to \prod_{\lambda\in\mathbf{K}} B_\lambda, \ \widehat{\ }(b) = (b|_{\widehat{q}=\lambda})_{\lambda\in\mathbf{K}}.$$

Then $\widehat{\ }$ is a $\mathbf{k}$-algebra homomorphism such that $\widehat{\ }(\widehat{q}) = (\lambda)_{\lambda\in\mathbf{K}}$ and $\widehat{\ }(a) = (a)_{\lambda\in\mathbf{K}}$ for all $a \in A$. Set

$$\widehat{B} = \widehat{\ }^{-1}(\gamma(B)).$$

It is clear that $\widehat{B}$ is a $\mathbf{k}$-subalgebra of $B_{\widehat{q}}$ and that there exists the composition of Γ and $\widehat{\ }$

$$\widehat{\Gamma} : \widehat{B} \xrightarrow{\widehat{\ }} \gamma(B) \xrightarrow{\Gamma} B_1, \quad \widehat{\Gamma}(b) = \Gamma(\widehat{\ }(b)). \tag{2}$$

Lemma 4. (1) *$(\widehat{q}-1)^{-1} \notin \widehat{B}$ and $A \subseteq \widehat{B}$.*
(2) *$\widehat{\Gamma}(z) = z, \widehat{\Gamma}(\widehat{q}) = 1$ and $\widehat{\Gamma}(a) = a$ for all $a \in A$.*
(3) *For any ideal I of $\widehat{B}$, $\widehat{\Gamma}(I)$ is a Poisson ideal of B_1.*

Proof. [5, Remark 3.2] and [9, Theorem 1.4]. □

LEMMA 5. *Let q be an element of* $\mathbf{K}$. *Then* $B_q = \widehat{B}$.

Proof. Since q can take any element of $\mathbf{K}$, q plays a role as a parameter taking values in $\mathbf{K}$ and thus q is equal to $\widehat{q}$ as parameters. Since q is an element of $\mathbf{K}$ and $B_q = B/(t-q)B \cong A[z;(q-1)\delta]$,

$$\begin{aligned} f \in B_q \Leftrightarrow f &= \sum_{i\geq 0} a_i(q)z^i \text{ for some } a_i(t) \in \mathbf{k}[t] \\ \Leftrightarrow f &= \left(\sum_{i\geq 0} a_i(t)z^i\right)|_{t=q} \\ \Leftrightarrow f &= \left(\sum_{i\geq 0} a_i(t)z^i\right)|_{t=\widehat{q}} \in \widehat{B}. \end{aligned}$$

Hence $B_q = \widehat{B}$. (cf., [5, Lemma 3.6]) □

Let R be an algebra. The spectrum of R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R equipped with the Zariski topology. The primitive spectrum, denoted by $\operatorname{Prim} R$, is the subspace of $\operatorname{Spec} R$ consisting of all primitive ideals of R. Similarly, let S be a Poisson algebra. The Poisson spectrum of S, denoted by $\mathrm{P.Spec}\, S$, is the set of all Poisson prime ideals of S equipped with the Zariski topology. The Poisson primitive spectrum of S, denoted by $\mathrm{P.Prim}\, S$, is the subspace of $\mathrm{P.Spec}\, S$ consisting of all Poisson primitive ideals of S. If S is noetherian then $\mathrm{P.Spec}\, S$ is a subspace of $\operatorname{Spec} S$ since Poisson prime is prime.

An ideal I of A is said to be δ-ideal if $\delta(I) \subseteq I$. A δ-ideal P is said to be δ-prime if, for any δ-ideals I and J, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

LEMMA 6. *The map*

$$\varphi : \operatorname{Spec} \widehat{B} \longrightarrow \mathrm{P.Spec}\, B_1, \quad \varphi(P) = \widehat{\Gamma}(P) \tag{3}$$

is a homeomorphism.

Proof. Let us find $\operatorname{Spec} B_q$ and $\mathrm{P.Spec}\, B_1$. Note that $B_q \cong A[z;(q-1)\delta]$ and $B_1 \cong A[z;\delta]_p$ by Lemma 2 and Lemma 1, that $\delta(A)B_q$ is an ideal of B_q and that $\delta(A)B_1$ is a Poisson ideal of B_1. Set

$$\begin{aligned} \operatorname{Spec}_1 B_q &= \{P \in \operatorname{Spec} B_q |\ \delta(A)B_q \subseteq P\} \\ \mathrm{P.Spec}_1\, B_1 &= \{P \in \mathrm{P.Spec}\, B_1 |\ \delta(A)B_1 \subseteq P\}. \end{aligned}$$

Since $B_q = \widehat{B}$ by Lemma 5 and $\widehat{\Gamma}(z) = z$, $\widehat{\Gamma}(a) = a$ for all $a \in A$ by Lemma 4(2), we have $\widehat{\Gamma}(\delta(A)B_q) = \delta(A)B_1$. Hence φ is bijective between $\operatorname{Spec}_1 B_q$ and $\mathrm{P.}\operatorname{Spec}_1 B_1$ since

$$B_q/\delta(A)B_q \cong (A/\delta(A)A)[z] \cong B_1/\delta(A)B_1.$$

By [3, Lemma 3.2, 3.3] and [7, 2.2],

$$\operatorname{Spec} B_q \setminus \operatorname{Spec}_1 B_q = \{IB_q | I \text{ is a } \delta\text{-prime ideal of } A \text{ such that } \delta(A) \nsubseteq I\},$$
$$\operatorname{Spec} B_1 \setminus \operatorname{Spec}_1 B_1 = \{IB_1 | I \text{ is a } \delta\text{-prime ideal of } A \text{ such that } \delta(A) \nsubseteq I\}.$$

Hence φ is a bijection between $\operatorname{Spec} B_q \setminus \operatorname{Spec}_1 B_q$ and $\mathrm{P.}\operatorname{Spec} B_1 \setminus \mathrm{P.}\operatorname{Spec}_1 B_1$ since $\widehat{\Gamma}(a) = a$ for all $a \in A$. It follows that φ in (3) is a homeomorphism from $\operatorname{Spec} B_q$ onto $\mathrm{P.}\operatorname{Spec} B_1$ by Lemma 5. □

Now we can prove the following theorem.

THEOREM 7. [3, Theorem 3.6] *The map (3) induces a homeomorphism from* $\operatorname{Spec} A[z;\delta]$ *onto* $\mathrm{P.}\operatorname{Spec} A[z;\delta]_p$ *such that its restriction to* $\operatorname{Prim} A[z;\delta]$ *is also a homeomorphism onto* $\mathrm{P.}\operatorname{Prim} A[z;\delta]_p$.

Proof. The map (3) induces a homeomorphism from $\operatorname{Spec} A[z;\delta]$ onto $\mathrm{P.}\operatorname{Spec} A[z;\delta]_p$ by Lemma 2, Lemma 5 and Lemma 6 since $\widehat{\Gamma}$ is a map preserving inclusions. Moreover, the restriction of (3) to $\operatorname{Prim} A[z;\delta]$ is also a homeomorphism onto $\mathrm{P.}\operatorname{Prim} A[z;\delta]_p$ by [3, Corollary 4.4]. □

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