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UTILIZING ISOTONE MAPPINGS UNDER GERAGHTY-TYPE CONTRACTION TO PROVE MULTIDIMENSIONAL FIXED POINT THEOREMS WITH APPLICATION

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Abstract. We study the existence and uniqueness of fixed point for isotone mappings of any number of arguments under Geraghty-type contraction on a complete metric space endowed with a partial order. As an application of our result we study the existence and uniqueness of the solution to a nonlinear Fredholm integral equation. Our results generalize, extend and unify several classical and very recent related results in the literature in metric spaces.

1. Introduction

After the appearance of the pioneering Banach contractive mapping principle and due to its possible applications, fixed point theory has become one of the most useful branches of nonlinear analysis with applications to very different settings including resolution of all kind of equations, image recovery, convex minimization, split feasibility and equilibrium problems.

In the last decades, fixed point theorems in partially ordered metric spaces have attracted much attention, especially after the works of Ran and Reurings [31], Nieto and Rodriguez-Lopez [30], Bhaskar and Lakshmikantham [5], Berinde and Borcut [2, 3], Choudhury and Kundu [6, 7], Karapinar [18, 19], Harjani and Sadarangani [14, 15] and many others.

In [13], Guo and Lakshmikantham introduced the notion of coupled fixed point and proved some related theorems for certain type of mappings. After this pioneering work, Gnana-Bhaskar and Lakshmikantham [5] reconsidered coupled fixed point in the context of partially ordered sets by defining the notion of mixed monotone mapping. In this outstanding paper, the authors proved the existence and uniqueness of

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coupled fixed points for mixed monotone mappings and also discussed the existence and uniqueness of solution for periodic boundary value problems. Following these initial papers, a significant number of papers on coupled fixed point theorems have been reported in different context including [1, $8-11$, 16, 17, $25-27$, 36, 38]..

Berinde and Borcut [3] extended the notion of coupled fixed point to tripled fixed point and established some tripled point results using mixed monotone property, which extend and generalized the results of Gnana-Bhaskar and Lakshmikantham [5]. Following it, Karapinar [20] improved this idea by defining the notion of quadruple fixed point. Recently, the concept of multidimensional fixed/coincidence point was introduced by Roldan et al. in [32] (see also [12, $21-24$, 28, $33-35$, 37, 40]), which is an extension of Berzig and Samet's notion given in [4].

In this paper, we establish the existence and uniqueness of fixed point for isotone mappings of any number of arguments under Geraghty-type contraction on a complete metric space endowed with a partial order. As an application of our result we study the existence and uniqueness of the solution to a nonlinear Fredholm integral equation. The results we obtain generalize, extend and unify several classical and very recent related results in the literature in metric spaces.

2. PRELIMINARIES

In order to fix the framework needed to state our main results, we recall the following notions. For simplicity, we denote from now on $X \times X \times ... \times X$ (n times) by X^n, where $n \in \mathbb{N}$ with $n \geq 2$ and X is a non-empty set. If elements x, y of a partially ordered set $(X, \preceq)$ are comparable (i.e. $x \preceq y$ or $y \preceq x$ holds), we will write $x \asymp y$. Let $\{A, B\}$ be a partition of the set $\Lambda_n = \{1, 2, ..., n\}$, that is, A and B are non-empty subsets of Λ_n such that $A \cup B = \Lambda_n$ and $A \cap B = \emptyset$. We will denote $\Omega_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$ and $\Omega'_{A,B} = \{\sigma : \Lambda_n \to \Lambda_n : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}$. Henceforth, let $\sigma_1, \sigma_2, ..., \sigma_n$ be n mappings from Λ_n into itself and let Υ be the $n-$tuple $(\sigma_1, \sigma_2, ..., \sigma_n)$. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. For brevity, $g(x)$ will be denoted by gx.

A partial order $\preceq$ on X can be extended to a partial order $\sqsubseteq$ on X^n in the following way. If $(X, \preceq)$ be a partially ordered space, $x, y \in X$ and $i \in \Lambda_n$, we will use the following notations:

$$x \preceq_i y \Rightarrow \begin{cases} x \preceq y, & \text{if } i \in A, \\ x \succeq y, & \text{if } i \in B. \end{cases} \tag{1}$$

Consider on the product space X^n the following partial order: for $Y = (y_1, y_2, ..., y_i, ..., y_n)$, $V = (v_1, v_2, ..., v_i, ..., v_n) \in X^n$,

$$Y \sqsubseteq V \Leftrightarrow y_i \preceq_i v_i. \tag{2}$$

Notice that $\sqsubseteq$ depends on A and B. We say that two points Y and V are comparable, if $Y \sqsubseteq V$ or $V \sqsubseteq Y$. Obviously, $(X^n, \sqsubseteq)$ is a partially ordered set.

Definition 1 ([23, 32, 35])**.** A point $(x_1, x_2, ..., x_n) \in X^n$ is called a $\Upsilon-$*fixed point* of the mapping $F : X^n \to X$ if

$$F(x_{\sigma_i(1)},\ x_{\sigma_i(2)},\ ...,\ x_{\sigma_i(n)}) = x_i,\ \text{for all } i \in \Lambda_n. \tag{3}$$

It is clear that the previous definition extend the notions of coupled, tripled, and quadruple fixed points. In fact, if we represent a mapping $\sigma : \Lambda_n \to \Lambda_n$ throughout its ordered image, that is, $\sigma = (\sigma(1), \sigma\ (2), ..., \sigma\ (n))$, then

(i) Gnana-Bhaskar and Lakshmikantham's coupled fixed points occur when $n = 2$, $\sigma_1 = (1, 2)$ and $\sigma_2 = (2, 1)$,

(ii) Berinde and Borcut's tripled fixed points are associated with $n = 3$, $\sigma_1 = (1, 2, 3)$, $\sigma_2 = (2, 1, 2)$ and $\sigma_3 = (3, 2, 1)$,

(iii) Karapinar's quadruple fixed points are considered when $n = 4$, $\sigma_1 = (1, 2, 3, 4)$, $\sigma_2 = (2, 3, 4, 1)$, $\sigma_3 = (3, 4, 1, 2)$ and $\sigma_4 = (4, 1, 2, 3)$.

These cases consider A as the odd numbers in $\{1, 2, ..., n\}$ and B as its even numbers. However, Berzig and Samet [4] use $A = \{1, 2, ..., m\}$, $B = \{m + 1, ..., n\}$ and arbitrary mappings.

Definition 2 ([32])**.** Let $(X, \preceq)$ be a partially ordered space. We say that F has the *mixed monotone property* if F is monotone non-decreasing in arguments of A and monotone non-increasing in arguments of B, that is, for all $x_1, x_2, ..., x_n, y, z \in X$ and all i

(4)

$$y \preceq z \Rightarrow F(x_1,\ ...,\ x_{i-1},\ y,\ x_{i+1},\ ...,\ x_n) \preceq_i F(x_1,\ ...,\ x_{i-1},\ z,\ x_{i+1},\ ...,\ x_n).$$

Definition 3 ([35, 40])**.** Let (X, d) be a metric space and define $\Delta_n, \rho_n : X^n \times X^n \to [0, +\infty)$, for $Y = (y_1, y_2, ..., y_n)$, $V = (v_1, v_2, ..., v_n) \in X^n$, by

$$\Delta_n(Y,\ V) = \frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i) \text{ and } \rho_n(Y,\ V) = \max_{1\le i\le n} d(y_i,\ v_i). \tag{5}$$

Then Δ_n and ρ_n are metric on X^n and (X, d) is complete if and only if (X^n, Δ_n) and (X^n, ρ_n) are complete. It is easy to see that

$$\Delta_n(Y^k,\ Y) \to 0 \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty) \tag{6}$$
$$\text{and } \rho_n(Y^k,\ Y) \to 0 \Leftrightarrow d(y_i^k,\ y_i) \to 0 \text{ (as } k \to \infty),\ i \in \Lambda_n,$$

where $Y^k = (y_1^k,\, y_2^k,\, ...,\, y_n^k)$ and $Y = (y_1,\, y_2,\, ...,\, y_n) \in X^n$.

Lemma 4 ([35, 39, 40])**.** *Let $(X,\, d,\, \preceq)$ be an ordered metric space and let $F : X^n \to X$ and $g : X \to X$ be two mappings. Let $\Upsilon = (\sigma_1,\, \sigma_2,\, ...,\, \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Define $F_\Upsilon,\, G : X^n \to X^n$, for all $y_1,\, y_2,\, ...,\, y_n \in X$, by*

$$F_\Upsilon(y_1,\ y_2,\ ...,\ y_n) = \begin{pmatrix} F(y_{\sigma_1(1)},\ y_{\sigma_1(2)},\ ...,\ y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)},\ y_{\sigma_2(2)},\ ...,\ y_{\sigma_2(n)}), \\ ...,\ F(y_{\sigma_n(1)},\ y_{\sigma_n(2)},\ ...,\ y_{\sigma_n(n)}) \end{pmatrix},$$
$$\text{and } G(y_1,\ y_2,\ ...,\ y_n) = (gy_1,\ gy_2,\ ...,\ gy_n). \tag{7}$$

(1) *If F has the mixed $(g,\, \preceq)-$monotone property, then F_Υ is monotone $(G,\, \sqsubseteq)-$non-decreasing.*

(2) *If F is $d-$continuous, then F_Υ is also Δ_n-continuous and ρ_n-continuous.*

(3) *If g is $d-$continuous, then G is Δ_n-continuous and ρ_n-continuous.*

(4) *A point $(y_1,\, y_2,\, ...,\, y_n) \in X^n$ is a $\Upsilon-$fixed point of F if and only if $(y_1,\, y_2,\, ...,\, y_n)$ is a fixed point of F_Υ.*

(5) *A point $(y_1,\, y_2,\, ...,\, y_n) \in X^n$ is a $\Upsilon-$coincidence point of F and g if and only if $(y_1,\, y_2,\, ...,\, y_n)$ is a coincidence point of F_Υ and G.*

(6) *If $(X,\, d,\, \preceq)$ is regular, then $(X^n,\, \Delta_n,\, \sqsubseteq)$ and $(X^n,\, \rho_n,\, \sqsubseteq)$ are also regular.*

The following definitions are usual in the field of fixed point theory.

Definition 5 ([5])**.** An ordered metric space $(X,\, d,\, \preceq)$ is said to be *non-decreasing-regular* (respectively, *non-increasing-regular*) if for every sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all n, we have that $x_n \preceq x$ (respectively, $x_n \succeq x$) for all n. $(X,\, d,\, \preceq)$ is said to be *regular* if it is both non-decreasing-regular and non-increasing-regular.

Definition 6 ([40])**.** Let $(X,\, \preceq)$ be a partially ordered set and T be a self-mapping on X^n. It is said that T has an *isotone property* if, for any $Y_1,\, Y_2 \in X^n$, we have

$$Y_1 \preceq Y_2 \Rightarrow T(Y_1) \preceq T(Y_2). \tag{8}$$

Remark 7. Note that if $n = 1$ in Definition 6, then T is a non-decreasing mapping (see [30]).

Definition 8 ([40]). An element $Y \in X^n$ is called a *fixed point* of the mapping $T : X^n \to X^n$ if $T(Y) = Y$.

3. Main Results

Let Θ denote the class of all functions $\theta : [0, +\infty) \to [0, 1)$ satisfying that for any sequence $\{s_n\}$ of non-negative real numbers $\theta(s_n) \to 1$ implies that $s_n \to 0$.

The following are examples of some functions belonging to Θ.

(1) $\theta(s) = k$ for all $s \geq 0$, where $k \in [0, 1)$.

(2) $\theta(s) = \begin{cases} \frac{\ln(1+s)}{s} & s > 0, \\ r \in [0,\ 1), & s = 0. \end{cases}$

(3) $\theta(s) = \begin{cases} \frac{\ln(1+ks)}{ks} & s > 0, \\ r \in [0,\ 1), & s = 0, \end{cases}$ where $k \in [0, 1)$.

Now, we will prove our main result.

Theorem 9. *Let$(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping for which there exists $\theta \in \Theta$ such that*

$$d(Tx,\ Ty) \leq \theta\left(d(x,\ y)\right) d(x,\ y), \tag{9}$$

for all $x, y \in X$ with $x \preceq y$. Suppose either

(a) T is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exists $x_0 \in X$ such that $x_0 \asymp Tx_0$, then T has a fixed point. Moreover, if for each $x, y \in X$ there exists $z \in X$ which is $\preceq -$comparable to x and y then the fixed point is unique.

Proof. Let $x_0 \in X$ be such that $x_0 \asymp Tx_0$. Take $x_1 \in X$ be such that $x_1 = Tx_0$, that is, $x_0 \asymp x_1$. Take $x_2 = Tx_1$, we have $Tx_0 \asymp Tx_1$, that is, $x_1 \asymp x_2$. Again, we have $Tx_1 \asymp Tx_2$. Proceeding by induction, we obtain a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} = Tx_n$ and $x_n \asymp x_{n+1}$ for each $n \geq 0$, that is,

$$x_0 \asymp x_1 \asymp x_2 ... \asymp x_n \asymp ..., \tag{10}$$

that is,

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq ... \text{ or } x_0 \succeq x_1 \succeq x_2 \succeq ... \succeq x_n \succeq ... \tag{11}$$

If $x_n = x_{n+1}$ for some n, then T has a fixed point and the proof of the existence of the fixed point is complete. Assume that $x_n \neq x_{n+1}$ for all n. Now, by contractive condition (9), we have

$$d(x_{n+1},\ x_{n+2}) = d(Tx_n,\ Tx_{n+1}) \le \theta\left(d(x_n,\ x_{n+1})\right) d(x_n,\ x_{n+1}), \tag{12}$$

which, by the fact that $\theta < 1$, implies

$$d(x_{n+1},\ x_{n+2}) < d(x_n,\ x_{n+1}),\ \text{for all } n \ge 0.$$

Thus the sequence $\{\delta_n\}_{n=0}^{\infty}$ given by

$$\delta_n = d(x_n,\ x_{n+1}),\ \text{for all } n \ge 0,$$

is decreasing. Hence there exists an $\delta \ge 0$ such that

$$\delta = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(x_n,\ x_{n+1}). \tag{13}$$

We claim that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then, from (12), we obtain that

$$\frac{d(x_{n+1},\ x_{n+2})}{d(x_n,\ x_{n+1})} \le \theta\left(d(x_n,\ x_{n+1})\right) < 1.$$

On taking limit as $n \to \infty$, we get

$$\theta\left(d(x_n,\ x_{n+1})\right) \to 1 \text{ as } n \to \infty.$$

Using the properties of function θ, we have

$$\delta_n = d(x_n,\ x_{n+1}) \to 0 \text{ as } n \to \infty.$$

which contradicts the assumption that $\delta > 0$. Hence

$$\delta = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{14}$$

We now prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $(X,\ d)$. Suppose, to the contrary, that the sequence $\{x_n\}_{n=0}^{\infty}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of $\{x_n\}_{n=0}^{\infty}$ with $n(k) > m(k) \ge k$ such that

$$d(x_{n(k)},\ x_{m(k)}) \ge \varepsilon. \tag{15}$$

We can choose $n(k)$ to be the smallest positive integer satisfying (15). Then

$$d(x_{n(k)-1},\ x_{m(k)}) < \varepsilon. \tag{16}$$

By (15), (16) and triangle inequality, we have

$$\begin{aligned} \varepsilon &\le r_k = d(x_{n(k)},\ x_{m(k)}) \\ &\le d(x_{n(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{m(k)}) \\ &< d(x_{n(k)},\ x_{n(k)-1}) + \varepsilon. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (14), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)}) = \varepsilon. \tag{17}$$

By the triangle inequality, we have

$$\begin{aligned} r_k &= d(x_{n(k)},\ x_{m(k)}) \\ &\leq d(x_{n(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{m(k)+1}) + d(x_{m(k)+1},\ x_{m(k)}) \\ &\leq \delta_{n(k)} + \delta_{m(k)} + d(Tx_{n(k)},\ Tx_{m(k)}) \\ &\leq \delta_{n(k)} + \delta_{m(k)} + \theta\left(d(x_{n(k)},\ x_{m(k)})\right) d(x_{n(k)},\ x_{m(k)}) \\ &\leq \delta_{n(k)} + \delta_{m(k)} + r_k. \end{aligned}$$

This shows that

$$r_k \leq \delta_{n(k)} + \delta_{m(k)} + \theta\left(d(x_{n(k)},\ x_{m(k)})\right) r_k \leq \delta_{n(k)} + \delta_{m(k)} + r_k.$$

On taking limit as $n \to \infty$ in the above inequality, by using (14) and (17), we get

$$\theta\left(d(x_{n(k)},\ x_{m(k)})\right) \to 1.$$

Using the properties of function θ, we obtain

$$d(x_{n(k)},\ x_{m(k)}) \to 0 \text{ as } k \to \infty,$$

which imply that

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(x_{n(k)},\ x_{m(k)}) = 0, \tag{18}$$

which contradicts with $\varepsilon > 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. As it is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x. \tag{19}$$

Suppose that (a) holds, that is, T is continuous. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx$, that is, x is a fixed point of T.

Suppose now that (b) holds. Since $x_n \to x$, $x_n \asymp x$, therefore by (9), we obtain $d(x_{n+1}, Tx) = d(Tx_n, Tx) \leq \theta\left(d(x_n,\ x)\right) d(x_n, x)$. On taking $n \to \infty$ in the above inequality and by using (19), we get $d(x, Tx) = 0$, that is, x is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Suppose T has another fixed point y. From the assumption, there exists $z \in X$ such that $x \asymp z$ and $y \asymp z$. If $z = x$ or $z = y$, it is trivial. We suppose that $z \neq x$ and $z \neq y$. Put $z_0 = z$ and choose $z_1 \in X$ such that $z_1 = Tz_0$. Then we have $z_0 \asymp x$, which implies that $Tz_0 \asymp Tx$, that is, $z_1 \asymp x$. Again, we have $Tz_1 \asymp Tx$, that is, $z_2 \asymp x$. Proceeding by induction, we obtain $z_{n+1} = Tz_n$ and $z_n \asymp x$. For definiteness we assume $x \neq z_n$ for all n. Similarly, we have $z_n \asymp y$ and $z_n \neq y$ for all n. By (9), we have

$$d(z_{n+1},\ x) = d(Tz_n,\ Tx) \leq \theta\,(d(z_n,\ x))\,d(z_n,\ x), \tag{20}$$

which, by the fact that $\theta < 1$, implies

$$d(z_{n+1},\ x) < d(z_n,\ x),\ \text{for all } n \geq 0.$$

Thus the sequence $\{d_n\}_{n=0}^{\infty}$ given by

$$d_n = d(z_n,\ x) \text{ for all } n \geq 0,$$

is decreasing. Hence there exists an $d \geq 0$ such that

$$d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d(z_n,\ x). \tag{21}$$

We claim that $d = 0$. Suppose, to the contrary, that $d > 0$. Then, from (20), we obtain that

$$\frac{d(z_{n+1},\ x)}{d(z_n,\ x)} \leq \theta\,(d(z_n,\ x)) < 1.$$

On taking limit as $n \to \infty$, we get

$$\theta\,(d(z_n,\ x)) \to 1 \text{ as } n \to \infty.$$

Using the properties of function θ, we have

$$d_n = d(z_n,\ x) \to 0 \text{ as } n \to \infty,$$

which contradicts the assumption that $d > 0$. Thus, we get $x = \lim_{n\to\infty} z_n$. Similarly, we can show that $y = \lim_{n\to\infty} z_n$. Thus $x = y$, that is, the fixed point of T is unique. □

Taking $\theta(s) = k$ with $k \in [0,\ 1)$ for all $s \in [0,\ \infty)$ in Theorem 9, we obtain the following corollary.

Corollary 10. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \to X$ be a non-decreasing mapping for which there exists $k \in [0,\ 1)$ such that*

$$d(Tx,\ Ty) \leq kd(x,\ y), \tag{22}$$

for all $x,\ y \in X$ with $x \preceq y$. Suppose either

(a) T is continuous or

(b) $(X,\ d,\ \preceq)$ is regular.

Next we give an $n-$dimensional fixed point theorem for mixed monotone mappings. For brevity, $(y_1, y_2, ..., y_n)$, $(v_1, v_2, ..., v_n)$ and $(y_0^1, y_0^2, ..., y_0^n)$ will be denoted by Y, V and Y_0 respectively. Consider the mapping $F_\Upsilon : X^n \to X^n$ defined by

$$F_\Upsilon(Y) = \left(\begin{array}{c} F(y_{\sigma_1(1)},\ y_{\sigma_1(2)},\ ...,\ y_{\sigma_1(n)}), \\ F(y_{\sigma_2(1)},\ y_{\sigma_2(2)},\ ...,\ y_{\sigma_2(n)}), \\ ...,\ F(y_{\sigma_n(1)},\ y_{\sigma_n(2)},\ ...,\ y_{\sigma_n(n)}) \end{array} \right),\ \text{for } Y \in X^n. \tag{23}$$

Under these conditions, the following properties hold:

Lemma 11. *Let $(X, d, \preceq)$ be a partially ordered metric space and let $F : X^n \to X$ be a mapping. Then*

(1) If there exists $y_0^1, y_0^2, ..., y_0^n \in X$ verifying $y_0^i \preceq_i F(y_0^{\sigma_i(1)}, y_0^{\sigma_i(2)}, ..., y_0^{\sigma_i(n)})$, for $i \in \Lambda_n$, then there exists $Y_0 \in X^n$ such that $Y_0 \sqsubseteq F_\Upsilon(Y_0)$.

(2) If F is a mixed monotone mapping, then F_Υ is an isotone mapping.

(3) If there exists $\theta \in \Theta$ such that

$$d(F(y_1,\ y_2,\ ...,\ y_n),\ F(v_1,\ v_2,\ ...,\ v_n)) \le \theta\left(\max_{1\le i\le n} d(y_i,\ v_i)\right) \max_{1\le i\le n} d(y_i,\ v_i), \tag{24}$$

for all $y_1, y_2, ..., y_n, v_1, v_2, ..., v_n \in X$ with $y_i \preceq_i v_i$, for $i \in \Lambda_n$, then

$$\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V)) \le \theta\left(\rho_n(Y,\ V)\right) \rho_n(Y,\ V), \tag{25}$$

for all $Y, V \in X^n$ with $Y \sqsubseteq V$.

(4) If for each $i \in \Lambda_n$ and $y_i, v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i, then there exists $Z \in X^n$ which is $\sqsubseteq -$comparable to Y and V.

Proof. (1) and (4) are obvious.

(2) Suppose that $Y \sqsubseteq V$ for $Y, V \in X^n$. By (2), we have $y_t \preceq v_t$ when $t \in A$ and $y_t \succeq v_t$ when $t \in B$. For each $i \in A$, we have $\sigma_i \in \Omega_{A,B}$. So $y_{\sigma_i(t)} \preceq v_{\sigma_i(t)}$, for each $i \in A$ and $y_{\sigma_i(t)} \succeq v_{\sigma_i(t)}$, for each $i \in B$. Thus, by the mixed monotone property of F, we have, for fixed $i \in A$,

$$\begin{aligned} &F(y_{\sigma_i(1)},\ ...,\ y_{\sigma_i(t-1)},\ y_{\sigma_i(t)},\ y_{\sigma_i(t+1)},\ ...,\ y_{\sigma_i(n)}) \\ &\preceq F(y_{\sigma_i(1)},\ ...,\ y_{\sigma_i(t-1)},\ v_{\sigma_i(t)},\ y_{\sigma_i(t+1)},\ ...,\ y_{\sigma_i(n)}), \end{aligned} \tag{26}$$

when $t \in A$. Similarly, if $t \in B$, then the inequality (26) holds for fixed $i \in A$. So, for fixed $i \in A$, inequality (26) holds for $t \in \Lambda_n$. From this, we have

$$\begin{aligned} F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}) &\preceq F(v_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}) \\ &\preceq F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}), \\ &\quad \ldots, \\ &\preceq F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ v_{\sigma_i(n)}). \end{aligned}$$

Thus

$$F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}) \preceq F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ v_{\sigma_i(n)}), \tag{27}$$

for $i \in A$. Similarly, we have

$$F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}) \succeq F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ v_{\sigma_i(n)}), \tag{28}$$

for $i \in B$. From (23), (27) and (28), we deduce that F_Υ is an isotone mapping.

(3) Suppose that $Y \sqsubseteq V$ for Y, $V \in X^n$. For fixed $i \in A$, we have $y_{\sigma_i(t)} \preceq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. From (24), we have

$$\begin{aligned} &d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ v_{\sigma_i(n)})) \\ &\leq \theta\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right) \max_{1\leq i\leq n} d(y_i,\ v_i), \end{aligned} \tag{29}$$

for all $i \in A$. Similarly, for fixed $i \in B$, we have $y_{\sigma_i(t)} \succeq_t v_{\sigma_i(t)}$ for $t \in \Lambda_n$. It follows from (29) that

$$\begin{aligned} &d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ v_{\sigma_i(n)})) \\ &\leq d(F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ \ldots,\ v_{\sigma_i(n)}),\ F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ \ldots,\ y_{\sigma_i(n)})) \\ &\leq \theta\left(\max_{1\leq i\leq n} d(y_i,\ v_i)\right) \max_{1\leq i\leq n} d(y_i,\ v_i), \end{aligned} \tag{30}$$

for all $i \in B$. By (2), (23), (29) and (30), we have

$$\rho_n(F_\Upsilon(Y),\ F_\Upsilon(V)) \leq \theta\left(\rho_n(Y,\ V)\right) \rho_n(Y,\ V),$$

for all Y, $V \in X^n$ with $Y \sqsubseteq V$. □

Theorem 12. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\Upsilon = (\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^n \to X$ be a mixed monotone mapping for which there exists $\theta \in \Theta$ satisfying (24). Also, suppose that either F is continuous or $(X,\ d,\ \preceq)$ is regular. If there exists $y_0^1,\ y_0^2,\ \ldots,\ y_0^n \in X$ verifying $y_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ \ldots,\ y_0^{\sigma_i(n)})$, for $i \in \Lambda_n$, then F has a $\Upsilon-$fixed point. Moreover, if for each $i \in \Lambda_n$ and*

y_i, $v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i. Then F has a unique $\Upsilon-$fixed point.

Proof. It is only necessary to apply Theorem 9 to the mappings $T = F_\Upsilon$ in the ordered metric space $(X^n,\ \rho_n,\ \sqsubseteq)$ taking into account all items of Lemma 4 and Lemma 11. □

Theorem 13. *Let $(X,\ \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\Upsilon = (\sigma_1,\ \sigma_2,\ ...,\ \sigma_n)$ be an $n-$tuple of mappings from Λ_n into itself verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega^{'}_{A,B}$ if $i \in B$. Let $F : X^n \to X$ be a mixed monotone mapping for which there exists $\theta \in \Theta$ such that*

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} d(F(y_{\sigma_i(1)},\ y_{\sigma_i(2)},\ ...,\ y_{\sigma_i(n)}),\ F(v_{\sigma_i(1)},\ v_{\sigma_i(2)},\ ...,\ v_{\sigma_i(n)})) \\ &\leq \theta\left(\frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i)\right)\left(\frac{1}{n}\sum_{i=1}^{n} d(y_i,\ v_i)\right), \end{aligned} \tag{31}$$

for all $y_1,\ y_2,\ ...,\ y_n,\ v_1,\ v_2,\ ...,\ v_n \in X$ with $y_i \preceq_i v_i$, for $i \in \Lambda_n$. Also, suppose that either F is continuous or $(X,\ d,\ \preceq)$ is regular. If there exists $y_0^1,\ y_0^2,\ ...,\ y_0^n \in X$ verifying $y_0^i \preceq_i F(y_0^{\sigma_i(1)},\ y_0^{\sigma_i(2)},\ ...,\ y_0^{\sigma_i(n)})$, for $i \in \Lambda_n$, then F has a $\Upsilon-$fixed point. Moreover, if for each $i \in \Lambda_n$ and $y_i,\ v_i \in X$ there exists $z_i \in X$ which is $\preceq_i -$comparable to y_i and v_i. Then F has a unique $\Upsilon-$fixed point.

Proof. Notice that the contractivity condition (24) means that

$$\Delta_n(F_\Upsilon(Y),\ F_\Upsilon(V)) \leq \theta\left(\Delta_n(Y,\ V)\right)\Delta_n(Y,\ V), \tag{32}$$

for all $Y, V \in X^n$ with $Y \sqsubseteq V$. Therefore it is only necessary to use Theorem 9 to the mappings $T = F_\Upsilon$ in the ordered metric space $(X^n,\ \Delta_n,\ \sqsubseteq)$ taking into account all items of Lemma 4 and Lemma 11.

In a similar way, we may state the results analog of Corollary 10 for Theorem 12 and Theorem 13. □

Example 14. Suppose that $X = \mathbb{R}$, equipped with the usual metric $d : X\times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$. Let $T : X \to X$ be defined as

$$Tx = \ln(1 + x),\ \text{for all } x \in X.$$

Define $\theta : [0, +\infty) \to [0, 1)$ as follows

$$\theta(s) = \begin{cases} \frac{\ln(1+s)}{s}, \ s > 0, \\ 0, \ s = 0. \end{cases}$$

First, we shall show that the contractive condition (9) holds for the mapping T. Let $x, y \in X$ such that $x \preceq y$, we have

$$\begin{aligned} d(Tx,\ Ty) &= |Tx - Ty| \\ &= |\ln(1+x) - \ln(1+y)| \\ &= \left|\ln \frac{1+x}{1+y}\right| \\ &= \left|\ln\left(1 + \frac{x-y}{1+y}\right)\right| \\ &\leq \ln(1 + |x-y|) \\ &\leq \frac{\ln(1+|x-y|)}{|x-y|} \times |x-y| \\ &\leq \frac{\ln(1+d(x,\ y))}{d(x,\ y)} \times d(x,\ y) \\ &\leq \theta(d(x,\ y))\, d(x,\ y). \end{aligned}$$

This shows that condition (9) holds with the function θ. In addition, all the other conditions of Theorem 9 are satisfied and $z = 0$ is a fixed point of T.

4. Applications

In this section, based on the results in [15], we propose an application to our results. Consider the integral equation

$$u(t) = \int_0^T K(t,\ s,\ u(s))ds + g(t),\ t \in [0,\ T], \tag{33}$$

where $T > 0$. We introduce the following space:

$$C[0,\ T] = \{u : [0,\ T] \to \mathbb{R} : u \text{ is continuous on } [0,\ T]\},$$

equipped with the metric

$$d(x,\ y) = \sup_{t\in[0,\ T]} |x(t) - y(t)|,\ \text{for each } x,\ y \in C[0,\ T].$$

It is clear that $(C[0,\ T],\ d)$ is a complete metric space. Furthermore, $C[0,\ T]$ can be equipped with the partial order $\preceq$ as follows: for $x, y \in C[0, T]$,

$$x \preceq y \Longleftrightarrow x(t) \leq y(t),\ \text{for each } t \in [0,\ T].$$

Due to [29], we know that $(C[0,\, T],\, d,\, \preceq)$ is regular.

Now, we state the main result of this section.

Theorem 15. *We assume that the following hypotheses hold:*
(i) $K : [0,\, T] \times [0,\, T] \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*
(ii) *for all* $s,\, t,\, u,\, v \in C[0,\, T]$ *with* $v \preceq u$, *we have*

$$K(t,\; s,\; v(s)) \leq K(t,\; s,\; u(s)),$$

(iii) *there exists a continuous function* $G : [0,\, T] \times [0,\, T] \to [0,\, +\infty)$ *such that*

$$|K(t,\; s,\; x) - K(t,\; s,\; y)| \leq G(t,\; s) \cdot \ln\left(1 + |x - y|\right),$$

for all $s,\, t \in C[0,\, T]$ *and* $x,\, y \in \mathbb{R}$ *with* $x \geq y$,
(iv) $\sup_{t\in[0,\, T]} \int_0^T G(t,\, s)^2 ds \leq \frac{1}{T}$.
Then the integral (33) *has a solution* $u^* \in C[0,\, T]$.

Proof. We, first, define $\theta : [0,\, +\infty) \to [0,\, 1)$ as follows

$$\theta(s) = \begin{cases} \frac{\ln(1+s)}{s}, & s > 0, \\ 0, & s = 0, \end{cases}$$

and define $T : C[0,\, T] \to C[0,\, T]$ by

$$Tu(t) = \int_0^T K(t,\; s,\; u(s))ds + g(t), \text{ for all } t \in [0,\; T] \text{ and } u \in C[0,\; T].$$

We first prove that T is non-decreasing. Assume that $v \preceq u$. From (ii), for all s, $t \in [0,\, T]$, we have $K(t,\, s,\, u(s)) \leq K(t,\, s,\, v(s))$. Thus, we get,

$$Tv(t) = \int_0^T K(t,\; s,\; v(s))ds + g(t) \leq \int_0^T K(t,\; s,\; u(s))ds + g(t) = Tu(t).$$

Now, for all $u,\, v \in C[0,\, T]$ with $v \preceq u$, due to (iii) and by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} & |Tu(t) - Tv(t)| \\ \leq\; & \int_0^T |K(t,\; s,\; u(s)) - K(t,\; s,\; v(s))|\, ds \\ \leq\; & \int_0^T G(t,\; s) \cdot \ln\left(1 + |u(s) - v(s)|\right) ds \\ \leq\; & \left(\int_0^T G(t,\; s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^T \left(\ln\left(1 + |u(s) - v(s)|\right)^2\right) ds\right)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$|Tu(t) - Tv(t)| \leq \left(\int_0^T G(t,\ s)^2 ds\right)^{\frac{1}{2}} \left(\int_0^T (\ln(1 + |u(s) - v(s)|))^2\, ds\right)^{\frac{1}{2}}. \tag{34}$$

Taking (iv) into account, we estimate the first integral in (34) as follows:

$$\left(\int_0^T G(t,\ s)^2 ds\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}}. \tag{35}$$

For the second integral in (34) we proceed in the following way:

$$\left(\int_0^T (\ln(1 + |u(s) - v(s)|))^2\, ds\right)^{\frac{1}{2}} \leq \sqrt{T} \cdot \ln(1 + d(u,\ v)). \tag{36}$$

Combining (34), (35) and (36), we conclude that

$$|Tu(t) - Tv(t)| \leq \ln(1 + d(u,\ v)).$$

It yields

$$d(Tu,\ Tv) \leq \frac{\ln(1 + d(u,\ v))}{d(u,\ v)} \times d(u,\ v),$$

which implies that

$$d(Tu,\ Tv) \leq \theta(d(u,\ v))\, d(u,\ v),$$

for all $u,\ v \in C[0,\ T]$ with $v \preceq u$. Hence, all hypotheses of Theorem 9 are satisfied. Thus, T has a fixed point $u^* \in C[0,\ T]$ which is a solution of (33). □

REFERENCES

1. H. Aydi, E. Karapinar & W. Shatanawi: Coupled fixed point results for $(\psi,\ \varphi)-$weakly contractive condition in ordered partial metric spaces. Comput. Math. Appl. **62** (2011), no. 12, 4449-4460.
2. M. Borcut & V. Berinde: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218** (2012), no. 10, 5929-5936.
3. V. Berinde & M. Borcut: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74** (2011), 4889-4897.
4. M. Berzig & B. Samet: An extension of coupled fixed point's concept in higher dimension and applications. Comput. Math. Appl. **63** (2012), no. 8, 1319-1334.
5. T.G. Bhaskar & V. Lakshmikantham: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65** (2006), no. 7, 1379-1393.
6. B.S. Choudhury & A. Kundu: A coupled coincidence point results in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73** (2010), 2524-2531.
7. B.S. Choudhury & A. Kundu: $(\psi,\ \alpha,\ \beta)$-weak contractions in partially ordered metric spaces. Appl. Math. Lett. **25** (2012), no. 1, 6-10.

8. B.S. Choudhury, N. Metiya & M. Postolache: A generalized weak contraction principle with applications to coupled coincidence point problems. Fixed Point Theory Appl. 2013:152.
9. B. Deshpande & A. Handa: Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations. Afr. Mat. **26** (2015), no. (3-4), 317-343.
10. B. Deshpande & A. Handa: Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces. Adv. Fuzzy Syst. Volume 2014, Article ID 348069.
11. H.S. Ding, L. Li & S. Radenovic: Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces. Fixed Point Theory Appl. 2012:96.
12. I.M. Erhan, E. Karapinar, A. Roldan & N. Shahzad: Remarks on coupled coincidence point results for a generalized compatible pair with applications. Fixed Point Theory Appl. 2014:207.
13. D. Guo & V. Lakshmikantham: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11** (1987), no. 5, 623-632.
14. J. Harjani & K. Sadarangani: Fixed point theorems for weakly contractive mappings in partially ordered sets. Nonlinear Anal. **71** (2009), 3403-3410.
15. J. Harjani & K. Sadarangani: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. **72** (2010), no. (3-4), 1188-1197.
16. X.Q. Hu: Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces. Fixed Point Theory Appl. 2011, Article ID 363716.
17. N.M. Hung, E. Karapinar & N.V. Luong: Coupled coincidence point theorem for O-compatible mappings via implicit relation. Abstr. Appl. Anal. 2012, Article ID 796964.
18. E. Karapinar: Quadruple fixed point theorems for weak ϕ-contractions. ISRN Mathematical Analysis 2011, Article ID 989423.
19. E. Karapinar & V. Berinde: Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces. Banach J. Math. Anal. **6** (2012), no. 1, 74-89.
20. E. Karapinar & N.V. Luong: Quadruple fixed point theorems for nonlinear contractions. Comput. Math. Appl. **64** (2012), no. 6, 1839-1848.
21. E. Karapinar & A. Roldan: A note on n-Tuplet fixed point theorems for contractive type mappings in partially ordered metric spaces. J. Inequal. Appl. 2013, Article ID 567.
22. E. Karapinar, A. Roldan, C. Roldan & J. Martinez-Moreno: A note on N-Fixed point theorems for nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. 2013, Article ID 310.

23. E. Karapinar, A. Roldan, J. Martinez-Moreno & C. Roldan: Meir-Keeler type multidimensional fixed point theorems in partially ordered metric spaces. Abstr. Appl. Anal. 2013, Article ID 406026.
24. E. Karapinar, A. Roldan, N. Shahzad & W. Sintunavarat: Discussion on coupled and tripled coincidence point theorems for $\phi-$contractive mappings without the mixed g-monotone property. Fixed Point Theory Appl. 2014, Article ID 92.
25. V. Lakshmikantham & L. Ciric: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70** (2009), no. 12, 4341-4349.
26. N.V. Luong & N.X. Thuan: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74** (2011), 983-992.
27. N.V. Luong & N.X. Thuan: Coupled fixed points in ordered generalized metric spaces and application to integro-differential equations. Comput. Math. Appl. **62** (2011), no. 11, 4238-4248.
28. S.A. Al-Mezel, H. Alsulami, E. Karapinar & A. Roldan: Discussion on multidimensional coincidence points via recent publications. Abstr. Appl. Anal. Volume 2014, Article ID 287492.
29. J.J. Nieto, R.L. Pouso & R. Rodriguez-Lopez: Fixed point theorems in partially ordered sets. Proc. Amer. Math. Soc. **132** (2007), no. 8, 2505-2517.
30. J.J. Nieto & R. Rodriguez-Lopez: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22** (2005), no. 3, 223-239.
31. A.C.M. Ran & M.C.B. Reurings: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc. **132** (2004), 1435-1443.
32. A. Roldan, J. Martinez-Moreno & C. Roldan: Multidimensional fixed point theorems in partially ordered metric spaces. J. Math. Anal. Appl. **396** (2012), 536-545.
33. A. Roldan & E. Karapinar: Some multidimensional fixed point theorems on partially preordered G^*-metric spaces under (ψ, φ)-contractivity conditions. Fixed Point Theory Appl. 2013, Article ID 158.
34. A. Roldan, J. Martınez-Moreno, C. Roldan & E. Karapinar: Multidimensional fixed-point theorems in partially ordered complete partial metric spaces under (ψ, φ)-contractivity conditions. Abstr. Appl. Anal. 2013, Article ID 634371.
35. A. Roldan, J. Martınez-Moreno, C. Roldan & E. Karapinar: Some remarks on multidimensional fixed point theorems. Fixed Point Theory **15** (2014), no. 2, 545-558.
36. B. Samet: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. **72** (2010), no. 12, 4508-4517.
37. B. Samet, E. Karapinar, H. Aydi & V.C. Rajic: Discussion on some coupled fixed point theorems. Fixed Point Theory Appl. 2013:50.
38. W. Shatanawi, B. Samet & M. Abbas: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. Math. Comput. Modelling **55** (2012), no. (3-4), 680-687.

39. S. Wang: Coincidence point theorems for G-isotone mappings in partially ordered metric spaces. Fixed Point Theory Appl. (2013), 1687-1812-2013-96.
40. S. Wang: Multidimensional fixed point theorems for isotone mappings in partially ordered metric spaces. Fixed Point Theory Appl. 2014:137.

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