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FILTER SPACES ON ECL-PREMONOIDS

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Abstract. In this paper, we introduce the notion of the $(L,*)$-filter spaces on ecl-premonoids. Moreover, we obtain various $(L,*)$-filters incuced by two $(L,*)$-filters and give their examples.

1. Introduction

Filter spaces are very useful tools in several area of mathematical structures with direct applications, both mathematical (e.g. topology, logic) and extramathematical (e.g. data mining, knowledge representation). In fuzzy set theory, Gäher [2,3] introduced the notions of fuzzy filters in a frame L. Höhle and Sostak [4] introduced the concept of L-filters for a complete quasimonoidal lattice L. Kim and Ko [8,9] introduced the images and preimages of L-filter bases on stsc quantales and developed $(L,*,\odot)$-quasiuniform convergence spaces on ecl-premonoid in Orpen's sense [10].

In this paper, we introduce the notion of the $(L,*)$-filter spaces on ecl-premonoids in Orpen's sense [10]. Moreover, we obtain various $(L,*)$-filters incuced by two $(L,*)$-filters and give their examples.

2. Preliminaries

In this paper, we consider complete lattices $(L,\leq,\perp,\top)$ with bottom element $\perp$ and top element $\top$.

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Definition 2.1 ([1, 4, 10])**.** A complete lattice $(L, \leq, \bot, \top)$ is called a *GL-monoid* $(L, \leq, *, \bot, \top)$ with a binary operation $* : L \times L \to L$ satisfying the following conditions:

(G1) $a * \top = a$, for all $a \in L$,

(G2) $a * b = b * a$, for all $a, b \in L$,

(G3) $a * (b * c) = (a * b) * c$, for all $a, b \in L$,

(G4) if $a \leq b$, there exists $c \in L$ such that $b * c = a$,

(G5) $a * \bigvee_{i\in\Gamma} b_i = \bigvee_{i\in\Gamma}(a * b_i)$.

We can define an implication operator:

$$a \Rightarrow b = \bigvee\{c \mid a * c \leq b\}.$$

Remark 2.2 ([1, 4, 10])**.** (1) A continuous t-norm $([0, 1], \leq, *)$ is a GL-monoid.

(2) A frame $(L, \leq, \wedge)$ is a GL-monoid.

Definition 2.3 ([1, 4, 10])**.** A complete lattice $(L, \leq, \bot, \top)$ is called a *cl-premonoid* $(L, \leq, \odot)$ with a binary operation $\odot : L \times L \to L$ satisfying the following conditions:

(CL1) $a \leq a \odot \top$ and $a \leq \top \odot a$, for all $a \in L$,

(CL2) if $a \leq b$ and $c \leq d$, then $a \odot c \leq b \odot d$,

(CL3) $a \odot \bigvee_{i\in\Gamma} b_i = \bigvee_{i\in\Gamma}(a \odot b_i)$ and $\bigvee_{j\in\Gamma} a_j \odot b = \bigvee_{j\in\Gamma}(a_j \odot b)$.

We can define an implication operator:

$$a \to b = \bigvee\{c \mid a \odot c \leq b\}.$$

Example 2.4. (1) Every GL-monoid $(L, \leq, *)$ is a cl-premonoid.

(2) Defines maps $\odot_i : [0, 1] \times [0, 1] \to [0, 1]$ as follows:

$$x \odot_1 y = x^{\frac{1}{p}} \cdot y^{\frac{1}{p}} (p \geq 1), x \odot_2 y = (x^p + y^p) \wedge 1 (p \geq 1).$$

Then $(L, \leq, \odot_i)$ is a cl-premonoid for $i = 1, 2$.

Definition 2.5 ([1, 4, 10])**.** A complete lattice $(L, \leq, \bot, \top)$ is called an *ecl-premonoid* $(L, \leq, \odot, *)$ with a GL-monoid $(L, \leq, *)$ and a cl-premonoid $(L, \leq, \odot)$ which satisfy the following condition:

(D) $(a \odot b) * (c \odot d) \leq (a * c) \odot (b * d)$, for all $a, b, c, d \in L$.

An ecl-premonoid $(L, \leq, \odot, *)$ is called an M-*ecl-premonoid* if it satisfies the following condition:

(M) $a \leq a \odot a$ for all $a \in L$.

Example 2.6. (1) Let $(L, \leq, *)$ be a GL-monoid and $(L, \leq, \wedge)$ is a cl-premonoid. Then $(L, \leq, \wedge, *)$ is an M-ecl-premonoid.

(2) Let $(L, \leq, *)$ be a GL-monoid. Then $(L, \leq, *, *)$ is an ecl-premonoid. If $* = \cdot$, $0.5 \not\leq 0.5 \cdot 0.5 = 0.25$. $(L, \leq, \cdot, \cdot)$ is not an M-ecl-premonoid.

(3) Let $(L, \leq, \cdot)$ be a GL-monoid. Define a map $\odot : [0,1] \times [0,1] \to [0,1]$ as $x \odot y = (x + y) \wedge 1$. Then $(L, \leq, \odot, \cdot)$ is not an M-cl-premonoid because

$$0.7 = (0.3 \odot 0.4) \cdot (0.5 \odot 0.7) \not\leq (0.3 \cdot 0.5) \odot (0.4 \cdot 0.7) = 0.15 + 0.28 = 0.43$$

(4) Let $(L, \leq, \cdot)$ be a GL-monoid. Define a map $\odot : [0,1] \times [0,1] \to [0,1]$ as $x \odot y = x^{\frac{1}{3}} \cdot y^{\frac{1}{3}}$. Then $(L, \leq, \odot, \cdot)$ is an M-cl-premonoid.

In this paper, we always assume that $(L, \leq, \odot, *)$ is an ecl-premonoid unless otherwise specified.

Lemma 2.7 ([1, 4, 9, 10])**.** *Let $(L, \leq, \odot, *)$ be an ecl-premonoid. For each $a, b, c, d, a_i, b_i \in L$ and for $\uparrow \in \{\rightarrow, \Rightarrow\}$, we have the following properties.*

(1) *If $b \leq c$, then $a \odot b \leq a \odot c$ and $a * b \leq a * c$.*

(2) *$a \odot b \leq c$ iff $a \leq b \rightarrow c$. Moreover, $a * b \leq c$ iff $a \leq b \Rightarrow c$.*

(3) *If $b \leq c$, then $a \uparrow b \leq a \uparrow c$ and $c \uparrow a \leq b \uparrow a$.*

(4) *$a \leq b$ iff $a \Rightarrow b = \top$.*

(5) *$a * b \leq a \odot b$, $a \rightarrow b \leq a \Rightarrow b$ and $a * (b \odot c) \leq (a * b) \odot c$.*

(6) *$(a \uparrow b) \odot (c \uparrow d) \leq (a \odot c) \uparrow (b \odot d)$.*

(7) *$(b \uparrow c) \leq (a \odot b) \uparrow (a \odot c)$.*

(8) *$(b \uparrow c) \leq (a \uparrow b) \uparrow (a \uparrow c)$ and $(b \uparrow a) \leq (a \uparrow c) \uparrow (b \uparrow c)$.*

(9) *$(b \rightarrow c) \leq (a \uparrow b) \rightarrow (a \uparrow c)$ and $(b \uparrow a) \leq (a \rightarrow c) \rightarrow (b \uparrow c)$*

(10) *$a_i \uparrow b_i \leq (\bigwedge_{i \in \Gamma} a_i) \uparrow (\bigwedge_{i \in \Gamma} b_i)$.*

(11) *$a_i \uparrow b_i \leq (\bigvee_{i \in \Gamma} a_i) \uparrow (\bigvee_{i \in \Gamma} b_i)$.*

(12) *$(c \uparrow a) * (b \rightarrow d) \leq (a \rightarrow b) \rightarrow (c \uparrow d)$.*

Definition 2.8 ([4, 10])**.** A mapping $\mathcal{F} : L^X \to L$ is called an $(L, *)$-*filter* on X if it satisfies the following conditions:

(F1) $\mathcal{F}(\bot_X) = \bot$ and $\mathcal{F}(\top_X) = \top$, where $\bot_X(x) = \bot, \top_X(x) = \top$ for $x \in X$.

(F2) $\mathcal{F}(f * g) \geq \mathcal{F}(f) * \mathcal{F}(g)$, for each $f, g \in L^X$,

(F3) if $f \leq g$, $\mathcal{F}(f) \leq \mathcal{F}(g)$.

An $(L, *)$-filter is called *stratified* if

(S) $\mathcal{F}(\alpha * f) \geq \alpha * \mathcal{F}(f)$ for each $f \in L^X$ and $\alpha \in L$.

The pair $(X, \mathcal{F})$ is called an (resp. a stratified)$(L, *)$-*filter space.* Let $F_*(X)$ (resp. $F_*^s(X)$) is a family of (resp. stratified) $(L, *)$-filters on X.

Example 2.9. (1) Define a map $[x] : L^X \to L$ as $[x](f) = f(x)$. Then $[x]$ is a stratified $(L,*)$-filter on X.

(2) Define a map inf : $L^X \to L$ as $\inf(f) = \bigwedge_{x\in X} f(x)$. Then inf is a stratified $(L,*)$-filter on X.

3. Filter Spaces on ecl-premonoids

Theorem 3.1. *For $\mathcal{F},\mathcal{G} \in F_*(X)$ and for $\diamond \in \{\odot, *\}$, we define $\mathcal{F}\diamond\mathcal{G}, \mathcal{F}\diamond_*\mathcal{G} : L^X \to L$ as follows:*

$$(\mathcal{F}\diamond\mathcal{G})(h) = \mathcal{F}(h)\diamond\mathcal{G}(h)$$

$$\mathcal{F}\diamond_*\mathcal{G}(h) = \bigvee\{\mathcal{F}(f)\diamond\mathcal{G}(g) \mid f*g \le h\}.$$

Then we have the following properties:

(1) *$\mathcal{F}\diamond\mathcal{G}$ is an $(L,*)$-filter on X for $\diamond\in\{\odot,*\}$.*

(2) *If $(L,\le,\odot,*)$ is an M-ecl-premonoid and $\mathcal{F},\mathcal{G}\in F^s_*(X)$, then $\mathcal{F}\odot\mathcal{G}\in F^s_*(X)$.*

(3) *If $f*g=\perp$ implies $\mathcal{F}(f)\odot\mathcal{G}(g)=\perp$, then $\mathcal{F}\odot_*\mathcal{G}\in F_*(X)$ is the filter finer than $\mathcal{F}$ and $\mathcal{G}$.*

(4) *If $f*g=\perp$ implies $\mathcal{F}(f)*\mathcal{G}(g)=\perp$, then $\mathcal{F}*_*\mathcal{G}$ is an $(L,*)$-filter on X which is the coarsest filter finer than $\mathcal{F}$ and $\mathcal{G}$.*

(5) *If $\mathcal{F}\in F^s_*(X)$ or $\mathcal{G}\in F^s_*(X)$, then $\mathcal{F}\diamond_*\mathcal{G}\in F^s_*(X)$ for $\diamond\in\{\odot,*\}$.*

(6) *If $\mathcal{F}\in F^s_*(X)$, then $\mathcal{F}*_*(\bigwedge_{x\in X}[x])=\mathcal{F}$.*

(7) *$\mathcal{F}*_*\mathcal{F}=\mathcal{F}$ and $(\mathcal{F}*_*\mathcal{G})*_*\mathcal{H}=\mathcal{F}*_*(\mathcal{G}*_*\mathcal{H})$.*

(8) *$(\mathcal{F}_1\odot\mathcal{F}_2)\diamond_*(\mathcal{G}_1\odot\mathcal{G}_2)\le(\mathcal{F}_1\diamond_*\mathcal{G}_1)\odot(\mathcal{F}_2\diamond_*\mathcal{G}_2)$ for $\diamond\in\{\odot,*\}$.*

Proof. (1) Since $\top=\top\odot\top$ and $\perp=\perp\odot\perp$ from (CL3), $(\mathcal{F}\odot\mathcal{G})(\top)=\top$ and $(\mathcal{F}\odot\mathcal{G})(\perp)=\perp$. For each $f,g\in L^X$,

$$\begin{aligned}&(\mathcal{F}\odot\mathcal{G})(f*g)=\mathcal{F}(f*g)\odot\mathcal{G}(f*g)\ge(\mathcal{F}(f)*\mathcal{F}(g))\odot(\mathcal{G}(f)*\mathcal{G}(g))\\&\ge(\mathcal{F}(f)\odot\mathcal{G}(f))*(\mathcal{F}(g)\odot\mathcal{G}(g))=(\mathcal{F}\odot\mathcal{G})(f)*(\mathcal{F}\odot\mathcal{G})(g).\end{aligned}$$

Hence $\mathcal{F}\odot\mathcal{G}$ is an $(L,*)$-filter. Similarly, $\mathcal{F}*\mathcal{G}$ is an $(L,*)$-filter.

(2) For each $f\in L^X$ and $\alpha\in L$,

$$\begin{aligned}&(\mathcal{F}\odot\mathcal{G})(\alpha*f)=\mathcal{F}(\alpha*f)\odot\mathcal{G}(\alpha*f)\ge(\alpha*\mathcal{F}(f))\odot(\alpha*\mathcal{G}(f))\\&\ge(\alpha\odot\alpha)*(\mathcal{F}(f)\odot\mathcal{G}(f))\ge\alpha*(\mathcal{F}\odot\mathcal{G})(f).\end{aligned}$$

Hence $\mathcal{F}\odot\mathcal{G}$ is a stratified $(L,*)$-filter.

(3) (F1) Since $f*g=\perp$ implies $\mathcal{F}(f)\odot\mathcal{G}(g)=\perp$, $(\mathcal{F}\odot_*\mathcal{G})(\perp_X)=\perp$.

(F2) is easy. (F3)

$$\begin{aligned}&(\mathcal{F}\odot_*\mathcal{G})(h_1)*(\mathcal{F}\odot_*\mathcal{G})(h_2)\\&=\bigvee\{\mathcal{F}(f_1)\odot\mathcal{G}(g_1)\mid f_1*g_1\leq h_1\}*\bigvee\{\mathcal{F}(f_2)\odot\mathcal{G}(g_2)\mid f_2*g_2\leq h_2\}\\&=\bigvee\{(\mathcal{F}(f_1)\odot\mathcal{G}(g_1))*(\mathcal{F}(f_2)\odot\mathcal{G}(g_2))\mid f_1*g_1\leq h_1,f_2*g_2\leq h_2\}\\&\leq\bigvee\{(\mathcal{F}(f_1)*\mathcal{F}(g_2))\odot(\mathcal{G}(g_1)*\mathcal{G}(g_2))\mid (f_1*g_1)*(f_2*g_2)\leq h_1\odot h_2\}\\&\leq\bigvee\{\mathcal{F}(f_1*f_2)\odot\mathcal{G}(g_1*g_2)\mid (f_1*f_2)*(g_1*g_2)\leq h_1*h_2\}\\&\leq(\mathcal{F}\odot_*\mathcal{G})(h_1*h_2).\end{aligned}$$

(4) By a similar method, $\mathcal{F}*_*\mathcal{G}$ is an $(L,*)$-filter on X.

If $\mathcal{F}\leq\mathcal{H}$ and $\mathcal{G}\leq\mathcal{H}$, then $\mathcal{F}*_*\mathcal{G}\leq\mathcal{H}$ from

$$\begin{aligned}&(\mathcal{F}*_*\mathcal{G})(h)=\bigvee\{\mathcal{F}(f)*\mathcal{G}(g)\mid f*g\leq h\}\\&\leq\bigvee\{\mathcal{H}(f)*\mathcal{H}(g)\mid f*g\leq h\}\leq\bigvee\{\mathcal{H}(f*g)\mid f*g\leq h\}\leq\mathcal{H}(h).\end{aligned}$$

(5) Let $\mathcal{F}\in F^s_*(X)$. Since $(a\odot\top)*(b\odot c)\leq(a*b)\odot(\top*c)$, we have

$$\begin{aligned}&\alpha*(\mathcal{F}\odot_*\mathcal{G})(h)=\alpha*\bigvee\{\mathcal{F}(f)\odot\mathcal{G}(g)\mid f*g\leq h\}\\&\leq(\alpha\odot\top)*\bigvee\{\mathcal{F}(f)\odot\mathcal{G}(g)\mid f*g\leq h\}\\&=\bigvee\{(\alpha*\mathcal{F}(f))\odot\mathcal{G}(g)\mid\alpha*f*g\leq\alpha*h\}\\&\leq\bigvee\{\mathcal{F}(\alpha*f)\odot\mathcal{G}(g)\mid\alpha*f*g\leq\alpha*h\}\leq(\mathcal{F}\odot_*\mathcal{G})(\alpha*h).\end{aligned}$$

Similarly, $\mathcal{F}*_*\mathcal{G}\in F^s_*(X)$.

(6) By (4), $\mathcal{F}*_*(\bigwedge_{x\in X}[x])\geq\mathcal{F}$. Moreover, we have

$$\begin{aligned}&(\mathcal{F}*_*(\textstyle\bigwedge_{x\in X}[x]))(h)=\bigvee\{\mathcal{F}(f)*(\textstyle\bigwedge_{x\in X}[x])(g)\mid f*g\leq h\}\\&=\bigvee\{\mathcal{F}(f)*\textstyle\bigwedge_{x\in X}g(x)\mid f*g\leq h\}\\&\leq\bigvee\{\mathcal{F}(f*\textstyle\bigwedge_{x\in X}g(x))\mid f*g\leq h\}\leq\mathcal{F}(h).\end{aligned}$$

(7) $\mathcal{F}*_*\mathcal{F}$ is finer than $\mathcal{F}$ from (4). It follows from:

$$\begin{aligned}(\mathcal{F}*_*\mathcal{F})(h)&=\bigvee\{\mathcal{F}(f)*\mathcal{F}(g)\mid f*g\leq h\}\\&\leq\bigvee\{\mathcal{F}(f*g)\mid f*g\leq h\}=\mathcal{F}(h),\end{aligned}$$

$$\begin{aligned}((\mathcal{F}*_*\mathcal{G})*_*\mathcal{H})(l)&=\bigvee\{(\mathcal{F}*_*\mathcal{G})(k)*\mathcal{H}(h)\mid k*h\leq l\}\\&=\bigvee\{\bigvee(\mathcal{F}(f)*\mathcal{G}(g))*\mathcal{H}(h)\mid f*g\leq k,k*h\leq l\}\\&=\bigvee\{(\mathcal{F}(f)*\mathcal{G}(g))*\mathcal{H}(h)\mid f*g*h\leq l\}\\&=\bigvee\{\mathcal{F}(f)*(\mathcal{G}(g)*\mathcal{H}(h))\mid f*g*h\leq l\}\\&=(\mathcal{F}*_*(\mathcal{G}*_*\mathcal{H}))(l).\end{aligned}$$

(8) For $\diamond=*$,

$$\begin{aligned}&((\mathcal{F}_1\odot\mathcal{F}_2)*_*(\mathcal{G}_1\odot\mathcal{G}_2))(h)\\&=\bigvee\{(\mathcal{F}_1\odot\mathcal{F}_2)(f)*(\mathcal{G}_1\odot\mathcal{G}_2)(g)\mid f*g\leq h\}\\&\leq\bigvee\{(\mathcal{F}_1(f)*\mathcal{G}_1(g))\odot(\mathcal{F}_2(f)*\mathcal{G}_2(g))\mid f*g\leq h\}\\&\leq\bigvee\{(\mathcal{F}_1*_*\mathcal{G}_1)(f*g)\odot(\mathcal{F}_2*_*\mathcal{G}_2)(f*g)\mid f*g\leq h\}\\&\leq(\mathcal{F}_1*_*\mathcal{G}_1)(h)\odot(\mathcal{F}_2*_*\mathcal{G}_2)(h).\end{aligned}$$

□

Example 3.2. Let $X = \{x, y\}$ be a set. Define a map $\odot, * : [0,1] \times [0,1] \to [0,1]$ as $a \odot b = \frac{ab}{a+b-ab}$ and $a * b = ab$. Then $(L = [0,1], \odot)$ is a cl-premonoid and $(L = [0,1], *)$ is a GL-monoid. It satisfies

(D) $(a \odot b) * (c \odot d) \leq (a * c) \odot (b * d)$, for all $a, b, c, d \in L$ from

$$\begin{aligned}&(a+b-ab)(c+d-cd) - ac - bd + abcd\\ &= bc + ad - abc - abd - acd - bcd + 2abcd\\ &= bc(1-a) + ad(1-b) - acd(1-b) - bcd(1-a)\\ &= bc(1-a)(1-d) + ad(1-b)(1-c) \geq 0.\end{aligned}$$

Hence $(L, \leq, \odot, *)$ is an ecl-premonoid. But it is not an M-ecl-premonoid because $a \odot a = \frac{a}{2-a} < a$ for $0 < a < 1$. Let $[x]$ and $[y]$ are stratified $(L,*)$-filters on X. Then $[x] \odot [y]$ is an $(L,*)$-filter on X and not a stratified $(L,*)$-filter on X because

$$([x] \odot [y])(0.7 * A) = 0.28 \odot 0.35 \not\geq 0.7 * ([x] \odot [y])(A) = 0.2$$

where $A(x) = 0.4, A(y) = 0.5$.

For $A_1 * A_2 = 0_X$ with $A_1(x) = 1, A_1(y) = 0$ and $A_2(x) = 0, A_1(y) = 1$, $[x](A_1) \odot [y](A_2) = 1 \neq 0$. Hence $[x] \odot_* [y]$ is not an $(L,*)$-filter.

Let $\phi : X \to Y$ be a function, $\mathcal{F} \in L^{(L^X)}$ L-filter on X and $\mathcal{G} \in L^{(L^Y)}$ L-filter on Y. Two functions $\phi^{\Rightarrow}(\mathcal{F}) : L^Y \to L$ and $\phi^{\Leftarrow}(\mathcal{G}) : L^X \to L$ are defined by

$$\phi^{\Rightarrow}(\mathcal{F})(g) = \mathcal{F}(\phi^{\leftarrow}(g)),$$
$$\phi^{\Leftarrow}(\mathcal{G})(f) = \bigvee\{\mathcal{G}(h) \mid \phi^{\leftarrow}(h) \leq f\}.$$

Definition 3.3. Let $\mathcal{F} \in F_*(X)$ and $\mathcal{G} \in F_*(Y)$. Then we define $\mathcal{F} \otimes_{\odot} \mathcal{G} : L^{X \times Y} \to L$ as follows:

$$\mathcal{F} \otimes_{\odot} \mathcal{G} = \pi_1^{\Leftarrow}(\mathcal{F}) \odot_* \pi_2^{\Leftarrow}(\mathcal{G})$$

where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

Theorem 3.4. *Let $\mathcal{F}, \mathcal{H} \in F_*(X)$ and $\mathcal{G} \in F_*(Y)$. Then we have the following properties:*

(1) *For $f \otimes g(x, y) = f(x) * g(y)$ and $u \in L^{X \times Y}$, we have*

$$\mathcal{F} \otimes_{\odot} \mathcal{G}(u) = \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}.$$

(2) $(\mathcal{G} \otimes_{\odot} \mathcal{F})^{-1} = \mathcal{F} \otimes_{\odot} \mathcal{G}$ *where* $(\mathcal{G} \otimes_{\odot} \mathcal{F})^{-1}(u) = (\mathcal{G} \otimes_{\odot} \mathcal{F})(u^{-1})$ *for* $u^{-1}(x, y) = u(y, x)$.

(3) $f \otimes g = \perp$ *implies* $\mathcal{F}(f) \odot \mathcal{G}(g) = \perp$ *iff* $\mathcal{F} \otimes_{\odot} \mathcal{G} \in F_*(X \times Y)$.

(4) *If* $\mathcal{F} \in F_*^s(X)$ *and* $\mathcal{G} \in F_*^s(Y)$, *then* $\mathcal{F} \otimes_{\odot} \mathcal{G} \in F_*^s(X \times Y)$.

(5) *For* $x \in X, \mathcal{F}, \mathcal{G} \in F_*(X)$, $(\mathcal{F} \otimes_{\odot} [x]) * (\mathcal{G} \otimes_{\odot} [x]) \leq (\mathcal{F} * \mathcal{G}) \otimes_{\odot} [x]$.

(6) *If* $(L, \leq, \odot, *)$ *is an M-ecl-premonoid and for* $x \in X, \mathcal{F}, \mathcal{G} \in F_*(X)$, $(\mathcal{F} \otimes_\odot [x]) \odot (\mathcal{G} \otimes_\odot [x]) \geq (\mathcal{F} \odot \mathcal{G}) \otimes_\odot [x]$.

(7) *For* $x \in X$, $\mathcal{F}, \mathcal{G} \in F_*(X)$, *we have* $(\mathcal{F} \otimes_\odot [x]) *_* (\mathcal{G} \otimes_\odot [x]) \leq (\mathcal{F} *_* \mathcal{G}) \otimes_\odot [x]$. *In particular,* $(\mathcal{F} \otimes_* [x]) *_* (\mathcal{G} \otimes_* [x]) = (\mathcal{F} *_* \mathcal{G}) \otimes_* [x]$.

(8) *For* $x \in X$ *and* $y \in Y$, $\pi_1^{\Rightarrow}([(x,y)]) = [x]$ *and* $\pi_2^{\Rightarrow}([(x,y)]) = [y]$.

(9) *For* $x \in X, \mathcal{F}, \mathcal{G} \in F_*^s(X)$ *and* $u \in L^{X \times X}$, $(\mathcal{F} \otimes_* [x])(u) = \mathcal{F}(u(-,x))$ *and* $(\mathcal{F} \otimes_* [x]) * (\mathcal{G} \otimes_* [x]) = (\mathcal{F} * \mathcal{G}) \otimes_* [x]$.

(10) *For* $x, y \in X$ *and* $u \in L^{X \times Y}$, $([x] \otimes_* [y])(u) = [x](u(-,y)) = u(x,y) = [(x,y)](u)$.

(11) *For* $x, y \in X$ *and* $u \in L^{X \times Y}$, $([x] \otimes_\odot [y])(u) \geq ([x] \otimes_* [y])(u) = [(x,y)](u)$.

(12) *For* $x \in X$, $\mathcal{F}, \mathcal{G} \in F_*^s(X)$ *and* $u \in L^{X \times X}$, *we have* $(\mathcal{F} \otimes_* [x]) *_* (\mathcal{G} \otimes_* [x])(u) = (\mathcal{F} *_* \mathcal{G})(u(-,x)) = ((\mathcal{F} *_* \mathcal{G}) \otimes_* [x])(u)$.

(13) $\mathcal{I}_x \otimes_\diamond \mathcal{I}_y \geq \mathcal{I}_{(x,y)}$ *for* $\diamond \in \{*, \odot\}$, *the equality holds, if* $a * b \neq \bot$ *for each* $a \neq \bot$ *and* $b \neq \bot$, *where*

$$\mathcal{I}_y(f) = \begin{cases} \top, & \text{if } f(y) \neq \bot, \\ \bot, & \text{if } f(y) = \bot. \end{cases}$$

(14) *Let* $\mathcal{F}, \mathcal{H} \in F_*(X)$. *Then* $(\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y) = (\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y$.

Proof. (1) From the definition of $\mathcal{F} \otimes_\odot \mathcal{G}$, we only show that $\bigvee\{\pi_1^{\Leftarrow}(\mathcal{F})(u_1) \odot \pi_2^{\Leftarrow}(\mathcal{G})(u_2) \mid u_1 * u_2 \leq u\} = \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}$. For each $f \otimes g = \pi_1^{\leftarrow}(f) * \pi_2^{\leftarrow}(g) \leq u$, $\mathcal{F}(f) \odot \mathcal{G}(g) \leq \pi_1^{\Leftarrow}(\mathcal{F})(\pi_1^{\leftarrow}(f)) \odot \pi_2^{\Leftarrow}(\mathcal{G})(\pi_2^{\leftarrow}(g))$. Hence, $\bigvee\{\pi_1^{\Leftarrow}(\mathcal{F})(u_1) \odot \pi_2^{\Leftarrow}(\mathcal{G})(u_2) \mid u_1 * u_2 \leq u\} \geq \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}$.

Suppose $\bigvee\{\pi_1^{\Leftarrow}(\mathcal{F})(u_1) \odot \pi_2^{\Leftarrow}(\mathcal{G})(u_2) \mid u_1 * u_2 \leq u\} \not\leq \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}$. Then there exist $u_1, u_2 \in L^{X \times Y}$ with $u_1 * u_2 \leq u$ such that $\pi_1^{\Leftarrow}(\mathcal{F})(u_1) \odot \pi_2^{\Leftarrow}(\mathcal{G})(u_2) \not\leq \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}$. From the definitions of $\pi_1^{\Leftarrow}(\mathcal{F})(u_1)$ and $\pi_2^{\Leftarrow}(\mathcal{G})(u_2)$, there exist $f \in L^X$ and $g \in L^Y$ with $\pi_1^{\leftarrow}(f) \leq u_1$ and $\pi_2^{\leftarrow}(g) \leq u_2$ such that $\pi_1^{\leftarrow}(f)(x,y) * \pi_2^{\leftarrow}(g)(x,y) = f(x) * g(y) \leq u(x,y)$ and

$$\mathcal{F}(f) \odot \mathcal{G}(g) \not\leq \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}.$$

It is a contradiction. Thus $\bigvee\{\pi_1^{\Leftarrow}(\mathcal{F})(u_1) \odot \pi_2^{\Leftarrow}(\mathcal{G})(u_2) \mid u_1 * u_2 \leq u\} \leq \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\}$.

(2) For $f \otimes g(x,y) = f(x) * g(y)$ and $u \in L^{X \times Y}$, we have

$$\begin{aligned}(\mathcal{G} \otimes_\odot \mathcal{F})^{-1}(u) &= (\mathcal{G} \otimes_\odot \mathcal{F})(u^{-1}) = \bigvee\{\mathcal{G}(g) \odot \mathcal{F}(f) \mid g \otimes f \leq u^{-1}\} \\ &= \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\} = \mathcal{F} \otimes_\odot \mathcal{G}(u).\end{aligned}$$

(3) (F2) For each $f_1 \otimes g_1 \leq u$ and $f_2 \otimes g_2 \leq v$, since $(f_1 * f_2) \otimes (g_1 * g_2) = (f_1 \otimes g_1) * (f_2 \otimes g_2) \leq u * v$, we have:

$$\begin{aligned}
&(\mathcal{F} \otimes_{\odot} \mathcal{G})(u) * (\mathcal{F} \otimes_{\odot} \mathcal{G})(v) \\
&= \bigvee\{\mathcal{F}(f_1) \odot \mathcal{G}(g_1) \mid f_1 \otimes g_1 \leq u\} * \bigvee\{\mathcal{F}(f_2) \odot \mathcal{G}(g_2) \mid f_2 \otimes g_2 \leq v\} \\
&\leq \bigvee\{(\mathcal{F}(f_1) \odot \mathcal{G}(g_1)) * (\mathcal{F}(f_2) \odot \mathcal{G}(g_2)) \mid (f_1 \otimes g_1) * (f_2 \otimes g_2) \leq u \odot v\} \\
&\leq \bigvee\{(\mathcal{F}(f_1) * \mathcal{F}(f_2)) \odot (\mathcal{G}(g_1) * \mathcal{G}(g_2)) \mid (f_1 * f_2) \otimes (g_1 * g_2) \leq u \odot v\} \\
&\leq \bigvee\{\mathcal{F}(f_1 * f_2) \odot \mathcal{G}(g_1 * g_2) \mid (f_1 * f_2) \otimes (g_1 * g_2) \leq u \odot v\} \\
&\leq (\mathcal{F} \otimes_{\odot} \mathcal{G})(u * v).
\end{aligned}$$

Other cases are easily proved.

(4) Let $f \otimes g = \perp$. Then $\mathcal{F}(f) * \mathcal{G}(g) \leq \mathcal{G}(\mathcal{F}(f) * g)$. Since $\mathcal{F}(f) * g(y) \leq \mathcal{F}(f * g(y)) = \mathcal{F}(\perp) = \perp$, we have $\mathcal{F}(f) * \mathcal{G}(g) \leq \mathcal{G}(\mathcal{F}(f) * g) = \perp$.

$$\begin{aligned}
&\alpha * (\mathcal{F} \otimes_{\odot} \mathcal{G})(u) = \alpha * \bigvee\{\mathcal{F}(f) \odot \mathcal{G}(g) \mid f \otimes g \leq u\} \\
&= \bigvee\{\alpha * (\mathcal{F}(f) \odot \mathcal{G}(g)) \mid f \otimes g \leq u\} \\
&\leq \bigvee\{(\alpha \odot \top) * (\mathcal{F}(f) \odot \mathcal{G}(g)) \mid f \otimes g \leq u\} \\
&\leq \bigvee\{(\alpha * \mathcal{F}(f)) \odot (\top * \mathcal{G}(g)) \mid (\alpha * f) \otimes g \leq \alpha * u\} \\
&\leq \bigvee\{\mathcal{F}(\alpha * f)) \odot \mathcal{G}(g) \mid (\alpha * f) \otimes g \leq \alpha * u\} \\
&\leq (\mathcal{F} \otimes_{\odot} \mathcal{G})(\alpha * u).
\end{aligned}$$

(5) Suppose there exists $u \in L^{X \times X}$ such that

$$((\mathcal{F} \otimes_{\odot} [x]) * (\mathcal{G} \otimes_{\odot} [x]))(u) \not\leq ((\mathcal{F} * \mathcal{G}) \otimes_{\odot} [x])(u).$$

There exist $f_i \in L^X, g_i \in L^X$ with $(f_i \otimes g_i) \leq u$ such that

$$(\mathcal{F}(f_1) \odot [x](g_1)) * (\mathcal{G}(f_2) \odot [x](g_2)) \not\leq ((\mathcal{F} * \mathcal{G}) \otimes_{\odot} [x])(u).$$

Since $(\mathcal{F}(f_1) \odot [x](g_1)) * (\mathcal{G}(f_2) \odot [x](g_2)) \leq (\mathcal{F}(f_1) * \mathcal{G}(f_2)) \odot ([x](g_1) * [x](g_2))$, we have

$$(\mathcal{F}(f_1) * \mathcal{G}(f_2)) \odot [x](g_1 * g_2) \not\leq ((\mathcal{F} * \mathcal{G}) \otimes_{\odot} [x])(u).$$

On the other hand, since $(f_1 \vee f_2) \otimes (g_1 * g_2) \leq (f_1 \otimes g_1) \vee (f_2 \otimes g_2) \leq u$,

$$\begin{aligned}
&((\mathcal{F} * \mathcal{G}) \otimes_{\odot} [x])(u) \geq (\mathcal{F} * \mathcal{G})(f_1 \vee f_2) \odot [x](g_1 * g_2) \\
&\geq \mathcal{F}(f_1) * \mathcal{G}(f_2) \odot [x](g_1 * g_2).
\end{aligned}$$

It is a contradiction. Hence the result holds.

(6) Suppose there exist $x \in X$ and $u \in L^{X \times X}$ such that

$$((\mathcal{F} \otimes_{\odot} [x]) \odot (\mathcal{G} \otimes_{\odot} [x])(u) \not\geq ((\mathcal{F} \odot \mathcal{G}) \otimes_{\odot} [x])(u).$$

There exist $f \in L^X, g \in L^X$ with $f \otimes g \leq u$ such that

$$((\mathcal{F} \otimes_{\odot} [x]) \odot (\mathcal{G} \otimes_{\odot} [x])(u) \not\geq (\mathcal{F} \odot \mathcal{G})(f) \odot [x](g).$$

On the other hand, since $(L,\le,\odot,*)$ is an M-ecl-premonoid,

$$\begin{aligned}&((\mathcal{F}\otimes_{\odot}[x])\odot(\mathcal{G}\otimes_{\odot}[x]))(u)\ge(\mathcal{F}\otimes_{\odot}[x])(u)\odot(\mathcal{G}\otimes_{\odot}[x])(u)\\&\ge\mathcal{F}(f)\odot\mathcal{G}(f)\odot[x](g)\odot[x](g)\ge\mathcal{F}(f)\odot\mathcal{G}(f)\odot[x](g).\end{aligned}$$

It is a contradiction. Hence $(\mathcal{F}\otimes_{\odot}[x])\odot(\mathcal{G}\otimes_{\odot}[x])\ge(\mathcal{F}\odot\mathcal{G})\otimes_{\odot}[x]$.

(7) For $x\in X$, $\mathcal{F},\mathcal{G}\in F_*(X)$,

$$\begin{aligned}&((\mathcal{F}\otimes_{\odot}[x])*_*(\mathcal{G}\otimes_{\odot}[x]))(u)\\&=\bigvee\{(\mathcal{F}\otimes_{\odot}[x])(u_1)*(\mathcal{G}\otimes_{\odot}[x])(u_2)\mid u_1*u_2\le u\}\\&=\bigvee\{(\mathcal{F}(f_1)\odot[x](g_1))*(\mathcal{G}(f_2)\odot[x](g_2))\mid f_i\otimes g_i\le u_i,u_1*u_2\le u\}\\&\le\bigvee\{(\mathcal{F}(f_1)*\mathcal{G}(f_2))\odot([x](g_1)*[x](g_2))\mid f_i\otimes g_i\le u_i,u_1*u_2\le u\}\\&\le\bigvee\{(\mathcal{F}*_*\mathcal{G})(f_1*g_1)\odot[x](g_1*g_2)\mid(f_1*f_2)\otimes(g_1*g_2)\le u\}\\&\le((\mathcal{F}*_*\mathcal{G})\otimes_{\odot}[x])(u).\end{aligned}$$

Similarly, $(\mathcal{F}\otimes_*[x])*_*(\mathcal{G}\otimes_*[x])\le(\mathcal{F}*_*\mathcal{G})\otimes_*[x]$.

Suppose there exists $u\in L^{X\times X}$ such that

$$((\mathcal{F}\otimes_*[x])*_*(\mathcal{G}\otimes_*[x]))(u)\not\ge((\mathcal{F}*_*\mathcal{G})\otimes_*[x])(u).$$

There exist $f,g\in L^X$ with $f\otimes g\le u$ such that

$$((\mathcal{F}\otimes_*[x])*_*(\mathcal{G}\otimes_*[x]))(u)\not\ge(\mathcal{F}*_*\mathcal{G})(f)*[x](g).$$

There exist $f_1,f_2\in L^X$ with $(f_1*f_2)\otimes g\le f\otimes g\le u$ such that

$$((\mathcal{F}\otimes_*[x])*_*(\mathcal{G}\otimes_*[x]))(u)\not\ge(\mathcal{F}(f_1)*\mathcal{G}(f_2))*[x](g).$$

There exist $g_1,g_2\in L^X$ with $g_1*g_2=g$ such that

$$\begin{aligned}&(f_1*f_2)\otimes g=(f_1*f_2)\otimes(g_1*g_2)\\&=(f_1\otimes g_1)*(f_2\otimes g_2)\le f\otimes g\le u.\end{aligned}$$

Thus, we have

$$\begin{aligned}&((\mathcal{F}\otimes_*[x])*_*(\mathcal{G}\otimes_*[x]))(u)\\&\ge(\mathcal{F}\otimes_*[x])(f_1\otimes g_1)*((\mathcal{G}\otimes_*[x])(f_2\otimes g_2)\\&\ge(\mathcal{F}(f_1)*[x](g_1))*(\mathcal{G}(f_2)*[x](g_2))=(\mathcal{F}(f_1)*\mathcal{G}(f_2))*[x](g).\end{aligned}$$

It is a contradiction. Hence $(\mathcal{F}\otimes_*[x])*_*(\mathcal{G}\otimes_*[x])=(\mathcal{F}*_*\mathcal{G})\otimes_*[x]$.

(8) $\pi_1^{\Rightarrow}([(x,y)])(h)=[(x,y)](\pi_1^{\leftarrow}(h))=h(\pi_1(x,y))=h(x)=[x](h)$.

(9) For $x\in X$ and $\mathcal{F}\in F^s_*(X)$,

$$\begin{aligned}&(\mathcal{F}\otimes_*[x])(u)=\bigvee\{\mathcal{F}(f)*[x](g)\mid f\otimes g\le u\}\\&=\bigvee\{\mathcal{F}(f)*g(x)\mid f\otimes g\le u\}\le\bigvee\{\mathcal{F}(g(x)*f)\mid f(-)\otimes g(x)\le u(-,x)\}\\&\le\mathcal{F}(u(-,x)).\end{aligned}$$

$$\begin{aligned}&(\mathcal{F}\otimes_*[x])(u)=\bigvee\{\mathcal{F}(f)*[x](g)\mid f\otimes g\le u\}\\&\ge\{\mathcal{F}(u(-,x))*1_X(x)\mid u(-,x)\otimes 1_X\le u(-,x)\}=\mathcal{F}(u(-,x)).\end{aligned}$$

$$\begin{aligned}&((\mathcal{F}\otimes_*[x])*(\mathcal{G}\otimes_*[x]))(u)=\mathcal{F}(u(-,x))*\mathcal{G}(u(-,x))\\&=(\mathcal{F}*\mathcal{G})(u(-,x))=((\mathcal{F}*\mathcal{G})\otimes_*[x])(u).\end{aligned}$$

(10) Since $[x] \in F_*^s(X)$, by (9), the results hold.

(11) It is easy from the definition of $\otimes_\odot$.

(12) For $x \in X$, $\mathcal{F}, \mathcal{G} \in F_*^s(X)$ and $u \in L^{X\times X}$,

$$\begin{array}{l}((\mathcal{F} \otimes_* [x]) *_* (\mathcal{G} \otimes_* [x]))(u)\\ = \bigvee\{(\mathcal{F} \otimes_* [x])(u_1) * (\mathcal{G} \otimes [x])(u_2) \mid u_1 * u_2 \le u\}\\ = \bigvee\{\mathcal{F}(u_1(-,x)) * \mathcal{G}(u_2(-,x)) \mid u_1(-,x) * u_2(-,x) \le u(-,x)\}\\ = (\mathcal{F} *_* \mathcal{G})(u(-,x)) = ((\mathcal{F} *_* \mathcal{G}) \otimes_* [x])(u).\end{array}$$

(13) Let $u(x,y) = f(x) * g(y) \ne \bot$ such that $\mathcal{I}_{(x,y)}(u) = \top$. Then $f(x) \ne \bot$ and $g(y) \ne \bot$. Hence $(\mathcal{I}_x \otimes_\odot \mathcal{I}_y)(u) \ge \mathcal{I}_x(f) \odot \mathcal{I}_y(g) = \top$. So, $\mathcal{I}_x \otimes_\odot \mathcal{I}_y \ge \mathcal{I}_{(x,y)}$. Let $(\mathcal{I}_x \otimes_\odot \mathcal{I}_y)(w) \ne \bot$. Then there exist $f(x) \ne \bot$ and $g(y) \ne \bot$ with $f \otimes g \le w$. Since $f(x) * g(y) \ne \bot$ for each $f(x) \ne \bot$ and $g(y) \ne \bot$, $\bot \ne f(x) * g(y) \le w(x,y)$. Hence $\mathcal{I}_{(x,y)}(w) = \top$. So, $\mathcal{I}_x \otimes_\odot \mathcal{I}_y \le \mathcal{I}_{(x,y)}$.

(14) Suppose there exist $y \in Y$ and $\bot_{X\times Y} \ne u \in L^{X\times Y}$ such that

$$((\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y))(u) \not\le ((\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y)(u).$$

There exist $f_i \in L^X, g_i \in L^Y$ with $(f_i \otimes g_i) \le u$ and $g_i(y) \ne \bot$ such that

$$(\mathcal{F}(f_1) * \mathcal{I}_y(g_1)) * (\mathcal{H}(f_2) * \mathcal{I}_y(g_2)) \not\le ((\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y)(u).$$

Thus, $\mathcal{F}(f_1) * \mathcal{H}(f_2) \not\le ((\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y)(u)$.

On the other hand, since $(f_1 \vee f_2) \otimes (g_1 \wedge g_2) \le (f_1 \otimes g_1) \vee (f_2 \otimes g_2) \le u$,

$$\begin{array}{ll}((\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y)(u) & \ge (\mathcal{F} * \mathcal{H})(f_1 \vee f_2) * \mathcal{I}_y(g_1 \wedge g_2)\\ & \ge \mathcal{F}(f_1) * \mathcal{H}(f_2).\end{array}$$

It is a contradiction. Hence $(\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y) \le (\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y$.

Suppose there exist $y \in Y$ and $\bot_{X\times Y} \ne u \in L^{X\times Y}$ such that

$$((\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y))(u) \not\ge ((\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y)(u).$$

There exist $f \in L^X, g \in L^Y$ with $f \otimes g \le u$ and $g(y) \ne \bot$ such that

$$((\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y))(u) \not\ge (\mathcal{F} * \mathcal{H})(f) * \mathcal{I}_y(g).$$

On the other hand,

$$\begin{array}{ll}((\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y))(u) & \ge (\mathcal{F} \otimes_* \mathcal{I}_y)(u) * (\mathcal{H} \otimes_* \mathcal{I}_y)(u)\\ & \ge \mathcal{F}(f) * \mathcal{H}(f).\end{array}$$

It is a contradiction. Hence $(\mathcal{F} \otimes_* \mathcal{I}_y) * (\mathcal{H} \otimes_* \mathcal{I}_y) \ge (\mathcal{F} * \mathcal{H}) \otimes_* \mathcal{I}_y$. □

Example 3.5. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$ be sets, $(L = [0,1], *)$ an GL-monoid with $a * b = a \cdot b$ and $f, g \in [0,1]^X$ as follows:

$$f(x_1) = 1, f(x_2) = 0.6, \quad g(x_1) = 0.5, g(x_2) = 1.$$

Define $([0,1], *)$-filters as $\mathcal{F} : [0,1]^X \to [0,1]$ and $\mathcal{G} : [0,1]^Y \to [0,1]$ as follows:

$$\mathcal{F}(h) = \begin{cases} 1, & \text{if } h = 1_X, \\ 0.4^n, & \text{if } f^n \leq h \not\geq f^{n-1}, n \in N \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathcal{G}(h) = \begin{cases} 1, & \text{if } h = 1_X, \\ 0.3^n, & \text{if } g^n \leq h \not\geq g^{n-1}, n \in N \\ 0, & \text{otherwise,} \end{cases}$$

where $k^n = k^{n-1} * k$ and $k^0 = 1$.

(1) Let $x \odot y = x^{\frac{1}{3}} \cdot y^{\frac{1}{3}}$. Since $h * k = \perp_{X \times Y}$ implies $\mathcal{F}(h) \odot \mathcal{G}(k) = \perp$, we obtain $([0,1], *)$-filters as $\mathcal{F} \otimes_{\odot} \mathcal{G} : [0,1]^X \to [0,1]$ as follows:

$$\mathcal{F} \otimes_{\odot} \mathcal{G}(h) = \begin{cases} 1, & \text{if } h = 1_X, \\ 0.4^n \odot 0.3^m, & \text{if } f^n * g^m \leq h \not\geq f^{n-1} * g^m, \\ & f^n * g^m \leq h \not\geq f^n * g^{m-1}, \\ 0, & \text{otherwise,} \end{cases}$$

where $k^{[n]} = k^{[n-1]} \odot k$ and $k^0 = 1$.

(2) For $u \in [0,1]^{X \times X}$ with

$$u(x_1, x_1) = 0.3, u(x_1, x_2) = 1, u(x_2, x_1) = 0.8, u(x_2, x_2) = 0.7,$$

For $1_X \otimes h \leq 0.9 * u$ with $h(x_1) = 0.27, h(x_2) = 0.63$, $\mathcal{F} \otimes_* \mathcal{G}(0.9 * u) = \mathcal{F}(1_X) * [x_2](h) = 0.63 = 0.9 * (\mathcal{F} \otimes_* \mathcal{G})(u) \neq \mathcal{F}(0.9 * u(-, x_2)) = 0$.

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