JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 1, February 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.1

JORDAN DERIVATIONS ON SEMIPRIME RINGS AND THEIR RADICAL RANGE IN BANACH ALGEBRAS

Byung Do Kim*

Abstract. Let R be a 3!-torsion free noncommutative semiprime ring, and suppose there exists a Jordan derivation $D : R \to R$ such that $D^2(x)[D(x), x] = 0$ or $[D(x), x]D^2(x) = 0$ for all $x \in R$. In this case we have $f(x)^5 = 0$ for all $x \in R$. Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $D^2(x)[D(x), x] \in \text{rad}(A)$ or $[D(x), x]D^2(x) \in \text{rad}(A)$ for all $x \in A$. In this case, we show that $D(A) \subseteq \text{rad}(A)$.

1. Introduction

Through out this paper, let R be an associative ring. We write $[x, y]$ for the commutator $xy - yx$ for all x, y in a ring. A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. we say that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. A additive mapping $D : R \to R$ is said to be *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. A additive mapping $D : R \to R$ is said to be *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. A will be a complex Banach algebra. Let $\text{rad}(R)$ denote the (*Jacobson*) *radical* of a ring R. And a ring R is said to be (*Jacobson*) *semisimple* if its Jacobson radical $\text{rad}(R)$ is zero.

On the other hand, let X be an element of a normed algebra. Then for every $a \in X$ the *spectral radius* of a, denoted by $r(a)$, is defined by $r(a) = \inf\{\|a^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: if a be an element of a normed algebra, then $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$ (see Bonsall and Duncan [1]).

Received June 15, 2017; Accepted September 30, 2017.
2010 Mathematics Subject Classification: Primary:16N60, 16W25, 17B40, 46H40.
Key words and phrases: prime and semiprime ring, derivation, Jordan derivation, Banach algebra, (Jacobson) radical.

A noncommutative version of Singer and Wermer's Conjecture states that every continuous linear derivation on a noncommutative Banach algebra maps the algebra into its radical.

Brešar [4] proved the following theorem:

Let d be a continuous derivation of a Banach algebra A. If $[d(x), x] \in$ rad(A) for all $x \in A$, then d maps A into rad(A).

The purpose of this paper is to prove the two results in the ring theory, and is also to apply them to the Banach algebra theory.

2. Preliminaries

In this section, we review the basic results in prime and semiprime rings. The following lemma is due to Chung and Luh [5].

LEMMA 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that*

$$\sum_{k=1}^{n} t^k y_k = 0 \text{ for all } t = 1, 2, \cdots, n.$$

Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.

In 1988, M. Brešar [2] proved theorem as follows.

THEOREM 2.2. [2, Theorem 1] *Let R be a 2-torsion free semiprime ring and let $'$: $R \to R$ be a Jordan derivation. In this case, $'$ is a derivation.*

Kim [7] proved the following theorem.

THEOREM 2.3. *Let R be a 3!-torsion free noncommutative semiprime ring and let $D : R \longrightarrow R$ be a Jordan derivation. Suppose that $[D(x), x]D(x)[D(x), x] = 0$ for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.*

We denote by $Q(A)$ the set of all quasinilpotent elements in A. The following theorem is due to M. Bresar [3].

THEOREM 2.4. *Let D be a bounded derivation of a Banach algebra A. Suppose that $[D(x), x] \in Q(A)$ for all $x \in A$. Then D maps A into rad(A).*

3. Main theorems

It is well-known that there exists a (inner) derivation D on semiprime rings with $D^3(x) = 0$ for all $x \in R$, but $D \neq 0$. We consider the case that the suitable conditions of torsion freeness of semiprime rings and the conditions with $D^2(x)[D(x), x] = 0$ or $[D(x), x]D^2(x) = 0$ for all $x \in R$.

We need the following notations. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer. When R is a ring, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x], f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$ for all $x, y \in R$ respectively. And we have the basic properties:

$$
\begin{aligned}
&B(x, y) = B(y, x), B(x, x) = 2f(x),\\
&B(x, x^2) = 2(f(x)x + xf(x)),\ [D^2(x), x] = F(x),\\
&B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z),\\
&B(x, yx) = B(x, y)x + 2yf(x) + [y, x]D(x),\\
&B(x, xy) = xB(x, y) + 2f(x)y + D(x)[y, x],\\
&B(x, yD(x)) = B(x, y)D(x) + yF(x) + D(y)f(x) + [y, x]D^2(x),\\
&B(x, D(x)y) = D(x)B(x, y) + F(x)y + f(x)D(y) + D^2(x)[y, x],
\end{aligned}
$$

for $x, y, z \in R$.

Theorem 3.1. *Let R be a $3!$-torsionfree noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D^2(x)[D(x), x] = 0$$

for all $x \in R$. Then we have $f(x)D(x)f(x) = 0$ for all $x \in R$.

Proof. By Theorem 2.2, we can see that D is a derivation on R. From the assumption,

$$D^2(x)f(x) = D^2(x)[D(x), x] = 0,\ x \in R. \tag{3.1}$$

Replacing $x + ty$ for x in (3.1), we have

$$
\begin{aligned}
&D^2(x + ty)[D(x + ty), x + ty]\\
&\equiv D^2(x)f(x) + t\{D^2(y)f(x) + D^2(x)B(x, y)\}\\
&\quad +t^2 I(x, y) + t^3 D^2(y)f(y)\\
&= 0, \quad x, y \in R, t \in S_2,
\end{aligned}
$$

where I denotes the term satisfying the above identity. From (3.1) and the above relation, we obtain

$$t\{D^2(y)f(x) + D^2(x)B(x,y)\} + t^2 I(x,y) = 0, \quad x, y \in R, t \in S_2.$$

Since R is 3!-torsionfree, by Lemma 2.1 the above relation yields

$$D^2(y)f(x) + D^2(x)B(x,y) = 0, \quad x, y \in R. \tag{3.2}$$

Let $y = x^2$ in (3.2). Then using (3.1), we have

$$\begin{aligned} &xD^2(x)f(x) + 2D(x)^2 f(x) + D^2(x)xf(x) + 2D^2(x)(f(x)x + xf(x)) \\ &= 2D(x)^2 f(x) - 3D^2(x)g(x) = 0, \quad x \in R. \end{aligned}$$

From (3.1) and the above relation, we get

$$2D(x)^2 f(x) - 3D^2(x)g(x) = 0, \quad x \in R. \tag{3.3}$$

On the other hand, we obtain from (3.1)

$$0 = [D^2(x)f(x), x] = F(x)f(x) + D^2(x)g(x), \quad x \in R. \tag{3.4}$$

Writing xy for y in (3.2),

$$\begin{aligned} &xD^2(y)f(x) + 2D(x)D(y)f(x) + D^2(x)yf(x) + D^2(x)xB(x,y) \\ &+2D^2(x)f(x)y + D^2(x)D(x)[y,x] = 0, \quad x, y \in R. \end{aligned}$$

Left multiplication of (3.2) by x leads to

$$xD^2(y)f(x) + xD^2(x)B(x,y) = 0, \quad x, y \in R.$$

From the two relations, we arrive at

$$\begin{aligned} &2D(x)D(y)f(x) + D^2(x)yf(x) + F(x)B(x,y) + 2D^2(x)f(x)y \\ &+D^2(x)D(x)[y,x] = 0, \quad x, y \in R. \end{aligned} \tag{3.5}$$

From (3.1) and (3.5), we get

$$\begin{aligned} &2D(x)D(y)f(x) + D^2(x)yf(x) + F(x)B(x,y) \\ &+D^2(x)D(x)[y,x] = 0, \quad x, y \in R. \end{aligned} \tag{3.6}$$

Let $y = x$ in (3.6). Then using (3.1), we have

$$2D(x)^2 f(x) + 3F(x)f(x) = 2D(x)^2 f(x) - 3D^2(x)g(x) = 0, \quad x, y \in R.$$

On the other hand, it follows from (3.3) and (3.4) that

$$2D(x)^2 f(x) + 3F(x)f(x) = 0, \quad x \in R. \tag{3.7}$$

Substituting $x+ty$ for x in (3.7), we arrive at

$$\begin{aligned}
&2D(x+ty)^2[D(x+ty),x+ty]+3[D^2(x+ty),x+ty]f(x+ty)\\
&\equiv D^2(x)f(x)+t\{2D(y)D(x)f(x)+2D(x)D(y)f(x)+2D(x)^2B(x,y)\\
&\quad +(3([D^2(y),x]+[D^2(x),y])f(x)+3F(x)B(x,y))\}+t^2J_1(x,y)\\
&\quad +t^3J_2(x,y)+t^4(2D(y)^2f(y)+3F(y)f(y))\\
&=0
\end{aligned}$$

for $x,y\in R$ and $t\in S_3$ where J_1 and J_2 denote the term satisfying the above identity. From (3.1) and the above relation, we obtain

$$\begin{aligned}
(3.8)\qquad & t\{2D(y)D(x)f(x)+2D(x)D(y)f(x)+2D(x)^2B(x,y)\\
&+(3([D^2(y),x]+[D^2(x),y])f(x)+3F(x)B(x,y))\}\\
&+t^2J_1(x,y)+t^3J_2(x,y)=0,\quad x,y\in R, t\in S_3.
\end{aligned}$$

Since R is 3!-torsion free, by Lemma 2.1 the relation (3.8) yields

$$\begin{aligned}
(3.9)\ & 2D(y)D(x)f(x)+2D(x)D(y)f(x)+2D(x)^2B(x,y)\\
&+\{3([D^2(y),x]+[D^2(x),y])f(x)+3F(x)B(x,y)\}=0,\quad x,y\in R.
\end{aligned}$$

Let $y=x^2$ in (3.9). Then using (3.1), we have

$$\begin{aligned}
(3.10)\ & 2(D(x)x+xD(x))D(x)f(x)+2D(x)(D(x)x+xD(x))f(x)\\
&+4D(x)^2(f(x)x+xf(x))+3([xD^2(x)+2D(x)^2+D^2(x)x,x]\\
&+[D^2(x),x^2])f(x)+6F(x)(f(x)x+xf(x))\\
&=0,\quad x\in R.
\end{aligned}$$

From (3.1) and (3.10), we get

$$\begin{aligned}
(3.11)\ & 2D(x)xD(x)f(x)+2xD(x)^2f(x)+2D(x)^2xf(x)\\
&+2D(x)xD(x)f(x)+4D(x)^2f(x)x+4D(x)^2xf(x)\\
&+3xF(x)f(x)+6(D(x)f(x)+f(x)D(x))f(x)+3F(x)xf(x)\\
&+3xF(x)f(x)+3F(x)xf(x)+6F(x)f(x)x+6F(x)xf(x)\\
&=0,\quad x\in R.
\end{aligned}$$

And from (3.1) and (3.12), it follows that

$$(3.12)\ 16f(x)D(x)f(x)+12D(x)f(x)^2+12[F(x),x]f(x)=0,\quad x\in R.$$

Since R is 3!-torsionfree, by Lemma 2.1 the relation (3.12) yields

$$(3.13)\qquad 4f(x)D(x)f(x)+3D(x)f(x)^2+3[F(x),x]f(x)=0,\quad x\in R.$$

Putting yx instead of y in (3.9), we obtain

$$\begin{aligned}&2D(y)xD(x)f(x)+2yD(x)^2f(x)+2D(x)D(y)xf(x)\\&+2D(x)yD(x)f(x)+2D(x)^2B(x,y)x+4D(x)^2yf(x)\\&+2D(x)^2[y,x]D(x)+3[D^2(y),x]xf(x)+6[D(y),x]D(x)f(x)\\&+6D(y)f(x)^2+3yF(x)f(x)+3[y,x]D^2(x)f(x)+3yF(x)f(x)\\&+3[D^2(x),y]xf(x)+3F(x)B(x,y)x+6F(x)yf(x)\\&+3F(x)[y,x]D(x)\\&=0,\quad x,y\in R.\end{aligned}$$

Right multiplication of (3.9) by x leads to

$$\begin{aligned}&2D(y)D(x)f(x)x+2D(x)D(y)f(x)x+2D(x)^2B(x,y)x\\&+3([D^2(y),x]+[D^2(x),y])f(x)x+3F(x)B(x,y)x=0,\quad x,y\in R.\end{aligned}$$

From the above two relations, we arrive at

$$\begin{aligned}(3.14)\;&-2D(y)(f(x)^2+D(x)g(x))+2yD(x)^2f(x)-2D(x)D(y)g(x)\\&+2D(x)yD(x)f(x)+4D(x)^2yf(x)+2D(x)^2[y,x]D(x)\\&-3[D^2(y),x]g(x)+6[D(y),x]D(x)f(x)+6D(y)f(x)^2\\&+3yF(x)f(x)+3[y,x]D^2(x)f(x)+3yF(x)f(x)-3[D^2(x),y]g(x)\\&+6F(x)yf(x)+3F(x)[y,x]D(x)=0,\quad x,y\in R.\end{aligned}$$

From (3.1), (3.7) and (3.14), we get

$$\begin{aligned}(3.15)\;&-2D(y)(f(x)^2+D(x)g(x))-2D(x)D(y)g(x)+2D(x)yD(x)f(x)\\&+4D(x)^2yf(x)+2D(x)^2[y,x]D(x)-3[D^2(y),x]g(x)\\&+6[D(y),x]D(x)f(x)+6D(y)f(x)^2+3yF(x)f(x)\\&-3[D^2(x),y]g(x)+6F(x)yf(x)+3F(x)[y,x]D(x)=0,\quad x,y\in R.\end{aligned}$$

Replace y by $D(x)y$ in (3.15), we have

$$\begin{aligned}&-2D(x)D(y)(f(x)^2+D(x)g(x))-2D^2(x)y(f(x)^2+D(x)g(x))\\&-2D(x)^2D(y)g(x)-2D(x)D^2(x)yg(x)+2D(x)^2yD(x)f(x)\\&+4D(x)^3yf(x)+2D(x)^3[y,x]D(x)+2D(x)^2f(x)yD(x)\\&-3D(x)[D^2(y),x]g(x)-3f(x)D^2(y)g(x)-6D^2(x)[D(y),x]g(x)\\&-6F(x)D(y)g(x)-3D^3(x)[y,x]g(x)+[D^3(x),x]yg(x)\\&+6D(x)[D(y),x]D(x)f(x)+6f(x)D(y)D(x)f(x)\\&+6D^2(x)[y,x]D(x)f(x)+6F(x)yD(x)f(x)+6D(x)D(y)f(x)^2\\&+6D^2(x)yf(x)^2+3D(x)yF(x)f(x)-3D(x)[D^2(x),y]g(x)\end{aligned}$$

$$-3[D^2(x),D(x)]yg(x)+6F(x)D(x)yf(x)+3F(x)D(x)[y,x]D(x)$$
$$+3F(x)f(x)yD(x)=0,\ \ x,y\in R.$$

Left multiplication of (3.15) by $D(x)$ leads to

$$-2D(x)D(y)(f(x)^2+D(x)g(x))-2D(x)^2D(y)g(x)$$
$$+2D(x)^2yD(x)f(x)+4D(x)^3yf(x)+2D(x)^3[y,x]D(x)$$
$$-3D(x)[D^2(y),x]g(x)+6D(x)[D(y),x]D(x)f(x)+6D(x)D(y)f(x)^2$$
$$+3D(x)yF(x)f(x)-3D(x)[D^2(x),y]g(x)+6D(x)F(x)yf(x)$$
$$+3D(x)F(x)[y,x]D(x)=0,\ \ x,y\in R.$$

From the above two relations, it follows that

$$(3.16)\quad -2D^2(x)y(f(x)^2+D(x)g(x))-2D(x)D^2(x)yg(x)$$
$$+2D(x)^2f(x)yD(x)-3f(x)D^2(y)g(x)-6D^2(x)[D(y),x]g(x)$$
$$-6F(x)D(y)g(x)-3D^3(x)[y,x]g(x)+[D^3(x),x]yg(x)$$
$$+6f(x)D(y)D(x)f(x)+6D^2(x)[y,x]D(x)f(x)+6F(x)yD(x)f(x)$$
$$+6D^2(x)yf(x)^2-3[D^2(x),D(x)]yg(x)+6[F(x),D(x)]yf(x)$$
$$+3[F(x),D(x)][y,x]D(x)+3F(x)f(x)yD(x)=0,\ \ x,y\in R.$$

Setting x for y in (3.15),

$$-2D(x)(f(x)^2+D(x)g(x))-2D(x)^2g(x)$$
$$+2D(x)xD(x)f(x)+4D(x)^2xf(x)$$
$$-3F(x)g(x)+6f(x)D(x)f(x)+6D(x)f(x)^2$$
$$+3xF(x)f(x)-3F(x)g(x)+6F(x)xf(x)$$
$$=0,\ \ x,y\in R.$$

From (3.7) and the above relation, we find that

$$(3.17)\quad 4D(x)f(x)^2-4D(x)^2g(x)+8f(x)D(x)f(x)+4D(x)^2xf(x)$$
$$-6F(x)g(x)+6F(x)xf(x)=0,\ \ x,y\in R.$$

From (3.7) and (3.17), it follows that

$$(3.18)\quad 8D(x)f(x)^2-4D(x)^2g(x)+12f(x)D(x)f(x)-6F(x)g(x)$$
$$+6[F(x),x]f(x)=0,\ \ x,y\in R.$$

Since R is 3!-torsionfree, by Lemma 2.1 the relation (3.18) yields

$$(3.19)\quad 4D(x)f(x)^2-2D(x)^2g(x)+6f(x)D(x)f(x)-3F(x)g(x)$$
$$+3[F(x),x]f(x)=0,\ \ x,y\in R.$$

Writing x for y in (3.15), we get

$$-2D(x)f(x)^2 - 2D(x)^2g(x) - 2D(x)^2g(x) + 2D(x)xD(x)f(x)$$
$$+4D(x)^2xf(x) - 3F(x)g(x) + 6f(x)D(x)f(x) + 6D(x)f(x)^2$$
$$+3xF(x)f(x) - 3F(x)g(x) + 6F(x)xf(x) = 0, \quad x, y \in R.$$

The above relation can be rewritten as

(3.20) $$6f(x)D(x)f(x) + 4D(x)f(x)^2 - 4D(x)^2g(x) - 2D(x)xD(x)f(x)$$
$$+4D(x)^2xf(x) - 6F(x)g(x) + 3xF(x)f(x) + 6F(x)xf(x)$$
$$= 0, \quad x, y \in R.$$

From (3.7) and (3.20), we have

(3.21) $$6f(x)D(x)f(x) + 4D(x)f(x)^2 - 4D(x)^2g(x) - 2f(x)D(x)f(x)$$
$$+4(f(x)D(x) + D(x)f(x))f(x) - 6F(x)g(x) + 6[F(x), x]f(x)$$
$$= 0, \quad x, y \in R.$$

(3.21) can be rewritten as

$$8f(x)D(x)f(x) + 8D(x)f(x)^2 - 4D(x)^2g(x) - 6F(x)g(x)$$
$$+6[F(x), x]f(x) = 0, \quad x, y \in R.$$

Since R is 3!-torsionfree, by Lemma 2.1 the above relation gives

(3.22) $$4f(x)D(x)f(x) + 4D(x)f(x)^2 - 2D(x)^2g(x) - 3F(x)g(x)$$
$$+3[F(x), x]f(x) = 0, \quad x, y \in R.$$

From (3.13) and (3.22), we get

(3.23) $$D(x)f(x)^2 - 2D(x)^2g(x) - 3F(x)g(x) = 0, \quad x \in R.$$

From (3.13) and the above relation, we obtain

(3.24)
$$2f(x)D(x)f(x) + D(x)f(x)^2 - 2D(x)^2g(x) - 3F(x)g(x) = 0, \quad x \in R.$$

From (3.23) and (3.24), we obtain

$$2f(x)D(x)f(x) = 0, \quad x \in R.$$

Since R is 3!-torsionfree, the above relation yields

$$f(x)D(x)f(x) = 0, \quad x \in R.$$

Hence by Theorem 2.3, we have

$$f(x)^5 = [D(x), x]^5 = 0, \quad x \in R.$$

This completes the proof. □

The following is similarly proved as in Theorem 3.1.

THEOREM 3.2. *Let R be a $3!$-torsion free noncommutative prime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$[D(x), x]D^2(x) = 0$$

for all $x \in R$. Then we have $f(x)D(x)f(x) = 0$ for all $x \in R$.

We have the following theorem from Theorem 3.1 and simple calculations.

The following theorem is proved by the same arguments as in the proof of J. Vukman's theorem [12].

THEOREM 3.3. *Let A be a Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D^2(x)[D(x), x] \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B. E. Johnson and A. M. Sinclair [6] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [8] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P,\ \hat{x} = x + P$. By the assumption that $D^2(x)[D(x), x] \in \mathrm{rad}(A),\ x \in A$, we obtain $(D_P^2(\hat{x}))[D_P(\hat{x}), \hat{x}] = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.1 are fulfilled. Let the factor prime Banach algebra A/P be noncommutative. Then by Theorem 3.1, we find that $[D_P(\hat{x}), \hat{x}]^5 = 0$ for all $\hat{x}] \in A/P$. Hence we have

$$r_P([D_P(\hat{x}), \hat{x}])^5 = r_P([D_P(\hat{x}), \hat{x}]^5) = 0 \quad \text{for all } \hat{x} \in A/P.$$

Hence $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. Then by using Theorem 2.4, we conclude that $D_P(\hat{x}) \in \mathrm{rda}(A/P) = (0)$ for all $\hat{x} \in A/P$. This implies that $D(x) + P = (0)$ for all $x \in A$ and all primitive ideals of A. That is, we get $D(x) \in P$ for all $x \in A$. Thus $D(A) \subseteq \mathrm{rad}(A)$. Now we consider the case that A/P is commutative. Then since A/P is a commutative Banach semisimple Banach algebra, from the result of B. E. Johnson and A. M. Sinclair [8], it follows that $D_P(\hat{x}) = 0,\ \hat{x} \in A/P$. And so, $D(x) \in P$ for all $x \in A$ and all primitive ideals of A. Hence $D(A) \subseteq \mathrm{rad}(A)$. Therefore in any case we obtain $D(A) \subseteq \mathrm{rad}(A)$. □

The following is similarly proved as in Theorem 3.3.

Theorem 3.4. *Let A be a (noncommutative) Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D^2(x) \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

Theorem 3.5. *Let A be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D^2(x)[D(x), x] = 0$$

for all $x \in A$. Then we have $D = 0$.

Proof. It suffices to prove the case that A is noncommutative. According to the result of B. E. Johnson and A. M. Sinclair [6], every linear derivation on a semisimple Banach algebra is continuous. A. M. Sinclair [8] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. From the given assumptions $D^2(x)[D(x), x] = 0,\ x \in A$, it follows that $(D_P^2(\hat{x}))[D_P(\hat{x}), \hat{x}] = 0,\ \hat{x} \in A/P$, since all the assumptions of Theorem 3.1 are fulfilled. Let the factor algebra A/P be noncommutative. Then by Theorem 3.1, we find that

$$r_P([D_P(\hat{x}), \hat{x}])^5 = r_P([D_P(\hat{x}), \hat{x}]^5) = 0 \text{ for all } \hat{x} \in A/P.$$

Hence $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. Then by using Theorem 2.4, we conclude that $D_P(\hat{x}) \in \mathrm{rda}(A/P) = (0)$ for all $\hat{x} \in A/P$. This implies that $D(x) + P = (0)$ for all $x \in A$ and all primitive ideals of A. That is, we get $D(x) \in P$ for all $x \in A$. Thus $D(A) \subseteq \mathrm{rad}(A) = (0)$. So $D = 0$. In the other case, we consider the case that A/P is commutative. Then Johnson and Sinclair [6] have proved that any linear derivation on a semisimple Banach algebra is continuous. Combining this result with the Singer- Wermer theorem, one obtains that there are no nonzero linear derivations on commutative semisimple Banach algebras. Hence in case A/P is commutative, we have $D_p = 0$ as well. That is, we obtain $D(x) \in P$ for all $x \in A$ and all primitive ideals of P of A. Hence we get $D(A) \subseteq \cap P$ for all primitive ideals P of A. Thus $D(A) \subseteq \mathrm{rad}(A)$. And since A is semisimple, $D = 0$. □

Similarly, we have the statement.

THEOREM 3.6. *Let A be a semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D^2(x) = 0$$

for all $x \in A$. Then we have $D = 0$.

As a special case of Theorem 3.8 we get the following result which characterizes commutative semisimple Banach algebras.

COROLLARY 3.7. *Let A be a semisimple Banach algebra. Suppose*

$$[[x, y], y][[x, y], x] = 0$$

for all $x, y \in A$. In this case, A is commutative.

COROLLARY 3.8. *Let A be a semisimple Banach algebra. Suppose*

$$[[x, y], x][[x, y], y] = 0$$

for all $x, y \in A$. In this case, A is commutative.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Berlin-Heidelberg-New York, 1973.

[2] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1003-1006.

[3] M. Brešar, *Derivations of noncommutative Banach algebras II*, Arch. Math. **63** (1994), 56-59.

[4] M. Brešar, *Commuting maps: a survey*, Taiwanese J. Math. **8** (2004), no. 3, 361-397.

[5] L. O. Chung and J. Luh, *Semiprime rings with nilpotent derivatives* , Canad. Math. Bull. **24** (1981), no. 4, 415-421.

[6] B. E. Johnson and A. M. Sinclair, *Continuity of derivations and a problem of Kaplansky*, Amer. J. Math. **90** (1968), 1067-1073.

[7] B. D. Kim, *On the derivations of semiprime rings and noncommutative Banach algebras*, Acta Mathematica Sinica, **16** (2000), no. 1, 21-28.

[8] A. M. Sinclair, *Jordan homohorphisms and derivations on semisimple Banach algebras*, Proc. Amer. Math. Soc. **24** (1970), 209-214.

[9] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. **129** (1955), 260-264.

[10] M. P. Thomas, *The image of a derivation is contained in the radical*, Annals of Math. **128** (1988), 435-460.

[11] J. Vukman, *A result concerning derivations in noncommutative Banach algebras*, Glasnik Mathematicki **26** (1991), no. 46, 83-88.

[12] J. Vukman, *On derivations in prime rings and Banach algebras*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 877-884.

*
Department of Mathematics
Gangneung-Wonju National University,
Gangneung 25457, Republic of Korea
E-mail: bdkim@gwnu.ac.kr