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EXAMPLES OF m-ISOMETRIC TUPLES OF OPERATORS ON A HILBERT SPACE

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Abstract. The m-isometry of a single operator in Agler and Stankus [3] was naturally generalized to the m-isometric tuple of several commuting operators by Gleason and Richter [22]. Some examples of m-isometric tuples including the recently much studied Arveson-Drury d-shift were given in [22]. We provide more examples of m-isometric tuples of operators by using sums of operators or products of operators or functions of operators. A class of m-isometric tuples of unilateral weighted shifts parametrized by polynomials are also constructed. The examples in Gleason and Richter [22] are then obtained by choosing some specific polynomials. This work extends partially results obtained in several recent papers on the m-isometry of a single operator.

1. Introduction

We first introduce the tuples of operators to be studied in this paper. Let

$$z = (z_1, \dots, z_d), \overline{z} = (\overline{z_1}, \dots, \overline{z_d}).$$

Let $p(z, \overline{z})$ be a polynomial of multivariables z and $\overline{z}$ of the form

$$p(z, \overline{z}) = \sum_{|\alpha|=0}^{m} \sum_{|\beta|=0}^{m} c_{\alpha\beta} \overline{z}^{\alpha} z^{\beta}, c_{\alpha\beta} \in \mathbb{C},$$

where $\alpha = (\alpha_1, \dots \alpha_d)$ and $\beta = (\beta_1, \dots \beta_d)$ are multi-indices, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $z^{\alpha} = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. Let H be a complex Hilbert space and $B(H)$ be the algebra of all bounded linear operators on H. Let $T = (T_1, \dots, T_d) \in B(H)^d$ be a tuple of commuting operators and $T^* = (T_1^*, \dots, T_d^*)$. As in the hereditary functional calculus by Agler [1], we define

$$p(T) = \sum_{|\alpha|=0}^{m} \sum_{|\beta|=0}^{m} c_{\alpha\beta} T^{*\alpha} T^{\beta}, c_{\alpha\beta} \in \mathbb{C},$$

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where $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$. An important property of the hereditary functional calculus is that if H_0 is a common invariant subspace of T, then

$$p(T|H_0) = P_{H_0} p(T)|H_0,$$

where P_{H_0} is the projection from H to H_0. The d-tuple T is an m-isometry for some positive integer m if

$$\begin{aligned}\Phi_m(T) &:= (\overline{z}\cdot z - 1)^m(T) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\left(\overline{z}\cdot z\right)^k(T) \\ &= \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\binom{k}{\alpha}T^{*\alpha}T^{\alpha} = 0,\end{aligned}$$

where

$$\overline{z}\cdot z = \sum_{i=1}^{d}\overline{z_i}\cdot z_i, \ \binom{k}{\alpha} = \frac{k!}{\alpha_1!\cdots\alpha_d!}.$$

This notion of an m-isometric d-tuple is a natural generalization of the m-isometry of a single operator in Agler and Stankus [3]. Basic properties of m-isometric d-tuples are first obtained by Gleason and Richter [22]. Some examples including the recently much studied Drury-Arveson d-shifts [7], [19] are shown to be m-isometric tuples.

Since the seminal papers by Agler and Stankus [3], [4] and [5], the theory of m-isometries has been well developed. The theory for m-isometries on Hilbert spaces has rich connections to Toeplitz operators, classical function theory, and other areas of mathematics. The work of Richter [36] and [37] on analytic 2-isometries has a connection with the invariant subspaces of the shift operator on the Dirichlet space. See also related papers [33], [34], [35], [40]. Recently the class of m-isometric operators have also been introduced on Banach spaces and metric spaces [9], [10], [14], [24], [25].

The extension of the theory of the m-isometry of a single operator to a tuple of several commuting operators has been slow, see two recent papers [29], [38]. Motivated by several recent papers on the m-isometry of a single operator, here we provide more examples of m-isometric tuples of operators, which will motivate and help the further study of general m-isometric tuples and their applications. The following notation is useful. Let $T = (T_1, \ldots, T_d) \in B(H)^d$. Then there is an associated elementary operator $\Omega_T : B(H) \to B(H)$ defined by

$$\Omega_T(X) = \sum_{i=1}^{d} T_i^* X T_i, \ X \in B(H). \tag{1}$$

We first note a recursive identity which is also (2.1) in [22]. It follows from

$$(\overline{z}\cdot z - 1)^m = \left[\sum_{i=0}^{d}\overline{z_i}(\overline{z}\cdot z - 1)^{m-1}z_i\right] - (\overline{z}\cdot z - 1)^{m-1}$$

that

$$\Phi_m(T) = (\overline{z} \cdot z - 1)^m(T) = \left[\sum_{i=0}^{d} T_i^* \Phi_{m-1}(T) T_i\right] - \Phi_{m-1}(T)$$
$$= \Omega_T(\Phi_{m-1}(T)) - \Phi_{m-1}(T). \tag{2}$$

Therefore if T is an m-isometry, then T is an n-isometry for any $n \geq m$. If T is an m-isometry but not an $(m-1)$-isometry, then T is said to be a strict m-isometry. Throughout the paper, we will study d-tuples of commuting operators unless otherwise stated. Let $Q = (Q_1, \dots Q_d) \in B(H)^d$. We say T is a tuple of double commuting operators if for $i \neq j,\ 1 \leq i, j \leq d$,

$$T_i T_j = T_j T_i, T_i^* T_j = T_j T_i^*.$$

We say T and Q are commuting if

$$T_i Q_j = Q_j T_i,\ 1 \leq i, j \leq d,$$

and T and Q are double commuting if additionally

$$T_i^* Q_j = Q_j T_i^*,\ 1 \leq i, j \leq d.$$

The outline of this paper is as follows. In Section 2 we show how to compute the hereditary functional calculus for a sum of two commuting tuples of operators. We then prove that the sum of an m-isometric tuple and a tuple of nilpotent operators of order n is an $(m + 2n - 2)$-isometry. In Section 3 we show how to compute the hereditary functional calculus for a product of two double commuting tuples of operators. We then study when such a product is an m-isometry. In Section 4, we prove that if $T \in B(H)^d$ is an m-isometry and $\varphi(z)$ is an automorphism of the unit ball of $\mathbb{C}^d$, then $\varphi(T) \in B(H)^d$ is also an m-isometry. The proofs of most results in Section 2, Section 3 and Section 4 are direct applications of the hereditary functional calculus by using multinomial formulas. In fact, these relatively simple proofs made the discovery of the results for tuples of operators possible, since the original proofs of the corresponding results for a single operator using combinatorics are much more involved [11], [12], [15], [16], [17], [20], [21]. In Section 5, we give examples of multivariable weighted shifts, which are m-isometries. Our examples are parametrized by polynomials (subject to some positivity conditions). Those polynomials appeared in characterizing m-isometric weighted shifts of a single variable by the author [24]. We prove a Berger-Shaw type result for such a tuple of m-isometric weighted shifts T. Namely, we show that $\Phi_{m-1}(T)$ is a positive trace class operator for $m > d + 1$. In the last section, we reformulate results for unilateral weighted shifts of single variable or multivariable as theorems for multiplication operators on weighted Hardy spaces of analytic functions in one variable or several variables. In particular, we immediately get the examples of Drury-Arveson d-shifts considered in [22] by choosing some special polynomials.

2. Sums of m-isometries and nilpotent operators

Lemma 2.1. *Let $T=(T_1,\dots,T_d)\in B(H)^d$ and $Q=(Q_1,\dots,Q_d)\in B(H)^d$. Assume T and Q commute. Then the following holds.*

$$\begin{aligned}&\Phi_n(T+Q)\\ &=\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{|\alpha|=j}\sum_{|\beta|=k-j}\binom{n}{k}\binom{k}{j}\binom{j}{\alpha}\binom{k-j}{\beta}(T^*+Q^*)^\beta Q^{*\alpha}\Phi_{n-k}(T)T^\alpha Q^\beta.\end{aligned}$$

Proof. We first prove an identity for polynomials of multivariables. Let

$$w=(w_1,\dots,w_d),\overline{w}=(\overline{w_1},\dots,\overline{w_d}),$$
$$z+w=(z_1+w_1,\dots,z_d+w_d).$$

Then

$$\begin{aligned}&([\overline{z}+\overline{w}]\cdot[z+w]-1)^n\\ &=(\overline{z}\cdot z-1+\overline{w}\cdot z+[\overline{z}+\overline{w}]\cdot w)^n\\ &=\sum_{k=0}^{n}\binom{n}{k}(\overline{z}\cdot z-1)^{n-k}(\overline{w}\cdot z+[\overline{z}+\overline{w}]\cdot w)^k\\ &=\sum_{k=0}^{n}\binom{n}{k}(\overline{z}\cdot z-1)^{n-k}\sum_{j=0}^{k}\binom{k}{j}(\overline{w}\cdot z)^j\left([\overline{z}+\overline{w}]\cdot w\right)^{k-j}\\ &=\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{|\alpha|=0}^{j}\sum_{|\beta|=0}^{j}\binom{n}{k}\binom{k}{j}\binom{j}{\alpha}\binom{k-j}{\beta}(\overline{z}\cdot z-1)^{n-k}\overline{w}^\alpha z^\alpha\left[\overline{z}+\overline{w}\right]^\beta w^\beta\\ &=\sum_{k=0}^{n}\sum_{j=0}^{k}\sum_{|\alpha|=j}\sum_{|\beta|=k-j}\binom{n}{k}\binom{k}{j}\binom{j}{\alpha}\binom{k-j}{\beta}\left[\overline{z}+\overline{w}\right]^\beta\overline{w}^\alpha(\overline{z}\cdot z-1)^{n-k}z^\alpha w^\beta,\end{aligned}$$

where in the second and third equalities we use binomial formula, in the fourth equality we use multinomial formula for $(\overline{w}\cdot z)^j$ and $\left([\overline{z}+\overline{w}]\cdot w\right)^{k-j}$, and in the last equality we just rearrange the terms so that the conjugate variables $\overline{z}$ and $\overline{w}$ are on the left. In the above identity, z is replaced by T, $\overline{z}$ is replaced by T^*, w is replaced by Q, $\overline{w}$ is replaced by Q^*, and we get the desired identity for operators. The commuting condition of T and Q is needed to put the involved operators in the desired order. □

When $d=1$, the above lemma is proved in Lemma 1 [28].

Let $Q\in B(H)^d$. We say Q is a nilpotent tuple of order n if $Q^\beta=0$ for any multi-index β with $|\beta|=n$ and $Q^\beta\neq 0$ for some multi-index β with $|\beta|=n-1$.

Theorem 2.2. *Let $T=(T_1,\dots,T_d)\in B(H)^d$ and $Q=(Q_1,\dots,Q_d)\in B(H)^d$. Assume A and Q commute. If T is an m-isometry and Q is a nilpotent tuple of order n, then $T+Q$ is an $(m+2n-2)$-isometry.*

Proof. We need to show $\Phi_{m+2n-2}(T+Q)=0$. Set $l=m+2n-2$. By Lemma 2.1,

$$\begin{aligned}&\Phi_l(T+Q)\\&=\sum_{k=0}^{l}\sum_{j=0}^{k}\sum_{|\alpha|=j}\sum_{|\beta|=k-j}\binom{l}{k}\binom{k}{j}\binom{j}{\alpha}\binom{k-j}{\beta}(T^*+Q^*)^\beta Q^{*\alpha}\Phi_{l-k}(T)T^\alpha Q^\beta.\end{aligned}$$

When $0\le k\le 2n-2$, we have $l-k\ge m$ and thus $\Phi_{l-k}(T)=0$. When $k\ge 2n-1$, then either $|\alpha|\ge n$ or $|\beta|\ge n$ since $|\alpha|+|\beta|=k$. In this case, either $Q^{*\alpha}=0$ or $Q^\beta=0$. In conclusion, $\Phi_l(T+Q)=0$ and $T+Q$ is an $(m+2n-2)$-isometry. □

Remark 2.3. By a similar argument, the only possible nonzero term in $\Phi_{l-1}(T+Q)$ is when $l-1-k=m-1$. Thus

$$\begin{aligned}&\Phi_{l-1}(T+Q)\\&=\delta\sum_{|\alpha|=n-1}\sum_{|\beta|=n-1}\binom{n-1}{\alpha}\binom{n-1}{\beta}(T^*+Q^*)^\beta Q^{*\alpha}\Phi_{m-1}(T)T^\alpha Q^\beta,\end{aligned}\tag{3}$$

where

$$\delta=\binom{l-1}{2n-2}\binom{2n-2}{n-1}.$$

Therefore $T+Q$ is a strict $(m+2n-2)$-isometry if and only if $\Phi_{l-1}(T+Q)$ as above is a nonzero operator.

If $Q_i=c_iQ_0$ for some complex numbers $\{c_i\}$ and a nilpotent operator Q_0 of order n, let

$$c=(c_1,\dots,c_d).$$

Then, by (3)

$$\begin{aligned}&\Phi_{l-1}(T+Q)\\&=\delta\sum_{|\alpha|=n-1}\sum_{|\beta|=n-1}\binom{n-1}{\alpha}\binom{n-1}{\beta}(T^*+Q^*)^\beta\overline{c}^\alpha Q_0^{*(n-1)}\Phi_{m-1}(T)T^\alpha Q^\beta\\&=\delta\sum_{|\beta|=n-1}\binom{n-1}{\beta}(T^*+Q^*)^\beta Q_0^{*(n-1)}\Phi_{m-1}(T)\left[\sum_{|\alpha|=n-1}\binom{n-1}{\alpha}\overline{c}^\alpha T^\alpha\right]Q^\beta\\&=\delta\sum_{|\beta|=n-1}\binom{n-1}{\beta}(T^*+Q^*)^\beta Q_0^{*(n-1)}\Phi_{m-1}(T)\left(\overline{c}\cdot T\right)^{n-1}Q^\beta\\&=\delta\sum_{|\beta|=n-1}\binom{n-1}{\beta}T^{*\beta}Q_0^{*(n-1)}\Phi_{m-1}(T)\left(\overline{c}\cdot T\right)^{n-1}c^\beta Q_0^{n-1}\\&=\delta Q_0^{*(n-1)}\left(\overline{c}\cdot T\right)^{*(n-1)}\Phi_{m-1}(T)\left(\overline{c}\cdot T\right)^{n-1}Q_0^{n-1},\end{aligned}\tag{4}$$

where the fourth equality follows from $(T^*+Q^*)^\beta Q_0^{*(n-1)}=T^{*\beta}Q_0^{*(n-1)}$ for $|\beta|=n-1$ since $Q_0^{*n}=0$.

When $d = 1$, the above theorem is first proved in [11] for $m = 1$; for $m \geq 1$, see also related work in [14], [28], and [32].

A simple example shows that the commuting condition of T and Q can not be removed from the above theorem.

Example 2.4. Let

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then T is an isometry and $Q^2 = 0$. Set $A = T + Q$. Since $A^2 = I$, by a direct computation,

$$\Phi_m(A) = (-2)^{m-1}(A^*A - I) \neq 0.$$

Therefore A is not an m-isometry for any $m \geq 1$.

There are some constructions that give rise to commuting tuples of operators. We state them as corollaries. The first construction uses block operator matrices.

Corollary 2.5. *Let $A = (A_1, \dots, A_d) \in B(H)^d$ be an m-isometry. Let $S = (S_1, \dots, S_d) \in B(H^{(n)})^d$ be defined by*

$$S_i = \begin{bmatrix} A_i & c_i I & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & c_i I \\ 0 & \ddots & 0 & A_i \end{bmatrix} \text{ on } H^{(n)} = H \oplus \cdots \oplus H, \tag{5}$$

where c_i is a complex number for each $i = 1, 2, \dots, n$ and $H^{(n)}$ is the sum of n-copies of H. Then S is an $(m+2n-2)$-isometry. Furthermore, $S = (S_1, \dots, S_d)$ is a strict $(m+2n-2)$-isometry if and only if $(\overline{c} \cdot A)^{*(n-1)} \, \Phi_{m-1}(A) \, (\overline{c} \cdot A)^{n-1}$ is not a zero operator.

Proof. We write S_i as the sum of a block diagonal operator and a nilpotent operator of order n as follows:

$$S_i = T_i + c_i J_n,$$

where

$$T_i = \begin{bmatrix} A_i & 0 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & A_i \end{bmatrix} \text{ and } J_n = \begin{bmatrix} 0 & I & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & I \\ 0 & \ddots & 0 & 0 \end{bmatrix}.$$

Then, by (4),

$$\Phi_{m+2n-3}(S) = \delta J_n^{*n-1} \, (\overline{c} \cdot T)^{*(n-1)} \, \Phi_{m-1}(T) \, (\overline{c} \cdot T)^{n-1} \, J_n^{n-1}.$$

So $\Phi_{m+2n-3}(S)$ is a block operator matrix with only possible nonzero entry

$$\delta\,(\overline{c}\cdot A)^{*(n-1)}\,\Phi_{m-1}(A)\,(\overline{c}\cdot A)^{n-1}$$

in the lower right corner. The proof is complete. □

Example 2.6. We use the same notation as in Corollary 2.5. Let $A = \lambda := (\lambda_1, \dots, \lambda_d)$, where $|\lambda_1|^2 + \cdots + |\lambda_d|^2 = 1$ and $c = (c_1, \dots, c_d)$ be such that $\overline{c}\cdot\lambda \neq 0$. Then the tuple S as in (5) is a strict $(2n-1)$-isometry whose joint point spectrum is the single point λ.

It has been shown by Lemma 3.2 in [22] that the joint approximate point spectrum of an m-isometry is in the boundary of the unit ball of $\mathbb{C}^d$. By using an infinite direct sum of examples here, we see that any compact subset of the boundary of the unit ball of $\mathbb{C}^d$ could be the spectrum of a strict $(2n-1)$-isometry for $n \geq 2$. When $d = 1$, it is proved in Lemma 1.21 and Proposition 1.23 of [3] that the spectrum of a strict $2m$-isometry is the closed unit disk for $m \geq 1$.

The more general construction uses tensor products of operators. Let K be another complex Hilbert space. Let $H \otimes K$ denote the tensor product Hilbert space of H and K.

Corollary 2.7. *Let $T = (T_1, \dots, T_d) \in B(H)^d$ be an m-isometry. Let $Q = (Q_1, \dots, Q_d) \in B(K)^d$ be a nilpotent tuple of order n. Then $T \otimes I_K + I_H \otimes Q := (T_1 \otimes I_K + I_H \otimes Q_1, \dots, T_d \otimes I_K + I_H \otimes Q_d) \in B(H \otimes K)^d$ is an $(m+2n-2)$-isometry.*

Proof. We note that $T \otimes I_K \in B(H \otimes K)^d$ is an m-isometry and $I_H \otimes Q \in B(H \otimes K)^d$ is a nilpotent tuple of order n. □

The above corollary in the case $d = 1$ is proved in [23] and it almost has a converse, see Theorem 12 in [23] for details.

When H is finite dimensional, in the single operator case, it is shown in [2] that any m-isometry is of the form $U + Q$, where U is a unitary matrix and Q is a nilpotent matrix commuting with U. A natural question is to study the structure of m-isometric tuples of operators on a finite dimensional Hilbert space.

3. Product of two tuples of operators

For two multivariables z and w and two tuples of operators T and S, set

$$z * w = (z_1 w_1, \dots, z_d w_d),\ T * S = (T_1 S_1, \dots, T_d S_d).$$

Lemma 3.1. *Assume $S \in B(H)^d$ is a tuple of double commuting operators. Assume $T, S \in B(H)^d$ are double commuting. Then the following holds.*

$$\Phi_n(T * S) = \sum_{k=0}^{n} \sum_{|\alpha|=k} \binom{n}{k}\binom{k}{\alpha} T^{*\alpha}\Phi_{n-k}(T)T^{\alpha}\prod_{i=1}^{d}\Phi_{\alpha_i}(S_i). \tag{6}$$

Proof. As in Lemma 2.1, we first prove an identity for polynomials of multi-variables z and w. Note that

$$
\begin{aligned}
&([\overline{z*w}]\cdot[z*w]-1)^n\\
&=\left(\sum_{i=1}^d \overline{z_i w_i} w_i z_i - 1\right)^n\\
&=\left(\sum_{i=1}^d \overline{z_i}\left[\overline{w_i}w_i-1\right]z_i+\sum_{i=1}^d \overline{z_i}z_i-1\right)^n\\
&=\sum_{k=0}^n \binom{n}{k}\left(\sum_{i=1}^d \overline{z_i}\left[\overline{w_i}w_i-1\right]z_i\right)^k(\overline{z}\cdot z-1)^{n-k}\\
&=\sum_{k=0}^n \binom{n}{k}\left\{\sum_{|\alpha|=k}\binom{k}{\alpha}\prod_{i=1}^d \overline{z_i}^{\alpha_i}\left[\overline{w_i}w_i-1\right]^{\alpha_i} z_i^{\alpha_i}\right\}(\overline{z}\cdot z-1)^{n-k}\\
&=\sum_{k=0}^n\sum_{|\alpha|=k}\binom{n}{k}\binom{k}{\alpha}\overline{z}^{\alpha}(\overline{z}\cdot z-1)^{n-k}z^{\alpha}\prod_{i=1}^d\left[\overline{w_i}w_i-1\right]^{\alpha_i},
\end{aligned}
$$

where the third equality follows from binomial formula and the fourth equality follows from multinomial formula. The desired formula follows again by replacing z by T, replacing $\overline{z}$ by T^*, replacing w_i by S_i and replacing $\overline{w_i}$ by S_i^*. The double commuting condition of T and S is needed to put $\Phi_{\alpha_i}(S_i)$ on the right side of T^α, and the double commuting condition of S is needed to write the product $\prod_{i=1}^d \Phi_{\alpha_i}(S_i)$. □

Theorem 3.2. *Assume $T \in B(H)^d$ is an m-isometric tuple of commuting operators. Assume $S \in B(H)^d$ is a tuple of double commuting operators such that each S_i is an n_i-isometry for $i = 1, \dots, d$. Assume also T and S are double commuting. Then $T * S$ is an $(m + \sum_{i=1}^d n_i - d)$-isometric tuple of operators.*

Proof. Let $n = \sum_{i=1}^d n_i$ and $l = (m + n - d)$. By previous lemma,

$$
\Phi_l(T*S)=\sum_{k=0}^l\sum_{|\alpha|=k}\binom{l}{k}\binom{k}{\alpha}T^{*\alpha}\Phi_{l-k}(T)T^{\alpha}\prod_{i=1}^d\Phi_{\alpha_i}(S_i).
$$

We need to show $\Phi_l(T * S) = 0$. When $0 \le k \le n - d$, we have $l - k \ge m$ and thus $\Phi_{l-k}(T) = 0$. When $k > n - d$, then one of the α_i satisfies $|\alpha_i| \ge n_i$ since $|\alpha| = k$. In this case, $\Phi_{\alpha_i}(S_i) = 0$. In conclusion $\Phi_l(T * S) = 0$ and $T * S$ is a l-isometry. □

Remark 3.3. By a similar argument, the only nonzero term in $\Phi_{l-1}(T * S)$ is when $k = n - d$ and $\alpha = (n_1 - 1, \ldots, n_d - 1)$. Thus

$$\Phi_{l-1}(T * S)$$
$$= \binom{l-1}{n-d}\binom{n-d}{\alpha}\left(\prod_{i=1}^{d} T_i^{*n_i-1}\right)\Phi_{m-1}(T)\left(\prod_{i=1}^{d} T_i^{n_i-1}\right)\left(\prod_{i=1}^{d}\Phi_{n_i-1}(S_i)\right).$$

Therefore $T * S$ is a strict l-isometry if and only $\Phi_{l-1}(T * S)$ as above is a nonzero operator.

When $d = 1$, the above theorem is proved in [12]. In fact it is proved on a Banach space in [12], so only commuting condition of T and S is needed. The Banach space analogue (without the double commuting conditions of course) of the above theorem will be presented elsewhere [27]. When $d = 1$, (6) is proved in Lemma 7 [28], and it is useful for deriving results for related operators such as hypercontractions. See Theorem 4.7 in [26]. A simple example shows that the commuting condition of T and S is needed.

Example 3.4. Let

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then T is an isometry, S is a strict 3-isometry and $TS \neq ST$. Note that

$$TS = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

The product TS is not an m-isometry for any $m \geq 1$ by Example 2.4.

When each S_i is an isometry, we have the following more precise result which can also be obtained directly from definition.

Corollary 3.5. *Assume $T \in B(H)^d$ is a strict m-isometric tuple of commuting operators. Assume $S \in B(H)^d$ is a tuple of double commuting operators such that each S_i is an isometry for $i = 1, \ldots, d$. Assume also T and S are double commuting. Then $T * S$ is a strict m-isometric tuple of operators.*

Proof. We note in this case $l = (m + \sum_{i=1}^{d} n_i - d) = (m + d - d) = m$ and

$$\Phi_{l-1}(T * S) = \Phi_{m-1}(T).$$

The proof is complete. □

For the tensor product of the two tuples of operators, we have the following corollary.

Corollary 3.6. *Assume $T \in B(H)^d$ is an m-isometric tuple of commuting operators. Assume $S \in B(K)^d$ is a tuple of double commuting operators such* *that each S_i is an n_i-isometry for $i = 1, \ldots, d$. Then $T \otimes S := (T_1 \otimes S_1, \ldots, T_d \otimes S_d) \in B(H \otimes K)^d$ is a $(m + \sum_{i=1}^{d} n_i - d)$-isometric tuple of operators.*

Proof. We apply the previous theorem to $T \otimes I_K$ and $I_H \otimes S$ in $B(H \otimes K)^d$ to get $T \otimes S = [T \otimes I_K] * [I_H \otimes S]$. □

When $d = 1$, the above corollary is proved in [20], which in turn confirms the conjecture in [15] and [16]. This conjecture is formulated in term of elementary operators acting on Hilbert-Schmidt operator ideals. See also [28] and [32] for related work. A converse of the above corollary for $d = 1$ is obtained by Theorem 7 in [23].

4. Automorphism of the unit ball and m-isometries

For a single operator $T \in B(H)$, if T is an m-isometry, then $\varphi(T)$ is also an m-isometry where $\varphi(z)$ is an appropriate inner function, see Theorem 2.10 in [26] for details. In this section we prove that if $T \in B(H)^d$ is an m-isometric tuple of commuting operators and $\varphi(z)$ is an automorphism of the unit ball of $\mathbb{C}^d$, then $\varphi(T) \in B(H)^d$ (to be defined precisely below) is also an m-isometry.

We first introduce notation. Let $\mathbb{C}^d$ be the d dimensional complex space and

$$B^d = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1 \right\}$$

be the unit ball of $\mathbb{C}^d$. Then an automorphism of B^d is a biholomorphic map of B^d onto B^d. Let $a = (a_1, \ldots, a_d) \in B^d$ and $\langle z, a \rangle$ is the inner product defined by

$$\langle z, a \rangle = z \cdot \overline{a} = \sum_{i=1}^{d} z_i \overline{a_i}.$$

The general form of an automorphism $\varphi(z)$ of B^d is

$$\varphi(z) = U \circ L_a(z), \ z \in B^d, \tag{7}$$

where U is a $d \times d$ unitary matrix and L_a is the automorphism taking the point $a \in B^d \setminus \{0\}$ to 0. More precisely,

$$w = L_a(z) = \frac{a - P_a(z) - \delta q_a(z)}{1 - \langle z, a \rangle}, \tag{8}$$

where

$$\delta = \sqrt{1 - |a|^2}, \ P_a(z) = \langle z, a \rangle \frac{a}{|a|^2}, \ q_a(z) = z - P_a(z).$$

Furthermore

$$|w|^2 - 1 = \frac{\left(1 - |a|^2\right)\left(|z|^2 - 1\right)}{|1 - \langle z, a \rangle|^2}. \tag{9}$$

Let $T \in B(H)^d$ be an m-isometry. Then the joint spectral radius $r(T) = 1$; see Proposition 3.1 in [22]. Therefore $I - \sum_{i=1}^{d} \overline{a_i} T_i$ is invertible. According to (8), we define $S = (S_1, \ldots, S_d) := L_a(T)$ by

$$S_i = \left[a_i I - \frac{a_i}{|a|^2} \left(\sum_{i=1}^{d} \overline{a_i} T_i \right) - \delta \left(T_i - \frac{a_i}{|a|^2} \sum_{i=1}^{d} \overline{a_i} T_i \right) \right] \left[I - \sum_{i=1}^{d} \overline{a_i} T_i \right]^{-1} \tag{10}$$

for $i = 1, \ldots, d$. It is clear the $S \in B(H)^d$ is a d-tuple of commuting operators. This definition of $L_a(T)$ is standard; see for example [18].

Lemma 4.1. *Let $T \in B(H)^d$ be an m-isometry. Let S be defined by* (10). *Then S is also an m-isometry.*

Proof. We write (9) as

$$\overline{w} \cdot w - 1 = \left(1 - |a|^2\right) \left(1 - \sum_{i=1}^{d} a_i \overline{z_i}\right)^{-1} (\overline{z} \cdot z - 1) \left(1 - \sum_{i=1}^{d} \overline{a_i} z_i\right)^{-1}.$$

Then

$$(\overline{w} \cdot w - 1)^m = \left(1 - |a|^2\right)^m \left(1 - \sum_{i=1}^{d} a_i \overline{z_i}\right)^{-m} (\overline{z} \cdot z - 1)^m \left(1 - \sum_{i=1}^{d} \overline{a_i} z_i\right)^{-m}.$$

Replacing w by S, $\overline{w}$ by S^*, z by T, $\overline{z}$ by T^*, we have the operator identity

$$\Phi_m(S) = \left(1 - |a|^2\right)^m \left[I - \sum_{i=1}^{d} a_i T_i^*\right]^{-m} \Phi_m(T) \left[I - \sum_{i=1}^{d} \overline{a_i} T_i\right]^{-m}.$$

Thus $\Phi_m(T) = 0$ if and only if $\Phi_m(S) = 0$. □

The next lemma takes care of the unitary matrix in formula (7) for the automorphism $\varphi(z)$.

Lemma 4.2. *Let $T \in B(H)^d$ be an m-isometry. Let $U = [u_{ij}]$ be a $d \times d$ unitary matrix. Let $S = (S_1, \ldots, S_d)$ be defined by*

$$\begin{bmatrix} S_1 \\ \vdots \\ S_d \end{bmatrix} = \begin{bmatrix} u_{11} & \cdots & u_{1d} \\ \vdots & \vdots & \vdots \\ u_{d1} & \cdots & u_{dd} \end{bmatrix} \begin{bmatrix} T_1 \\ \vdots \\ T_d \end{bmatrix}.$$

Then S is also an m-isometry.

Proof. We first show that the two elementary operators Ω_T and Ω_S defined on $B(H)$ as in (1) are the same. Let $X \in B(H)$. Then

$$\Omega_S(X) = \sum_{i=1}^{d} S_i^* X S_i = \sum_{i=1}^{d} \left(\sum_{k=1}^{d} \overline{u_{ik}} T_k^* \right) X \left(\sum_{j=1}^{d} u_{ij} T_j \right)$$

$$= \sum_{i=1}^{d}\left(\sum_{k=1}^{d}\sum_{j=1}^{d}\overline{u_{ik}}u_{ij}T_k^*XT_j\right) = \sum_{k=1}^{d}\sum_{j=1}^{d}\left(\sum_{i=1}^{d}\overline{u_{ik}}u_{ij}\right)T_k^*XT_j$$

$$= \sum_{k=1}^{d}\sum_{j=1}^{d}\delta_{kj}T_k^*XT_j = \sum_{k=1}^{d}T_k^*XT_k = \Omega_T(X), \tag{11}$$

where the fifth equality follows from the assumption that U is unitary and δ_{kj} are Kronecker notations such that $\delta_{kj} = 0$ for $k \neq j$ and $\delta_{kj} = 1$ for $k = j$.

Now we prove $\Phi_m(S) = \Phi_m(T)$ by using induction. For $m = 1$, we have

$$\Phi_1(S) = \Omega_S(I) - I = \Omega_T(I) - I = \Phi_1(T).$$

Assume $\Phi_{m-1}(S) = \Phi_{m-1}(T)$ holds for $m - 1$. Then by (2) and (11),

$$\begin{aligned}\Phi_m(S) &= \Omega_S(\Phi_{m-1}(S)) - \Phi_{m-1}(S)\\ &= \Omega_T(\Phi_{m-1}(T)) - \Phi_{m-1}(T)\\ &= \Phi_m(T).\end{aligned}$$

The lemma follows from the above formula. □

Combining the previous two lemmas, we have the following theorem.

Theorem 4.3. *Let $T \in B(H)^d$ be an m-isometry. Let $\varphi(z)$ be an automorphism of the unit ball of $\mathbb{C}^d$. Then $\varphi(T) \in B(H)^d$ is also an m-isometry. In particular, $(\lambda_1 T_1, \ldots, \lambda_d T_d)$ is an m-isometry if each complex number λ_i satisfies $|\lambda_i| = 1$ for $i = 1, 2, \ldots, d$.*

5. Multivariable unilateral weighted shifts

In this section, we provide examples of multivariable weighted shifts that are m-isometries. We first develop some basic properties of m-isometric tuples. These properties and recent characterizations of m-isometries of one variable weighted shifts suggest a class of m-isometric tuple of multivariable weighted shifts.

We first introduce some notations. Let $T \in B(H)^d$. Set

$$\Psi_k(T) := (\overline{z} \cdot z)^k (T) = \sum_{|\alpha|=k}\binom{k}{\alpha}T^{*\alpha}T^{\alpha}.$$

The following lemma is equivalent to Lemma 2.2 in [22] with different notation. Here we give a rather trivial proof.

Lemma 5.1. *Let $T \in B(H)^d$. The following holds.*

$$\Psi_n(T) = \sum_{k=0}^{n}\binom{n}{k}\Phi_k(T).$$

Proof. Note that

$$\begin{aligned}(\overline{z} \cdot z)^n &= (\overline{z} \cdot z - 1 + 1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (\overline{z} \cdot z - 1)^k .\end{aligned}$$

The lemma follows by applying the hereditary functional calculus. □

In the single operator case, $\Psi_n(T) = T^{*n}T^n$ is called the symbol of T in [3], and there is a formula for this symbol; see Equation (1.3) in [3]. Here for $T \in B(H)^d$, we also call $\Psi_n(T)$ the symbol of T and the formula below for $\Psi_n(T)$ is called the reproducing formula for $\Psi_n(T)$. This reproducing formula follows directly from the above lemma and is used in several proofs in [22]. Here, to emphasize the importance of this formula, we state it as a proposition and view it as a characterization of m-isometries.

Proposition 5.2. *Let $T \in B(H)^d$. Then T is an m-isometry if and only if for $n \geq m$,*

$$\Psi_n(T) = \sum_{k=0}^{m-1} \binom{n}{k} \Phi_k(T). \tag{12}$$

Proof. Assume T is an m-isometry, that is $\Phi_k(T) = 0$ for $k \geq m$. Then by previous lemma

$$\Psi_n(T) = \sum_{k=0}^{n} \binom{n}{k} \Phi_k(T) = \sum_{k=0}^{m-1} \binom{n}{k} \Phi_k(T).$$

Assume (12) holds. Then

$$\Psi_m(T) = \sum_{k=0}^{m-1} \binom{m}{k} \Phi_k(T).$$

But by previous lemma

$$\Psi_m(T) = \sum_{k=0}^{m} \binom{m}{k} \Phi_k(T).$$

Subtracting above two formulas, we have

$$0 = \binom{m}{m} \Phi_m(T) = \Phi_m(T).$$

Therefore T is an m-isometry. □

To use formula (12), it is helpful to evaluate this formula on a vector $h \in H$. Hence we introduce the following notation:

$$\Psi_n(T, h) := \langle \Psi_n(T)h, h \rangle = \sum_{|\alpha|=n} \binom{n}{\alpha} \|T^\alpha h\|^2 , \tag{13}$$

$$\Phi_n(T,h) := \langle \Phi_n(T)h, h\rangle = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\Psi_k(T,h)$$

$$(14) \qquad = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{|\alpha|=n}\binom{n}{\alpha}\|T^\alpha h\|^2 .$$

The above proposition can be reformulated as follows: T is an m-isometry if and only if for $n \geq m, h \in H$,

$$(15) \qquad \Psi_n(T,h) = \sum_{k=0}^{m-1}\binom{n}{k}\Phi_k(T,h).$$

It turns out (from Lemma 5.1) the above formula is automatically true (T does not have to be an m-isometry) for $0 \leq n \leq m-1$ if one interprets $\binom{n}{k} = 0$ for $n < k$. We state this formally as a lemma.

Lemma 5.3. *Let $T \in B(H)^d$. For $h \in H$, $\Psi_n(T,h)$, and $\Phi_n(T,h)$ defined as in* (13) *and* (14)*, the unique polynomial $P(x)$ interpolating*

$$\{(k, \Psi_k(T,h)), 0 \leq k \leq m-1\}$$

is

$$P(x) = \sum_{k=0}^{m-1}\binom{x}{k}\Phi_k(T,h),$$

where for a real number x,

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

We also note the right side of the above formula is a polynomial of degree less than or equal to $m-1$. The right side is of degree $m-1$ for some $h \in H$ precisely when T is a strict m-isometry. To set up and motivate our result, we recall the following characterization of an m-isometry of a single operator in [30]. Although this observation seems to be a slight change of perspective to (15), it has proved to be very powerful; see several nice results following from this characterization [30]. This characterization essentially follows from the combinatorial fact below, see [30] for a reference of this fact. Let Z denote the set of integers and Z_+ denote the set of nonnegative integers.

Lemma 5.4. *Let $\{a_n\}_{n\in Z_+}$ be a sequence of real numbers. Then*

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}a_{j+k} = 0 \text{ for } j \geq 0$$

if and only if there exists a polynomial $P(x)$ of degree less than or equal to $m-1$ such that $a_n = P(n)$. In this case $P(x)$ is the unique polynomial interpolating $\{(k, a_k), 0 \leq k \leq m-1\}$.

Theorem 5.5 ([30]). *Let $T \in B(H)$. For any $h \in H$, set $a_n := \|T^n h\|^2$. Then T is an m-isometry if and only if for each $h \in H$, there exists a polynomial $P_h(x)$ of degree less than or equal to $m-1$, such that $a_n = P_h(n)$ for $n \in Z_+$.*

Proof. We include a short proof of one direction just using the definition and Lemma 5.4. If T is an m-isometry, then by definition (14) with $d = 1$,

$$\sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} \left\|T^k h\right\|^2 = 0.$$

Since h is arbitrary, for $j \geq 0$, replacing h by $T^j h$, we have

$$\sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} \left\|T^k h\right\|^2 = \sum_{k=0}^{m-1} (-1)^{m-k} \binom{m}{k} a_{j+k} = 0.$$

By Lemma 5.4, there exists a polynomial $P_h(x)$ of degree less than or equal to $m-1$ such that $a_n = P_h(n)$. For the proof of the other direction, see the proof of the next theorem. □

We now state the analogous characterization for m-isometric tuples. The proof is slightly more subtle.

Theorem 5.6. *Let $T \in B(H)^d$. Then T is an m-isometry if and only if for each $h \in H$, there exists a polynomial $P_h(x)$ of degree less than or equal to $m-1$, such that $\Psi_n(T, h) = P_h(n)$ for $n \in Z_+$.*

Proof. Assume T is an m-isometry. By (15), for each $h \in H$, there exists a polynomial $P_h(x)$ of degree less than or equal to $m-1$ such that $\Psi_n(T, h) = P_h(n)$.

Conversely, assume for each $h \in H$, there exists a polynomial $P_h(x)$ of degree less than or equal to $m-1$ such that $\Psi_n(T, h) = P_h(n)$. Then by Lemma 5.4, this $P_h(x)$ is the unique polynomial $P(x)$ interpolating

$$\{(k, \Psi_k(T, h)), 0 \leq k \leq m-1\}.$$

Now by Lemma 5.3,

$$P_h(x) = \sum_{k=0}^{m-1} \binom{x}{k} \Phi_k(T, h).$$

Therefore for $n \geq 0$,

$$\Psi_n(T, h) = P_h(n) = \sum_{k=0}^{m-1} \binom{n}{k} \Phi_k(T, h).$$

Now by Proposition 5.2 or equivalently (15), T is an m-isometry. □

One observation for a tuple of weighted shifts to be an m-isometry is that it only requires (15) to hold for basis vectors. We first introduce a tuple of d-variables unilateral weighted shifts. Let

$$Z_+^d = \{\alpha = (\alpha_1, \ldots \alpha_d) : \alpha_i \in Z_+,\ 1 \leq i \leq d\}.$$

We write $\alpha \geq 0$ if $\alpha \in Z_+^d$. Let $\varepsilon_i = (0, \ldots, 1, \ldots, 0)$ be the multi-index having 1 at i-th component and 0 elsewhere, and let 0 be the multi-index $(0, 0, , \ldots, 0)$. Let $l^2(Z_+^d)$ be the complex Hilbert space with standard bases $\{e_\alpha, \ \alpha \in Z_+^d\}$. Let $\{w_{\alpha,i}, \ \alpha \in Z_+^d, \ i = 1, \ldots, d\}$ be a bounded set of complex numbers such that

$$w_{\alpha,i} w_{\alpha+\varepsilon_i,j} = w_{\alpha,j} w_{\alpha+\varepsilon_j,i}, \alpha \in Z_+^d, 1 \leq i, j \leq d. \tag{16}$$

Definition 5.7. A tuple of d-variables unilateral weighted shifts is a family of d bounded operators on $l^2(Z_+^d), T = (T_1, \ldots, T_d)$ defined by

$$T_i e_\alpha = w_{\alpha,i} e_{\alpha+\varepsilon_i}, \alpha \in Z_+^d, \ i = 1, \ldots, d. \tag{17}$$

The condition (16) on $w_{\alpha,i}$ implies that T is a tuple of commuting operators. Note also

$$T_i^* e_\alpha = \overline{w_{\alpha-\varepsilon_i,i}} e_{\alpha-\varepsilon_i} \text{ if } \alpha_i \geq 1, \ i = 1, \ldots, d$$
$$T_i^* e_\alpha = 0 \text{ if } \alpha_i = 0, \ i = 1, \ldots, d.$$

Examples and characterizations of m-isometric weighted shifts of a single variable have been given in [6], [13], [17] and [21]. The author [24] gives a unified approach for both bilateral and unilateral shifts of a single variable on l_p spaces. Even though unilateral shifts are the focus of the most research, bilateral shifts are important because they provide examples of invertible m-isometries [17]. It seems more difficult to treat them uniformly in the case of several weighted shifts. Here we only discuss unilateral weighted shifts, and we will often say weighted shifts instead of unilateral weighted shifts.

Next we give a characterization of an m-isometric tuple of unilateral weighted shifts. This characterization is not as complete as for weighted shifts of a single variable (see [13] and [24]) because of the intrinsic freedom in tuples of weighted shifts. We assume all the weights $w_{\alpha,i} \neq 0$. By Corollary 3 in [31], in this case T is unitarily equivalent to weighted shifts with positive weights. Thus we further assume all weights $w_{\alpha,i} > 0$.

When T is a unilateral or bilateral weighted shift of a single variable, if T is an m-isometry, then all weights of T are nonzero; see Proposition 2.2 in [24]. But in the case of T being a tuple of weighted shifts, this is obviously not true. For example, if $T = (T_1, T_2) = (0, T_2)$, where T_2 is an m-isometric weighted shift, then T is an m-isometric tuple of two weighted shifts.

Lemma 5.8. *Let $T = (T_1, \ldots, T_d)$ be a tuple of unilateral weighted shifts. Then T is an m-isometry if and only if for all $n \geq m$ and $\beta \in Z_+^d$,*

$$\Psi_n(T, e_\beta) = \sum_{k=0}^{m-1} \binom{n}{k} \Phi_k(T, e_\beta). \tag{18}$$

Proof. Let $h = \sum_{\beta} c_\beta e_\beta \in l^2(Z_+^d)$. Since for any $\alpha \in Z_+^d$, T^α maps orthogonal vectors into orthogonal vectors, we have

$$\|T^\alpha h\|^2 = \left\| T^\alpha \left(\sum_\beta c_\beta e_\beta \right) \right\|^2 = \left\| \sum_\beta c_\beta T^\alpha e_\beta \right\|^2$$
$$= \sum_\beta |c_\beta|^2 \|T^\alpha e_\beta\|^2 .$$

Thus by the defining formulas (13) and (14), for all $n \geq 0$,

$$\Psi_n(T, h) = \sum_\beta |c_\beta|^2 \Psi_n(T, e_\beta),$$
$$\Phi_n(T, h) = \sum_\beta |c_\beta|^2 \Phi_n(T, e_\beta).$$

Therefore (15) holds for all $h \in l^2(Z_+^d)$ if and only if it holds for all $e_\beta, \beta \in Z_+^d$. □

Combining Theorem 5.6 and Lemma 5.8, we have the following characterization of an m-isometric tuple of weighted shifts.

Theorem 5.9. *Let $T = (T_1, \ldots, T_d)$ be a tuple of unilateral weighted shifts. Assume all the weights $w_{\alpha,i} \neq 0$. Then T is an m-isometry if and only if for all $\beta \in Z_+^d$, there exists a polynomial $P_\beta(x)$ of degree less than or equal to $m - 1$ such that*

$$P_\beta(n) = \Psi_n(T, e_\beta) := \sum_{|\alpha|=n} \binom{n}{\alpha} \|T^\alpha e_\beta\|^2 , \; n \in Z_+. \tag{19}$$

In the case of a single weighted shift T being an m-isometry ($d = 1$), since

$$\|T^\alpha e_\beta\|^2 = \frac{\left\|T^{\alpha+\beta} e_0\right\|^2}{\left\|T^\beta e_0\right\|^2},$$

only one polynomial $P_\beta(x)$ for $\beta = 0$ is needed. Furthermore this polynomial completely determines all the weights. See Theorem 3.4 in [13], Theorem 2.9 and Corollary 4.6 in [24]. Of course the polynomial $P_0(x)$ has to satisfy the positivity condition $P_0(n) > 0$ for all $n \geq 0$. Let P_+^{m-1} be the set of polynomials $P(x)$ degree $m - 1$ such that $P(0) = 1$ and $P(n) > 0$ for all $n > 0$. For a real number b, let $[b]$ denote the integer part of b. The following result from Proposition 4.8 [24] gives a description of P_+^{m-1}.

Proposition 5.10. *The polynomial $P(x) \in P_+^{m-1}$ if and only if $m - 1 = 2m_1 + 2m_2 + m_3$,*

$$P(x) = \prod_{i=1}^{m_1} (x - a_i)(x - \overline{a_i}) \prod_{i=1}^{m_2} (x - b_{2i-1})(x - b_{2i}) \prod_{i=1}^{m_3} (x + c_i) \tag{20}$$

for some complex numbers $a_i,\ i=1,\dots,m_1$, positive numbers b_{2i-1}, b_{2i} such that $[b_{2i-1}]=[b_{2i}]$, $i=1,\dots,m_2, c_i>0$, $i=1,\dots,m_2$ and all a_i, b_{2i-1}, b_{2i} are not integers, but c_i could be integers.

Theorem 5.11. *Let $T=(T_1,\dots,T_d)$ be a tuple of unilateral weighted shifts with weights*

$$w_{\alpha,i}^2=\frac{(\alpha_i+1)\,P(|\alpha|+1)}{(|\alpha|+d)\,P(|\alpha|)},\ \alpha\in Z_+^d,\ i=1,\dots,d,$$

where $P(x)\in P_+^{m-1}$, then T is a strict m-isometry.

Proof. We first verify that $w_{\alpha,i}$ satisfies (16) so that T is a tuple of commuting weighted shifts. Note that for $\alpha\in Z_+^d,\ 1\le i,j\le d$,

$$w_{\alpha,i}^2 w_{\alpha+\varepsilon_i,j}^2=\frac{(\alpha_i+1)\,P(|\alpha|+1)}{(|\alpha|+d)\,P(|\alpha|)}\frac{(\alpha_j+1)\,P(|\alpha|+2)}{(|\alpha|+1+d)\,P(|\alpha|+1)},$$

$$w_{\alpha,j}^2 w_{\alpha+\varepsilon_j,i}^2=\frac{(\alpha_j+1)\,P(|\alpha|+1)}{(|\alpha|+d)\,P(|\alpha|)}\frac{(\alpha_i+1)\,P(|\alpha|+2)}{(|\alpha|+1+d)\,P(|\alpha|+1)}.$$

Thus,

$$w_{\alpha,i}w_{\alpha+\varepsilon_i,j}=w_{\alpha,j}w_{\alpha+\varepsilon_j,i}.$$

Note that

$$\begin{aligned}
&T_1^{\alpha_1}e_0\\
&=w_{0,1}w_{\varepsilon_1,1}w_{2\varepsilon_1,1}\cdots w_{(\alpha_1-1)\varepsilon_1,1}e_{\alpha_1\varepsilon_1}\\
&=\left[\frac{1\cdot P(1)}{d\cdot P(0)}\frac{2\cdot P(2)}{(d+1)\cdot P(1)}\frac{3\cdot P(3)}{(d+2)\cdot P(2)}\cdots\frac{\alpha_1\cdot P(\alpha_1)}{(d+\alpha_1-1)\cdot P(\alpha_1-1)}\right]^{1/2}e_{\alpha_1\varepsilon_1}\\
&=\left[\frac{(d-1)!\alpha_1!P(|\alpha_1|)}{(|\alpha_1|+d-1)!P(0)}\right]^{1/2}e_{\alpha_1\varepsilon_1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
T^\alpha e_0=T_d^{\alpha_d}\cdots T_1^{\alpha_1}e_0&=\left[\frac{(d-1)!\alpha_1!P(|\alpha_1|)}{(|\alpha_1|+d-1)!P(0)}\right]^{1/2}T_d^{\alpha_d}\cdots T_2^{\alpha_2}e_{\alpha_1\varepsilon_1}\\
&=\left[\frac{(d-1)!\alpha_1!P(|\alpha_1|)}{(|\alpha_1|+d-1)!P(0)}\frac{\alpha_2!P(|\alpha_1|+|\alpha_2|)}{(|\alpha_1|+d)\cdots(|\alpha_1|+d+|\alpha_2|-1)\,P(|\alpha_1|)}\right]^{1/2}\\
&\quad\cdot T_d^{\alpha_d}\cdots T_3^{\alpha_3}e_{\alpha_1\varepsilon_1+\alpha_2\varepsilon_2}\\
&=\left[\frac{(d-1)!\alpha_1!\alpha_2!P(|\alpha_1|+|\alpha_2|)}{(|\alpha_1|+|\alpha_2|+d-1)!P(0)}\right]^{1/2}T_d^{\alpha_d}\cdots T_3^{\alpha_3}e_{\alpha_1\varepsilon_1+\alpha_2\varepsilon_2}\\
&=\left[\frac{(d-1)!\alpha_1!\alpha_2!\cdots\alpha_d!P(|\alpha|)}{(|\alpha|+d-1)!P(0)}\right]^{1/2}e_\alpha=\left[\frac{(d-1)!\alpha!P(|\alpha|)}{(|\alpha|+d-1)!P(0)}\right]^{1/2}e_\alpha.
\end{aligned}$$

Thus for any $\beta\in Z_+^d$,

(21) $\quad \Psi_n(T,e_\beta)$

$$
\begin{aligned}
&= \sum_{|\alpha|=n} \binom{n}{\alpha} \|T^\alpha e_\beta\|^2 = \sum_{|\alpha|=n} \binom{n}{\alpha} \frac{\left\|T^{\alpha+\beta} e_0\right\|^2}{\left\|T^\beta e_0\right\|^2} \\
&= \sum_{|\alpha|=n} \binom{n}{\alpha} \frac{(d-1)!(\alpha+\beta)!}{(|\alpha|+|\beta|+d-1)!} \frac{P(|\alpha|+|\beta|)}{P(0)} \frac{(|\beta|+d-1)!}{(d-1)!\beta!} \frac{P(0)}{P(|\beta|)} \\
&= \frac{P(n+|\beta|)\,(|\beta|+d-1)!}{P(|\beta|)\,(n+|\beta|+d-1)!\beta!} \sum_{|\alpha|=n} \binom{n}{\alpha} (\alpha+\beta)! \\
&= \frac{P(n+|\beta|)}{P(|\beta|)},
\end{aligned}
$$

where in the last equality we use the identity

$$\sum_{|\alpha|=n} \binom{n}{\alpha} (\alpha+\beta)! = \frac{(n+|\beta|+d-1)!\beta!}{(|\beta|+d-1)!},$$

which will be proved in the next lemma. By Theorem 5.9, T is an m-isometry. ☐

Lemma 5.12. *The following identity holds.*

$$\sum_{|\alpha|=n} \binom{n}{\alpha} (\alpha+\beta)! = \frac{(n+|\beta|+d-1)!\beta!}{(|\beta|+d-1)!} = \binom{n+|\beta|+d-1}{n} n!\beta!, \tag{22}$$

where α and β are multi-indices (or compositions) with d parts.

Proof. Looking at the left side of (22), we build a collection of objects by following these three steps:

1. Write down the numbers $1, \dots, \beta_1$ (in consecutive order), then a ",", then the numbers $\beta_1+1, \dots, \beta_1+\beta_2$, and another ",", and so on. This creates d strings of consecutive integers with strings separated by commas.

2. Place each one of the numbers $|\beta|+1, \dots, |\beta|+n+1$ at the end of any of the d consecutive strings. The choices made in this step account for the summand and the multinomial coefficient $\binom{n}{\alpha}$ in the left side of (22).

3. Permute each of the d strings. This accounts for the $(\alpha+\beta)!$ term in the left side of (22).

The number of ways to complete these three steps is the left side of (22).

Looking at the right side of (22), we can create these same objects by considering how to fill $n+|\beta|+d-1$ positions with either integers or commas. These steps give a second way to build the same objects:

1. Select n out of the total $n+|\beta|+d-1$ positions. This accounts for the binomial coefficient in the right side of (22).

2. In the selected positions, write a permutation of the n integers $|\beta|+1, \dots, |\beta|+n+1$. This accounts for the $n!$ in the right side of (22).

3. In the remaining positions, write a permutation of $1, \dots, \beta_1$, then a ",", then a permutation of $\beta_1+1, \dots, \beta_1+\beta_2$, and another ",", and so on. This accounts for the $\beta!$ in the right side of (22).

The number of ways to complete these three steps is the right side of (22).

This argument shows that the right and left sides of (22) count the same collection of objects and so the identity is proved. □

Inspired by the Berger-Shaw result [8] that the commutator of a finitely cyclic hyponormal operator is of trace class, Agler and Stankus proved in Proposition 1.24 [3] and Proposition 10.6 [5] that if T is an m-isometry on a Hilbert space for even m and T is finitely cyclic, then $\Phi_{m-1}(T)$ is a compact operator. An example of a cyclic 3-isometry T with non compact $\Phi_2(T)$ was given in [5]. For a single m-isometric weighted shift T, the author showed in Theorem 3.4 [24] that $\Phi_{m-1}(T)$ is not in the trace class when $m = 2$. However, it is in the von Neumann-Schatten class r for any $r > 1$, and $\Phi_{m-1}(T)$ is in the trace class when $m \geq 3$. We have the following analogue for tuples of weighted shifts as in Theorem 5.11. We assume $m \geq 2$ to avoid triviality.

Theorem 5.13. *Let $T = (T_1, \ldots, T_d)$ be a tuple of unilateral weighted shifts with weights*

$$w_{\alpha,i}^2 = \frac{(\alpha_i + 1)\, P(|\alpha| + 1)}{(|\alpha| + d)\, P(|\alpha|)}, \ \alpha \in Z_+^d, \ i = 1, \ldots, d, \tag{23}$$

where $P(x) \in P_+^{m-1}$. Then $\Phi_{m-1}(T)$ is a positive trace class operator if and only if $m \geq d + 2$. When $m = d + 1$, $\Phi_{m-1}(T)$ is in von Neumann-Schatten class r for any $r > 1$. When $1 \leq m \leq d$, $\Phi_{m-1}(T)$ is a compact operator.

The proof is similar to the proof of Theorem 3.4 [24], but notations are different. Slight simplifications are made here. We need two lemmas. The following lemma is known as Euler's finite difference theorem in some books and a short proof is given in Lemma 3.3 [24].

Lemma 5.14. *The following equalities hold.*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^i = 0 \ \textit{for}\ i = 0, 1, \ldots, n-1 \ \textit{and} \ \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n = n!.$$

Lemma 5.15. *Let $T = (T_1, \ldots, T_d)$ be a tuple of unilateral weighted shifts with weights as in (23). Then for all $\beta \in Z_+^d$,*

$$\Phi_{m-1}(T, e_\beta) = \frac{\Phi_{m-1}(T, e_0)}{P(|\beta|)}. \tag{24}$$

Proof. Write the polynomial $P(x)$ as

$$P(x) = b_{m-1} x^{m-1} + b_{m-2} x^{m-2} + \cdots + b_1 x + 1.$$

Then for $\beta \in Z_+^d$, polynomial $P(x + |\beta|)$ is

$$P(x + |\beta|) = a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \cdots + a_1 x + a_0,$$

where a_j are constants depending on $|\beta|$ for $0 \le j \le m-2$, but $a_{m-1} = b_{m-1}$ is a constant independent of β. By (14) and (21),

$$\begin{aligned}
\Phi_{m-1}(T, e_\beta) &= \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \Psi_k(T, e_\beta) \\
&= \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \frac{P(k+|\beta|)}{P(|\beta|)} \\
&= \frac{1}{P(|\beta|)} \sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} \left[\sum_{j=0}^{m-1} a_j k^j \right] \\
&= \frac{1}{P(|\beta|)} \sum_{j=0}^{m-1} a_j \left[\sum_{k=0}^{m-1} (-1)^{m-1-k} \binom{m-1}{k} k^j \right] = \frac{a_{m-1}(m-1)!}{P(|\beta|)},
\end{aligned}$$

where last equality follows from Lemma 5.14. By setting $\beta = 0$ and noting $P(0) = 1$, we actually have $a_{m-1}(m-1)! = \Phi_{m-1}(T, e_0)$. The proof is complete. □

Proof of Theorem 5.13. It was proved in Proposition 2.3 [22] that if S is an m-isometric tuple, then $\Phi_{m-1}(S)$ is a positive operator. By Lemma 5.15,

$$\begin{aligned}
&\sum_{\beta \ge 0} \langle \Phi_{m-1}(T) e_\beta, e_\beta \rangle \\
&= \sum_{\beta \ge 0} \Phi_{m-1}(T, e_\beta) = \sum_{\beta \ge 0} \frac{\Phi_{m-1}(T, e_0)}{P(|\beta|)} \\
&= \Phi_{m-1}(T, e_0) \sum_{|\beta|=0}^{\infty} \frac{\binom{|\beta|+d-1}{d-1}}{P(|\beta|)},
\end{aligned}$$

where, for each β, since $1/P(|\beta|)$ only depends on $|\beta|$, there are $\binom{|\beta|+d-1}{d-1}$ terms of $1/P(|\beta|)$. Since

$$\binom{|\beta|+d-1}{d-1} \approx |\beta|^{d-1} \text{ and } P(|\beta|) \approx |\beta|^{m-1},$$

the series $\sum_{\beta \ge 0} \langle \Phi_{m-1}(T) e_\beta, e_\beta \rangle$ is convergent if and only if $m \ge d+2$. The operator $\Phi_{m-1}(T)$ is a diagonal operator since

$$\langle \Phi_{m-1}(T) e_\beta, e_\gamma \rangle = 0$$

for any $\beta, \gamma \in Z_+^d$ such that $\beta \ne \gamma$. By (24),

$$\langle \Phi_{m-1}(T) e_\beta, e_\beta \rangle = \frac{\Phi_{m-1}(T, e_0)}{P(|\beta|)} \to 0 \text{ as } |\beta| \to \infty.$$

Therefore $\Phi_{m-1}(T)$ is a compact operator. □

6. Multiplication operators on weighted Hardy spaces of several variables

In this section, we reformulate results for unilateral weighted shifts of one variable or several variables as theorems for multiplication operators on weighted Hardy spaces of analytic functions of one variable or several variables. In particular, we immediately get the examples of m-isometries in Theorem 4.2 of [22] by simply choosing some special polynomials.

We first introduce the weighted Hardy spaces of d variables. The paper [39] is the classical reference for weighted Hardy spaces of one variable.

Definition 6.1. Let $\left\{v_\alpha, \alpha \in Z_+^d\right\}$ be a set of strictly positive numbers with $v_0 = 1$. Then, let

$$H^2(v) = \left\{ f(z) = \sum_{\alpha \geq 0} f_\alpha z^\alpha : \|f\|_v^2 = \sum_{\alpha \geq 0} |f_\alpha|^2 v_\alpha^2 < \infty \right\}.$$

Clearly $H^2(v)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{\alpha \geq 0} f_\alpha \overline{g_\alpha} v_\alpha^2,$$

and $\left\{z^\alpha, \alpha \in Z_+^d\right\}$ forms an orthogonal basis for $H^2(v)$ which, in general, is not orthonormal since

$$\|z^\alpha\|_v = v_\alpha.$$

Let $M_z = (M_{z_1}, \ldots, M_{z_d})$ be the tuple of the multiplication operators defined by

$$M_{z_i} f(z) = z_i f(z), \; f \in H^2(v), \; i = 1, \ldots, d.$$

Then by Proposition 8 in [31], M_z is unitarily equivalent to a tuple of d-variables unilateral weighted shifts with weights

$$\left\{ w_{\alpha,i} = \frac{v_{\alpha+\varepsilon_i}}{v_\alpha}, \alpha \in Z_+^d, i = 1, \ldots, d \right\}.$$

Conversely, a tuple of d-variables unilateral weighted shifts with weights $w_{\alpha,i}$ is unitarily equivalent to M_z on $H^2(v)$, where v_α is defined recursively by

$$v_0 = 1, \; v_{\alpha+\varepsilon_i} = w_{\alpha,i} v_\alpha, \; \alpha \geq 0, \; i = 1, \ldots, d.$$

Now Theorem 5.9 can be reformulated as the following.

Theorem 6.2. *The tuple M_z on $H^2(v)$ is an m-isometry if and only if for all $n \geq m$ and all $\beta \in Z_+^d$*

$$\sum_{|\alpha|=n} \binom{n}{\alpha} \frac{v_{\alpha+\beta}^2}{v_\beta^2} = \sum_{k=0}^{m-1} \binom{n}{k} \left[\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \sum_{|\alpha|=i} \binom{i}{\alpha} \frac{v_{\alpha+\beta}^2}{v_\beta^2} \right].$$

Equivalently, M_z is an m-isometry if and only if for each $\beta \in Z_+^d$, there exists a polynomial $P_\beta(x)$ of degree less than or equal to $m-1$, such that

$$\sum_{|\alpha|=n} \binom{n}{\alpha} \frac{v_{\alpha+\beta}^2}{v_\beta^2} = P_\beta(n) \text{ for } n \in Z_+. \tag{25}$$

Proof. We still use e_0 to denote the function of constant 1 in $H^2(v)$ and e_β to denote z^β. Note that

$$\begin{aligned}\Psi_n(M_z, e_\beta) &= \sum_{|\alpha|=n} \binom{n}{\alpha} \|M_z^\alpha e_\beta\|^2 \\ &= \sum_{|\alpha|=n} \binom{n}{\alpha} \frac{\left\|M_z^{\alpha+\beta} e_0\right\|^2}{\|M_z^\alpha e_0\|^2} = \sum_{|\alpha|=n} \binom{n}{\alpha} \frac{v_{\alpha+\beta}^2}{v_\beta^2}.\end{aligned}$$

Now (25) is (19) by using (14). □

Theorem 5.11 can be reformulated as the following.

Theorem 6.3. *Let $P(x)$ be a polynomial of degree $m-1$ such that $P(n) > 0$ for all $n \in Z_+$. Set*

$$v_\alpha^2 = \frac{(d-1)!\alpha! P(|\alpha|)}{(|\alpha|+d-1)! P(0)}, \ \alpha \in Z_+^d. \tag{26}$$

Then M_z on $H^2(v)$ is a strict m-isometry.

Proof. Let $T = (T_1, \dots, T_d)$ be the tuple of unilateral weighted shifts with weights

$$w_{\alpha,i}^2 = \frac{(\alpha_i + 1)\, P(|\alpha|+1)}{(|\alpha|+d)\, P(|\alpha|)}, \ \alpha \in Z_+^d, \ i = 1, \dots, d,$$

where $P(x)$ is a polynomial of degree $m-1$. Then by using $v_0 = 1$ and $v_{\alpha+\varepsilon_i} = w_{\alpha,i} v_\alpha$ recursively, we have the formula for v_α^2. □

When $d = 1$, the above two theorems are the same. See Theorem 3.4 in [13], Theorem 2.9 and Corollary 4.6 in [24].

Theorem 6.4. *Let $d = 1$. Then M_z is an m-isometry on $H^2(v)$ if and only if for all $n \geq m$,*

$$v_n^2 = \sum_{k=0}^{m-1} \binom{n}{k} \left[\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} v_i^2 \right].$$

Equivalently, M_z is a strict m-isometry if and only if there exists a polynomial $P(x)$ of degree equal to $m-1$ such that

$$v_n^2 = \frac{P(n)}{P(0)} \text{ for } n \in Z_+.$$

The above theorem says if M_z is an m-isometry on $H^2(v)$, then $v_1, v_2, \ldots, v_{m-1}$ completely determine all weights v_n. In the weighted shift case, $w_0, w_1, \ldots, w_{m-2}$ completely determine all weights w_n; see Section 4 in [24] for some discussions on which weights $w_0, w_1, \ldots, w_{m-2}$ are possible to generate an m-isometric unilateral weighted shift.

When $d = 1$ and $v_n^2 = n + 1$, then $H^2(v)$ is the Dirichlet space. The fact that Dirichlet shift is an 2-isometry plays an important role in studying the invariant subspaces of Dirichlet shift, see [36] and [37] for details.

The weighted Hardy space $H^2(v)$ is also a reproducing kernel Hilbert space with reproducing kernel $K(z, w)$ defined by

$$K(z,w) = \sum_{\alpha \geq 0} \frac{z^\alpha \overline{w}^\alpha}{v_\alpha^2}.$$

The reproducing property of $K(z, w)$ is

$$\langle f(z), K(z,w) \rangle = f(w) \text{ for } f \in H^2(v).$$

For v_α as in (26),

$$\begin{aligned} K(z,w) &= \sum_{\alpha \geq 0} \frac{z^\alpha \overline{w}^\alpha}{v_\alpha^2} = \sum_{\alpha \geq 0} \frac{(|\alpha| + d - 1)! P(0) z^\alpha \overline{w}^\alpha}{(d-1)! \alpha! P(|\alpha|)} \\ &= \sum_{n=0}^{\infty} \frac{(n+d-1)! P(0)}{(d-1)! P(n)} \sum_{|\alpha| = n} \frac{z^\alpha \overline{w}^\alpha}{\alpha!} \\ &= \sum_{n=0}^{\infty} \frac{(n+d-1)! P(0)}{(d-1)! n! P(n)} \sum_{|\alpha| = n} \frac{n! z^\alpha \overline{w}^\alpha}{\alpha!} \\ &= \sum_{n=0}^{\infty} \frac{(n+d-1)! P(0)}{(d-1)! n! P(n)} \left[z \cdot \overline{w} \right]^n . \end{aligned} \tag{27}$$

If $d \geq m$, let $a = d - m + 1 \geq 1$. If we set

$$P(n) = (n + d - 1) \cdots (n + d - m + 1), \tag{28}$$

then the degree of $P(x)$ is $m - 1$ (if $m = 1$, $P(n) = P(0) = 1$), and

$$\begin{aligned} K(z,w) &= \sum_{n=0}^{\infty} \frac{(n+d-1)! P(0)}{(d-1)! n! (n+d-1) \cdots (n+d-m+1)} \left[z \cdot \overline{w} \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(n+d-1)! (d-1) \cdots (d-m+1)}{(d-1)! n! (n+d-1) \cdots (n+d-m+1)} \left[z \cdot \overline{w} \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(n+a-1)!}{(a-1)! n!} \left[z \cdot \overline{w} \right]^n = \frac{1}{\left(1 - z \cdot \overline{w}\right)^a}. \end{aligned}$$

Therefore if $v_n^2 = P(n)/P(0)$, where $P(n)$ is given by (28), then $H^2(v)$ is the space $\mathcal{K}_{a,d}$ (for two positive integers a and d), which is the space of analytic functions on the ball B^d with reproducing kernel

$$K(z,w) = \frac{1}{(1 - z \cdot \overline{w})^a}.$$

It is also clear from the above discussion that if $d < a$, we can not find a polynomial $P(x)$ such that $K(z,w)$ given in (27) is of the form $1/(1 - z \cdot \overline{w})^a$ for some positive integer a. We have the following result which is contained in Theorem 4.2 [22].

Corollary 6.5 ([22]). *The tuple M_z on $\mathcal{K}_{a,d}$ is a strict m-isometry for some $m \geq 1$ if and only if $d \geq a$. In this case $m = d - a + 1$.*

In Theorem 4.2 [22], $d \geq a$ is assumed and the strictness is not stated.

Since $m = d - a + 1 \geq d + 1$ is impossible, Theorem 5.13 tells us $\Phi_{m-1}(M_z)$ on $\mathcal{K}_{a,d}$ is a compact operator, but not in von Neumann-Schatten class r for any $r \geq 1$. Here we assume $m \geq 2$ to avoid triviality.

The space $\mathcal{K}_{1,d}$ is denoted by H^2_d, which is now called Drury-Arveson space, and M_z on H^2_d has played a role in the dilation theory of row contractions.

It is natural to ask if M_z on the Dirichlet space of d-variables is an m-isometry. Let $\mathcal{D}_d$ denote the holomorphic Dirichlet space on B^d with reproducing kernel

$$K(z,w) = -\frac{1}{z \cdot \overline{w}} \ln(1 - z \cdot \overline{w}).$$

It is quite surprising that we can again apply Theorem 5.11. In fact, this time we will use the equivalent Theorem 6.3. Note that

$$\begin{aligned} K(z,w) &= -\frac{1}{z \cdot \overline{w}} \ln(1 - z \cdot \overline{w}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} [z \cdot \overline{w}]^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{|\alpha|=n} \frac{n! z^\alpha \overline{w}^\alpha}{\alpha!} \\ &= \sum_{\alpha \geq 0} \frac{|\alpha|! z^\alpha \overline{w}^\alpha}{(|\alpha|+1)\, \alpha!} = \sum_{\alpha \geq 0} \frac{z^\alpha \overline{w}^\alpha}{v_\alpha^2}, \end{aligned}$$

where

$$v_\alpha^2 = \frac{\alpha!\,(|\alpha|+1)}{|\alpha|!} = \frac{(d-1)!\alpha! P(|\alpha|)}{(|\alpha|+d-1)! P(0)}$$

$$\text{for } P(|\alpha|) = (|\alpha|+1)\,[(|\alpha|+d-1) \cdots (|\alpha|+2)\,(|\alpha|+1)]\,.$$

The degree of $P(x)$ above is d.

Corollary 6.6. *The tuple M_z on the Dirichlet space $\mathcal{D}_d$ is a strict $(d+1)$-isometry.*

By Theorem 5.13, $\Phi_d(M_z)$ on $\mathcal{D}_d$ is in von Neumann-Schatten class r for any $r > 1$.

We remark that it comes to our attention the more general Banach space version of Theorem 5.6 is Theorem 3.3 in a by Hoffman and Mackey [29]. Our treatment (on a Hilbert space) seems to be more direct and concise.

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