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AN APPROACH FOR HYPERSURFACE FAMILY WITH COMMON GEODESIC CURVE IN THE 4D GALILEAN SPACE $\mathbf{G}_4$

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Abstract. In the present study, we derive the problem of constructing a hypersurface family from a given isogeodesic curve in the 4D Galilean space $\mathbf{G}_4$. We obtain the hypersurface as a linear combination of the Frenet frame in $\mathbf{G}_4$ and examine the necessary and sufficient conditions for the curve as a geodesic curve. Finally, some examples related to our method are given for the sake of clarity.

1. Introduction

Curves on a surface which locally yield the minimal distance between any two points are of great interest. These curves are said to be geodesics which play an important role in differential geometry and theoretical physics, particularly in General Relativity, where gravity can be declared not a force but a significance of a curved spacetime geometry in which the source of curvature is the stress-energy tensor. Therefore, for instance, the path of a planet orbiting around a star is the projection of a geodesic of the curved $4D$ spacetime geometry around the star onto $3D$ space. Geodesics also are curves along which geodesic curvature vanishes. Geodesics have been studied the subject of many studies in a diversity of applications, such as the designing industry of shoes, tent manufacturing, cutting and painting path [4, 5, 8].

Generally, the aim of mostly studies about geodesics is to set up a family of surfaces passing a given geodesic curve and show it as a linear combination of the marching-scale functions and the Frenet vectors. Based on that, there have been various researches on this subject in 3-dimensional Euclidean and non-Euclidean space [1, 9, 10, 12, 14, 16].

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Besides, for the differential geometry of surface and hypersurface, there exists a rising interest in 4-dimensional space [2, 11]. Also, in [3], Bayram and Kasap gave the hypersurfaces family from a given common geodesic curve.

In this paper, we investigate the parametric representation of hypersurface family passing a given isogeodesic curve, i.e., both a geodesic and a parameter curve in 4-dimensional Galilean space $\mathbf{G}_4$. The remainder of our paper is given as four sections. Firstly, we mainly give the background. Secondly, we give the parametric representations of a hypersurface family passing a given geodesic curve and provide the necessary and sufficient condition for that curve as a geodesic curve on the given hypersurface. Subsequently, we introduce three types of the marching-scale functions. Finally, we give some examples and figures are plotted for the sake of clarity of our method.

2. Preliminaries

The Galilean space $\mathbf{G}_3$ is a 3-dimensional complex projective space P_3. The absolute figure of the Galilean space comprise of $\{w, f, I\}$ in which w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f.

The analyze of mechanics of plane-parallel motions reduces to the examine of a geometry of the 3-dimensional space with $\{x, y, t\}$, is investigated by the motion formula in [15]. It is defined that the 4D Galilean geometry, which examines all properties invariant under motions of objects in the space, is even complex. In an other words, it could be considered as the properties of 4-dimensional space with coordinates that are invariant under the general Galilean transformations in [15].

Let $z = (z_1, z_2, z_3, z_4)$ and $t = (t_1, t_2, t_3, t_4)$ be two vectors in $\mathbf{G}_4$. The Galilean scalar product in $\mathbf{G}_4$ is given by

$$\langle z, t\rangle = \begin{cases} z_1 t_1, & \text{if } z_1 \neq 0 \text{ or } t_1 \neq 0 \\ z_2 t_2 + z_3 t_3 + z_4 t_4, & \text{if } z_1 = 0 \text{ and } t_1 = 0 \end{cases}. \tag{1}$$

Let $z = (z_1, z_2, z_3, z_4)$, $t = (t_1, t_2, t_3, t_4)$ and $u = (u_1, u_2, u_3, u_4)$ be vectors in $\mathbf{G}_4$. Then the cross product in $\mathbf{G}_4$ is given as follows:

$$z \wedge t \wedge u = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ z_1 & z_2 & z_3 & z_4 \\ t_1 & t_2 & t_3 & t_4 \\ u_1 & u_2 & u_3 & u_4 \end{vmatrix}, \tag{2}$$

where e_i, $2 \leq i \leq 4$, are the standard basis vectors.

A curve $r : I \to \mathbf{G}_4$ is an arbitrary curve in $\mathbf{G}_4$ is given by

$$r(t) = (f(t), g(t), h(t), l(t)),$$

where $f(t), g(t), h(t)$ and $l(t)$ are smooth functions on $I \subset \mathbb{R}$. Let r be a curve in $\mathbf{G}_4$, parametrized by the Galilean invariant arc length s, is given by

$$r(s) = (s, g(s), h(s), l(s)).$$

For the curve r, the Frenet vectors are given in the following forms

$$\begin{aligned}
t(s) &= r'(s) = \left(1, g'(s), h'(s), l'(s)\right), \\
n(s) &= \frac{r''(s)}{\kappa(s)} = \frac{1}{\kappa(s)}\left(0, g''(s), h''(s), l''(s)\right), \\
b(s) &= \frac{1}{\tau(s)}\left(0, (\frac{1}{\kappa(s)}g''(s))', (\frac{1}{\kappa(s)}h''(s))', (\frac{1}{\kappa(s)}l''(s))'\right), \\
e(s) &= \mu t(s) \wedge n(s) \wedge b(s),
\end{aligned}$$

where μ equals ± 1 such that the determinant $|t, n, b, e| = 1$ and $\kappa(s), \tau(s)$ and $\sigma(s)$ are the first, second and third curvature of $r(s)$ which is given by, respectively,

$$\begin{aligned}
\kappa(s) &= \sqrt{g''(s)^2 + h''(s)^2 + l''(s)^2}, \\
\tau(s) &= \sqrt{\langle n'(s), n'(s)\rangle}, \\
\sigma(s) &= \sqrt{\langle b'(s), e(s)\rangle}.
\end{aligned} \tag{3}$$

The vectors $t(s), n(s), b(s)$ and $e(s)$ are called the tangent, principal normal, first binormal, and second binormal vector of r, respectively.

On the other hand, Frenet formulas can be given as [13]

$$\begin{aligned}
t'(s) &= \kappa(s)n(s), \\
n'(s) &= \tau(s)b(s), \\
b'(s) &= -\tau(s)n(s) + \sigma(s)e(s), \\
e'(s) &= -\sigma(s)b(s).
\end{aligned} \tag{4}$$

Let $R(s, \varkappa, \rho)$ be a hypersurface in $\mathbf{G}_4$. The isotropic normal vector field η of R is defined as follows [6]

$$\eta(s, \varkappa, \rho) = R_s \wedge R_\varkappa \wedge R_\rho,$$

where $R_s = \frac{\partial R(s,\varkappa,\rho)}{\partial s}, R_\varkappa = \frac{\partial R(s,\varkappa,\rho)}{\partial \varkappa}$ and $R_\rho = \frac{\partial R(s,\varkappa,\rho)}{\partial \rho}$.

3. Hypersurface Family with Common Geodesic Curve

A curve $r(s)$ on a hypersurface $R(s,\varkappa,\rho)$ in $\mathbf{G}_4$ is said to be an isoparametric curve if it is a parameter curve, that is, there exists a pair of parameters $\varkappa_0$ and ρ_0 such that $r(s)=R(s,\varkappa_0,\rho_0)$. Also the curve $r(s)$ on the hypersurface $R(s,\varkappa,\rho)$ is geodesic iff the principal normal vector $n(s)$ of $r(s)$ is everywhere parallel to the isotropic normal vector $\eta(s,\varkappa,\rho)$ of the hypersurface $R(s,\varkappa,\rho)$. Then, a given curve $r(s)$ is called an isogeodesic of the hypersurface R if it is both a geodesic and an isoparametric curve on R.

Let $R = R(s,\varkappa,\rho)$ be a parametric hypersurface through the arc-length parametrized curve $r(s)$ in $\mathbf{G}_4$. The hypersurface is defined by

$$\begin{aligned} R(s,\varkappa,\rho) &= r(s)+[\alpha(s,\varkappa,\rho)\,t(s)+\beta(s,\varkappa,\rho)\,n(s) \\ &\quad +\gamma(s,\varkappa,\rho)\,b(s)+\delta(s,\varkappa,\rho)\,e(s)], \\ L_1 &\leq s\leq L_2,\ T_1\leq\varkappa\leq T_2 \text{ and } Q_1\leq\rho\leq Q_2, \end{aligned} \tag{5}$$

where $\alpha(s,\varkappa,\rho)$, $\beta(s,\varkappa,\rho)$, $\gamma(s,\varkappa,\rho)$ and $\delta(s,\varkappa,\rho)$ are smooth functions. These functions are said to be the marching-scale functions.

Our aim is to provide necessary and sufficient conditions for the given curve $r(s)$ to be an isogeodesic curve on a hypersurface $R=R(s,\varkappa,\rho)$.

Firstly, let $r(s)$ be a curve on the hypersurface R in $\mathbf{G}_4$. If $r(s)$ is an isoparametric curve on this surface, then a parameter $\varkappa_0\in[T_1,T_2]$ and $\rho_0\in[Q_1,Q_2]$ should be existed such that $r(s)=R(s,\varkappa_0,\rho_0)$, $L_1\leq s\leq L_2$, that is,

$$\begin{aligned} \alpha(s,\varkappa_0,\rho_0) &= \beta(s,\varkappa_0,\rho_0)=\gamma(s,\varkappa_0,\rho_0)=\delta(s,\varkappa_0,\rho_0)=0, \\ L_1 &\leq s\leq L_2,\ \varkappa_0\in[T_1,T_2] \text{ and } \rho_0\in[Q_1,Q_2]. \end{aligned} \tag{6}$$

Secondly, $r(s)$ on the hypersurface $R(s,\varkappa,\rho)$ is a geodesic if and only if $n(s)\,\|\,\eta(s,\varkappa_0,\rho_0)$.

Now, the normal vector $\eta(s,\varkappa_0,\rho_0)$ can be found by calculating the cross product of the partial derivatives and using (4) as follows:

$$\begin{aligned} \frac{\partial R(s,\varkappa,\rho)}{\partial s} &= (1+\frac{\partial\alpha(s,\varkappa,\rho)}{\partial s})t(s)+(\alpha(s,\varkappa,\rho)\,\kappa(s)+\frac{\partial\beta(s,\varkappa,\rho)}{\partial s} \\ &\quad -\gamma(s,\varkappa,\rho)\,\tau(s))n(s)+(\beta(s,\varkappa,\rho)\,\tau(s)+\frac{\partial\gamma(s,\varkappa,\rho)}{\partial s} \\ &\quad -\delta(s,\varkappa,\rho)\,\sigma(s))b(s)+(\gamma(s,\varkappa,\rho)\,\sigma(s)+\frac{\partial\delta(s,\varkappa,\rho)}{\partial s})e(s), \end{aligned} \tag{7}$$

(8)
$$\frac{\partial R(s,\varkappa,\rho)}{\partial \varkappa} = \frac{\partial \alpha(s,\varkappa,\rho)}{\partial \varkappa} t(s) + \frac{\partial \beta(s,\varkappa,\rho)}{\partial \varkappa} n(s) + \frac{\partial \gamma(s,\varkappa,\rho)}{\partial \varkappa} b(s) + \frac{\partial \delta(s,\varkappa,\rho)}{\partial \varkappa} e(s)$$
and
(9)
$$\frac{\partial R(s,\varkappa,\rho)}{\partial \rho} = \frac{\partial \alpha(s,\varkappa,\rho)}{\partial \rho} t(s) + \frac{\partial \beta(s,\varkappa,\rho)}{\partial \rho} n(s) + \frac{\partial \gamma(s,\varkappa,\rho)}{\partial \rho} b(s) + \frac{\partial \delta(s,\varkappa,\rho)}{\partial \rho} e(s).$$

Remark 3.1. Since

$$\begin{aligned} \alpha(s,\varkappa_0,\rho_0) &= \beta(s,\varkappa_0,\rho_0) = \gamma(s,\varkappa_0,\rho_0) = \delta(s,\varkappa_0,\rho_0) = 0, \\ L_1 &\leq s \leq L_2,\ \varkappa_0 \in [T_1,T_2] \text{ and } \rho_0 \in [Q_1,Q_2], \end{aligned}$$

through the arc-length parametrized curve $r(s)$, by the definition of partial differentiation, we have

$$\begin{aligned} \frac{\partial \alpha(s,\varkappa_0,\rho_0)}{\partial s} &= \frac{\partial \beta(s,\varkappa_0,\rho_0)}{\partial s} = \frac{\partial \gamma(s,\varkappa_0,\rho_0)}{\partial s} = \frac{\partial \delta(s,\varkappa_0,\rho_0)}{\partial s} = 0, \\ L_1 &\leq s \leq L_2,\ \varkappa_0 \in [T_1,T_2] \text{ and } \rho_0 \in [Q_1,Q_2]. \end{aligned} \tag{10}$$

Then, from (5), we obtain

$$\begin{aligned} \eta(s,\varkappa_0,\rho_0) &= \frac{\partial R(s,\varkappa_0,\rho_0)}{\partial s} \wedge \frac{\partial R(s,\varkappa_0,\rho_0)}{\partial \varkappa} \wedge \frac{\partial R(s,\varkappa_0,\rho_0)}{\partial \rho} \\ &= \varphi_1(s,\varkappa_0,\rho_0)\, t(s) - \varphi_2(s,\varkappa_0,\rho_0)\, n(s) \\ &\quad + \varphi_3(s,\varkappa_0,\rho_0)\, b(s) - \varphi_4(s,\varkappa_0,\rho_0)\, e(s). \end{aligned} \tag{11}$$

We need to calculate the functions $\varphi_i(s,\varkappa_0,\rho_0)$, $1 \leq i \leq 4$.

Using (5) and taking account of Remark 3.1, we have

$$\begin{aligned} \varphi_1(s,\varkappa_0,\rho_0) &= 0, \\ \varphi_2(s,\varkappa_0,\rho_0) &= \frac{\partial \gamma(s,\varkappa,\rho)}{\partial \varkappa} \frac{\partial \delta(s,\varkappa,\rho)}{\partial \rho} - \frac{\partial \gamma(s,\varkappa,\rho)}{\partial \rho} \frac{\partial \delta(s,\varkappa,\rho)}{\partial \varkappa}, \\ \varphi_3(s,\varkappa_0,\rho_0) &= \frac{\partial \beta(s,\varkappa,\rho)}{\partial \varkappa} \frac{\partial \delta(s,\varkappa,\rho)}{\partial \rho} - \frac{\partial \beta(s,\varkappa,\rho)}{\partial \rho} \frac{\partial \delta(s,\varkappa,\rho)}{\partial \varkappa}, \\ \varphi_4(s,\varkappa_0,\rho_0) &= \frac{\partial \beta(s,\varkappa,\rho)}{\partial \varkappa} \frac{\partial \gamma(s,\varkappa,\rho)}{\partial \rho} - \frac{\partial \beta(s,\varkappa,\rho)}{\partial \rho} \frac{\partial \gamma(s,\varkappa,\rho)}{\partial \varkappa}. \end{aligned} \tag{12}$$

So, $n(s) \| \eta(s,\varkappa_0,\rho_0)$ if and only if

$$\varphi_2(s,\varkappa_0,\rho_0) \neq 0, \varphi_3(s,\varkappa_0,\rho_0) = 0 \text{ and } \varphi_4(s,\varkappa_0,\rho_0) = 0.$$

Hence, the necessary and sufficient conditions for the hypersurface $R(s,\varkappa,\rho)$ to have the curve $r(s)$ in $\mathbf{G}_4$ as an isogeodesic curve can be given with the following theorem.

Theorem 3.2. *Let $R(s,\varkappa,\rho)$ be a hypersurface having a curve $r(s)$ in $\mathbf{G}_4$. Then $r(s)$ is an isogeodesic curve on the hypersurface R if and only if*

$$\alpha(s,\varkappa_0,\rho_0)=\beta(s,\varkappa_0,\rho_0)=\gamma(s,\varkappa_0,\rho_0)=\delta(s,\varkappa_0,\rho_0)=0,$$

$$\varphi_2(s,\varkappa_0,\rho_0)\neq 0,\varphi_3(s,\varkappa_0,\rho_0)=0 \text{ and } \varphi_4(s,\varkappa_0,\rho_0)=0$$

satisfied, where $L_1\leq s\leq L_2$, $\varkappa_0\in[T_1,T_2]$ and $\rho_0\in[Q_1,Q_2]$.

We call the set of hypersurfaces satisfying Theorem 3.2 an isogeodesic hypersurface family.

Marching-scale Functions

For $L_1\leq s\leq L_2$, $T_1\leq\varkappa\leq T_2$ and $Q_1\leq\rho\leq Q_2$, we will define three different above mentioned types of the marching-scale functions.

Type A. Let marching-scale functions be

$$\begin{aligned}\alpha(s,\varkappa,\rho) &= \lambda(s)X(\varkappa,\rho),\\ \beta(s,\varkappa,\rho) &= \mu(s)Y(\varkappa,\rho),\\ \gamma(s,\varkappa,\rho) &= \nu(s)Z(\varkappa,\rho),\\ \delta(s,\varkappa,\rho) &= \xi(s)W(\varkappa,\rho),\end{aligned}\tag{13}$$

where

$$\lambda(s),\mu(s),\nu(s),\xi(s),X(\varkappa,\rho),Y(\varkappa,\rho),Z(\varkappa,\rho),W(\varkappa,\rho)\in C^1$$

and $\lambda(s),\mu(s),\nu(s)$ and $\xi(s)$ are not identically zero.

Hence, $r(s)$ is an isogeodesic curve on $R(s,\varkappa,\rho)$ if and only if

$$\begin{aligned}&X(\varkappa_0,\rho_0)=Y(\varkappa_0,\rho_0)=Z(\varkappa_0,\rho_0)=W(\varkappa_0,\rho_0),\\ &\nu(s)\neq 0 \text{ and } \xi(s)\neq 0\\ &\text{and } \frac{\partial Z(s,\varkappa_0,\rho_0)}{\partial\varkappa}\frac{\partial W(s,\varkappa_0,\rho_0)}{\partial\rho}-\frac{\partial Z(s,\varkappa_0,\rho_0)}{\partial\rho}\frac{\partial W(s,\varkappa_0,\rho_0)}{\partial\varkappa}\neq 0,\\ &\mu(s)=0 \text{ or } \frac{\partial Y(s,\varkappa_0,\rho_0)}{\partial\varkappa}\frac{\partial W(s,\varkappa_0,\rho_0)}{\partial\rho}-\frac{\partial Y(s,\varkappa_0,\rho_0)}{\partial\rho}\frac{\partial W(s,\varkappa_0,\rho_0)}{\partial\varkappa}=0,\\ &\mu(s)=0 \text{ or } \frac{\partial Y(s,\varkappa_0,\rho_0)}{\partial\varkappa}\frac{\partial Z(s,\varkappa_0,\rho_0)}{\partial\rho}-\frac{\partial Y(s,\varkappa_0,\rho_0)}{\partial\rho}\frac{\partial Z(s,\varkappa_0,\rho_0)}{\partial\varkappa}=0\end{aligned}\tag{14}$$

satisfied.

To simplify (14), we can write $\nu(s) \neq 0$ and $\xi(s) \neq 0$,

$$\begin{gathered} X(\varkappa_0, \rho_0) = Y(\varkappa_0, \rho_0) = Z(\varkappa_0, \rho_0) = W(\varkappa_0, \rho_0), \\ \frac{\partial Z(s, \varkappa_0, \rho_0)}{\partial \varkappa} \frac{\partial W(s, \varkappa_0, \rho_0)}{\partial \rho} - \frac{\partial Z(s, \varkappa_0, \rho_0)}{\partial \rho} \frac{\partial W(s, \varkappa_0, \rho_0)}{\partial \varkappa} \neq 0, \\ \mu(s) = 0 \text{ or } \frac{\partial Y(s, \varkappa_0, \rho_0)}{\partial \varkappa} = \frac{\partial Y(s, \varkappa_0, \rho_0)}{\partial \rho} = 0, \\ \varkappa_0 \in [T_1, T_2], \ \rho_0 \in [Q_1, Q_2]. \end{gathered} \tag{15}$$

Type B. Let marching-scale functions be

$$\begin{aligned} \alpha(s, \varkappa, \rho) &= \lambda(s, \varkappa) X(\rho), \\ \beta(s, \varkappa, \rho) &= \mu(s, \varkappa) Y(\rho), \\ \gamma(s, \varkappa, \rho) &= \nu(s, \varkappa) Z(\rho), \\ \delta(s, \varkappa, \rho) &= \xi(s, \varkappa) W(\rho), \end{aligned} \tag{16}$$

where $\lambda(s, \varkappa), \mu(s, \varkappa), \nu(s, \varkappa), \xi(s, \varkappa), X(\rho), Y(\rho), Z(\rho), W(\rho) \in C^1$. Thus, $r(s)$ is an isogeodesic curve on $R(s, \varkappa, \rho)$ if and only if

$$\begin{gathered} \lambda(s, \varkappa_0) X(\rho_0) = \mu(s, \varkappa_0) Y(\rho_0) = \nu(s, \varkappa_0) Z(\rho_0) = \xi(s, \varkappa_0) W(\rho_0) = 0, \\ \frac{\partial \nu(s, \varkappa_0)}{\partial \varkappa} \xi(s, \varkappa_0) Z(\rho_0) \frac{dW(\rho_0)}{d\rho} - \nu(s, \varkappa_0) \frac{\partial \xi(s, \varkappa_0)}{\partial \varkappa} W(\rho_0) \frac{dZ(\rho_0)}{d\rho} \neq 0, \\ Y(\rho_0) = \mu(s, \varkappa_0) = 0 \text{ or } \frac{dY(\rho_0)}{d\rho} = Y(\rho_0) = 0 \text{ or } \frac{dY(\rho_0)}{d\rho} = \frac{\partial \mu(s, \varkappa_0)}{\partial \varkappa} = 0, \\ \varkappa_0 \in [T_1, T_2], \ \rho_0 \in [Q_1, Q_2] \end{gathered} \tag{17}$$

satisfied.

Type C. Let marching-scale functions be

$$\begin{aligned} \alpha(s, \varkappa, \rho) &= \lambda(s, \rho) X(\varkappa), \\ \beta(s, \varkappa, \rho) &= \mu(s, \rho) Y(\varkappa), \\ \gamma(s, \varkappa, \rho) &= \nu(s, \rho) Z(\varkappa), \\ \delta(s, \varkappa, \rho) &= \xi(s, \rho) W(\varkappa), \end{aligned} \tag{18}$$

where $\lambda(s, \rho), \mu(s, \rho), \nu(s, \rho), \xi(s, \rho), X(\varkappa), Y(\varkappa), Z(\varkappa), W(\varkappa) \in C^1$. Therefore, $r(s)$ is an isogeodesic curve on $R(s, \varkappa, \rho)$ if and only if

$$\begin{gathered} \lambda(s, \rho_0) X(\varkappa_0) = \mu(s, \rho_0) Y(\varkappa_0) = \nu(s, \rho_0) Z(\varkappa_0) = \xi(s, \rho_0) W(\varkappa_0) = 0, \\ \nu(s, \rho_0) \frac{\partial \xi(s, \rho_0)}{\partial \rho} \frac{dZ(\varkappa_0)}{d\varkappa} W(\varkappa_0) - \frac{\partial \nu(s, \rho_0)}{\partial \rho} \xi(s, \rho_0) Z(\varkappa_0) \frac{dW(\varkappa_0)}{d\varkappa} \neq 0, \end{gathered} \tag{19}$$

$$Y(\varkappa_0) = \mu(s,\rho_0) = 0 \text{ or } \frac{dY(\varkappa_0)}{d\varkappa} = Y(\varkappa_0) = 0 \text{ or } \frac{dY(\varkappa_0)}{d\varkappa} = \frac{\partial\mu(s,\rho_0)}{\partial\rho} = 0,$$
$$\varkappa_0 \in [T_1, T_2],\ \rho_0 \in [Q_1, Q_2]$$

satisfied.

Example 3.3. Let $r(s)$ be a curve given by parametrization

$$r(s) = \left(s, \cos s, \sqrt{2}\sin s, \cos s\right).$$

It is easy to calculate that

$$\begin{aligned} t &= \left(1, -\sin s, \sqrt{2}\cos s, -\sin s\right), \\ n &= \frac{1}{\sqrt{2}}\left(0, -\cos s, -\sqrt{2}\sin s, -\cos s\right), \\ b &= \frac{1}{\sqrt{2}}\left(0, \sin s, -\sqrt{2}\cos s, \sin s\right), \\ e &= \frac{1}{\sqrt{2}}(0, -1, 0, 1). \end{aligned}$$

Now, we obtain the hypersurface family with the isogeodesic curve $r(s)$ for three different types of the marching-scale functions.

Marching-scale functions of Type A : Let us choose

$$\begin{aligned} \lambda(s) &= \mu(s) = \nu(s) = \xi(s) = 1, \\ X(\varkappa,\rho) &= \rho(\varkappa - \varkappa_0)(\rho - \rho_0), \\ Y(\varkappa,\rho) &= 0, \\ Z(\varkappa,\rho) &= \rho(\varkappa - \varkappa_0), \\ W(\varkappa,\rho) &= (\rho - \rho_0), \end{aligned}$$

where $\varkappa_0 \in [0,1]$, $0 \le s \le 2\pi$ and from (14) we take $\rho_0 \neq 0$. So, we get

$$\begin{aligned} \alpha(s,\varkappa,\rho) &= \rho(\varkappa - \varkappa_0)(\rho - \rho_0), \\ \beta(s,\varkappa,\rho) &= 0, \\ \gamma(s,\varkappa,\rho) &= \rho(\varkappa - \varkappa_0), \\ \delta(s,\varkappa,\rho) &= (\rho - \rho_0) \end{aligned}$$

and using (6) and Frenet vectors, then we get the hypersurface which is a member of hypersurface family as follows

$$R(s,\varkappa,\rho)=\begin{pmatrix} s+\rho(\varkappa-\varkappa_0)(\rho-\rho_0), \\ \cos s-\rho(\varkappa-\varkappa_0)(\rho-\rho_0)\sin s+\frac{1}{\sqrt{2}}\rho(\varkappa-\varkappa_0)\sin s-\frac{1}{\sqrt{2}}(\rho-\rho_0), \\ \sqrt{2}\sin s+\sqrt{2}\rho(\varkappa-\varkappa_0)(\rho-\rho_0)\cos s-\rho(\varkappa-\varkappa_0)\cos s, \\ \cos s-\rho(\varkappa-\varkappa_0)(\rho-\rho_0)\sin s+\frac{1}{\sqrt{2}}\rho(\varkappa-\varkappa_0)\sin s+\frac{1}{\sqrt{2}}(\rho-\rho_0) \end{pmatrix},$$

where $0\leq s\leq 2\pi$, $0\leq \varkappa_0\leq 1$. The position of the curve $r(s)$ can be set on the hypersurface by changing the parameters $\varkappa_0$ and ρ_0 . Let us take $\varkappa_0=0$ and $\rho_0=\frac{1}{2}$. Now $r(s)$ is again an isogeodesic on the hypersurface $R(s,\varkappa,\rho)$ and the equation of the hypersurface becomes

$$R(s,\varkappa,\rho)=\begin{pmatrix} s+\rho\varkappa(\rho-\frac{1}{2}), \\ \cos s-\rho\varkappa(\rho-\frac{1}{2})\sin s+\frac{1}{\sqrt{2}}\rho\varkappa\sin s-\frac{1}{\sqrt{2}}(\rho-\frac{1}{2}), \\ \sqrt{2}\sin s+\sqrt{2}\rho\varkappa(\rho-\frac{1}{2})\cos s-\rho\varkappa\cos s, \\ \cos s-\rho\varkappa(\rho-\frac{1}{2})\sin s+\frac{1}{\sqrt{2}}\rho\varkappa\sin s+\frac{1}{\sqrt{2}}(\rho-\frac{1}{2}) \end{pmatrix}.$$

The principle step for visualization 4D is projecting (parallel or perspective) the geometric objects in 4-space into the 3-space. Thus, we yield a three-dimensional volume. Furthermore, in practice the problem of visualizing and approximating three-dimensional data, commonly referred to as scalar fields. The graph of a function $f(\mathbf{x},\mathbf{y},\mathbf{z}):U\subset\mathbb{R}^3\to\mathbb{R}$, U is open, is a special type of parametric hypersurface the parametrization $(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}=f(\mathbf{x},\mathbf{y},\mathbf{z}))$ in 4-space. For further information about visualization of four-dimensional space, we refer to [17, 6, 7]. So, if we (parallel) project the hypersurface $R(s,\varkappa,\rho)$ into the $\mathbf{z}=0$ subspace and setting $\varkappa=\frac{1}{2}$, the surface is given by

$$R_{\mathbf{z}}(s,\rho)=\begin{pmatrix} s+\frac{1}{2}\rho(\rho-\frac{1}{2}), \\ \cos s-\frac{1}{2}\rho(\rho-\frac{1}{2})\sin s+\frac{1}{2\sqrt{2}}\rho\sin s-\frac{1}{\sqrt{2}}(\rho-\frac{1}{2}), \\ \cos s-\frac{1}{2}\rho(\rho-\frac{1}{2})\sin s+\frac{1}{2\sqrt{2}}\rho\sin s+\frac{1}{\sqrt{2}}(\rho-\frac{1}{2}) \end{pmatrix},$$

where $0\leq s\leq 2\pi$ and $0\leq\rho\leq 1$, in 3-space drawn in Figure 1-Type A.

Marching-scale functions of Type B : Let us take

$$\begin{aligned} \nu(s,\varkappa) &= (s+\varkappa),\xi(s,\varkappa)=s(\varkappa-\varkappa_0), \\ X(\rho) &= Y(\rho)=0, \\ Z(\rho) &= (\rho-\rho_0),W(\rho)\equiv 1, \end{aligned}$$

where $\varkappa_0\in[0,1]$, $\rho_0\in[0,1]$ and $\pi\leq s\leq 3\pi$. Then, we obtain

$$\begin{aligned}\alpha(s,\varkappa,\rho) &= 0,\\ \beta(s,\varkappa,\rho) &= 0,\\ \gamma(s,\varkappa,\rho) &= (s+\varkappa)(\rho-\rho_0),\\ \delta(s,\varkappa,\rho) &= s(\varkappa-\varkappa_0),\end{aligned}$$

and using (6) and Frenet vectors, the hypersurface satisfies

$$R(s,\varkappa,\rho)=\begin{pmatrix} s,\cos s+\frac{1}{\sqrt{2}}(s+\varkappa)(\rho-\rho_0)\sin s-\frac{1}{\sqrt{2}}s(\varkappa-\varkappa_0),\\ \sqrt{2}\sin s-(s+\varkappa)(\rho-\rho_0)\cos s,\\ \cos s+\frac{1}{\sqrt{2}}(s+\varkappa)(\rho-\rho_0)\sin s+\frac{1}{\sqrt{2}}s(\varkappa-\varkappa_0)\end{pmatrix},$$

where $\pi\leq s\leq 3\pi$, $0\leq\varkappa_0\leq 1$ and $0\leq\rho_0\leq 1$. Then, $R(s,\varkappa,\rho)$ is a member of the isogeodesic hypersurface family having the curve $r(s)$ as an isogeodesic.

If $\varkappa_0=1$ and $\rho_0=0$, then the hypersurface R is being

$$R(s,\varkappa,\rho)=\begin{pmatrix} s,\cos s+\frac{1}{\sqrt{2}}(s+\varkappa)\rho\sin s-\frac{1}{\sqrt{2}}s(\varkappa-1),\\ \sqrt{2}\sin s-(s+\varkappa)\rho\cos s,\\ \cos s+\frac{1}{\sqrt{2}}(s+\varkappa)\rho\sin s+\frac{1}{\sqrt{2}}s(\varkappa-1)\end{pmatrix}.$$

Thus, if we (parallel) project the hypersurface $R(s,\varkappa,\rho)$ into the $\mathbf{w}=0$ subspace and fixing $\rho=\dfrac{1}{8}$, the surface is given by

$$R_{\mathbf{w}}\left(s,\varkappa,\frac{1}{8}\right)=\begin{pmatrix} s,\cos s+\frac{1}{8\sqrt{2}}(s+\varkappa)\sin s-\frac{1}{\sqrt{2}}s(\varkappa-1),\\ \sqrt{2}\sin s-\dfrac{1}{8}(s+\varkappa)\cos s)\end{pmatrix},\tag{20}$$

where $\pi\leq s\leq 3\pi$, $0\leq\varkappa\leq 1$, in 3-space illustrated in Figure 1-Type B.

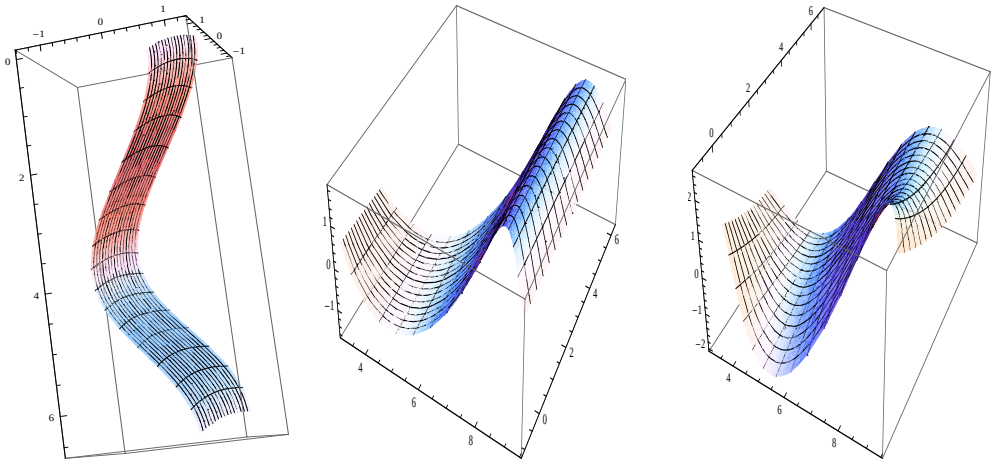
Marching-scale functions of Type C: Consider

$$\begin{aligned}\nu(s,\rho) &= s(\rho-\rho_0),\xi(s,\rho)=(s+\rho+1),\\ X(\varkappa) &= Y(\varkappa)=0,\\ Z(\varkappa) &= \varkappa^2, W(\varkappa)\equiv(\varkappa-\varkappa_0),\end{aligned}$$

where $\rho_0\in[0,1]$ and $\pi\leq s\leq 3\pi$ and from (14) we take $\varkappa_0\neq 0$.

Then, we obtain

$$\begin{aligned}\alpha(s,\varkappa,\rho) &= 0,\\ \beta(s,\varkappa,\rho) &= 0,\\ \gamma(s,\varkappa,\rho) &= s(\rho-\rho_0)\varkappa^2,\\ \gamma(s,\varkappa,\rho) &= (s+\rho+1)(\varkappa-\varkappa_0)\end{aligned}$$



(a) Type A (b) Type B (c) Type C

Figure 1. Projection of a member of the hypersurface family with marching-scale functions and its isogeodesic.

and using (6) and Frenet vectors, the hypersurface can be found as follows:

$$R(s,\varkappa,\rho)=\begin{pmatrix} s,\cos s+\frac{1}{\sqrt{2}}s(\rho-\rho_0)\varkappa^2\sin s-\frac{1}{\sqrt{2}}(s+\rho+1)(\varkappa-\varkappa_0),\\ \sqrt{2}\sin s-s(\rho-\rho_0)\varkappa^2\cos s,\\ \cos s+\frac{1}{\sqrt{2}}s(\rho-\rho_0)\varkappa^2\sin s+\frac{1}{\sqrt{2}}(s+\rho+1)(\varkappa-\varkappa_0)\end{pmatrix}.$$

Then $R(s,\varkappa,\rho)$ is a member of the isogeodesic hypersurface family.
Setting $\varkappa_0=1$ and $\rho_0=0$. Then, the hypersurface R becomes

$$R(s,\varkappa,\rho)=\begin{pmatrix} s,\cos s+\frac{1}{\sqrt{2}}s\rho\varkappa^2\sin s-\frac{1}{\sqrt{2}}(s+\rho+1)(\varkappa-1),\\ \sqrt{2}\sin s-s\rho\varkappa^2\cos s,\\ \cos s+\frac{1}{\sqrt{2}}s\rho\varkappa^2\sin s+\frac{1}{\sqrt{2}}(s+\rho+1)(\varkappa-1)\end{pmatrix}.$$

Hence, if we (parallel) project the hypersurface $R(s,\varkappa,\rho)$ into the $\mathbf{w}=0$ subspace and fixing $\rho=\dfrac{1}{4}$, the surface is given by

$$R_{\mathbf{w}}\left(s,\varkappa,\frac{1}{4}\right)=\begin{pmatrix} s,\cos s+\frac{1}{4\sqrt{2}}s\varkappa^2\sin s-\frac{1}{\sqrt{2}}(s+\frac{5}{4})(\varkappa-1),\\ \sqrt{2}\sin s-s\frac{1}{4}\varkappa^2\cos s\end{pmatrix},\tag{21}$$

where $\pi\le s\le 3\pi$, $0\le\varkappa\le 1$, in 3-space plotted in Figure 1-Type C.

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