J. Korean Math. Soc. **55** (2018), No. 1, pp. 211–224
https://doi.org/10.4134/JKMS.j170181
pISSN: 0304-9914 / eISSN: 2234-3008

GLOBAL EXISTENCE FOR A PARTIALLY LINEAR 3D EULER FLOW

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Abstract. We consider a certain three dimensional Euler flow with infinite energy, which is sometimes called the columnar or two and half dimensional flow. We prove the global smoothness of such flow in $\mathbb{R}^3$ when the initial data is in some Sobolev or Besov spaces and $\partial_3 u_3$ is nonnegative.

1. Introduction

We study the three dimensional incompressible Euler equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0 \tag{1}$$

in $\mathbb{R}^3$ with the assumption

$$u_1 = u_1(x_1, x_2, t),\ u_2 = u_2(x_1, x_2, t),\ u_3 = x_3\gamma(x_1, x_2, t) + \varphi(x_1, x_2, t) \tag{2}$$

for some scalar fields $\gamma, \varphi : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$. Here, u is the velocity field, and p is the pressure.

It is well known that (1) is well-posed in various spaces of fields with finite energy (See [3], [10], and [11]). However, it remains open whether an energy finite smooth solution of the 3D Euler equations for a given smooth initial data exists globally or blows up in a finite time. Solutions of the form (2) have infinite energy and, therefore, must be considered as a different one from those of finite energy. However, they still satisfy the 3D Euler equations and hence may enhance our understanding of the local behavior of the Euler flow with finite energy.

The flows of the type (2) have been first studied in [14] when $\gamma = 0$ and in [7] when γ depends only on time $\gamma = \gamma(t)$ for both the Navier-Stokes and the Euler equations. It is known that there exists a finite time blow-up solution for (1)-(2) both in $\mathbb{R}^2$ and in a flat torus ([4], [8]). Also, the global existence and blow-up of the flow (2) for the Navier-Stokes equations is known when the

Received March 9, 2017; Revised June 5, 2017; Accepted October 25, 2017.
2010 *Mathematics Subject Classification.* Primary 35A01, 35Q35, 35M99.
Key words and phrases. columnar flow, incompressible Euler equations, Besov spaces.
This work was financially supported by Samsung, SSTF-BA1502-02.

initial data is in $W^{1,p}$ over the whole domain ([12]). In this paper, we consider the global existence of the smooth solution of (1) of the type (2) under an additional condition. That is the nonnegativeness of γ_0.

Plugging (2) into (1), one can derive

$$-\partial_3 p = c(t)x_3 + d(t) \tag{3}$$

for some $c(t)$ and $d(t)$ depending only on t. And, at the same time, one can reduce (1) to

$$\begin{aligned} \frac{\partial \widetilde{u}}{\partial t} + \widetilde{u}\cdot\nabla\widetilde{u} + \nabla p &= 0,\\ \frac{\partial \gamma}{\partial t} + \widetilde{u}\cdot\nabla\gamma + \gamma^2 &= c(t),\\ \frac{\partial \varphi}{\partial t} + \widetilde{u}\cdot\nabla\varphi + \varphi\gamma &= d(t),\\ \nabla\cdot\widetilde{u} &= -\gamma, \end{aligned} \tag{4}$$

where $\widetilde{u} \equiv (u_1, u_2)$.

It is not easy to control the pressure in the formulation (4), due to the lack of the divergence free condition. Therefore, we will consider a system of the derivatives of the velocity instead of (4) and analyze it. We assume $c(t) = 0$ and $d(t) = 0$. This assumption is reasonable. In fact, it was proved rigorously for the same type of the Navier-Stokes flow in the Sobolev space setting in [12]. We remark that $B^s_{2,2}(\mathbb{R}^2) = H^s(\mathbb{R}^2)$. Hence, our result implies local and global existence in Sobolev spaces.

From now on, we present a more favorable formulation of (4). The vorticity of the flow (2) is given by $(x_3\gamma_2 + \varphi_2, -x_3\gamma_1 - \varphi_1, u_{2,1} - u_{1,2})$ and if we denote the third component of the vorticity by ω, then ω satisfies [16]:

$$\frac{\partial \omega}{\partial t} + \widetilde{u}\cdot\nabla\omega = \gamma\omega. \tag{5}$$

Using Helmholtz-Hodge decomposition and the relations

$$\begin{aligned} \nabla\cdot\widetilde{u} &= -\gamma,\\ \nabla\times\widetilde{u} &= \omega, \end{aligned}$$

$\widetilde{u}$ can be expressed formally by

$$\widetilde{u} = -\nabla(\Delta^{-1}\gamma) + \nabla^{\perp}(\Delta^{-1}\omega), \tag{6}$$

with $\nabla^{\perp} = (-\partial_2, \partial_1)$.

Since we are considering the Besov spaces $B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}$ for $s > 0$, γ and ω may decay slowly near infinity, hence the exact meaning of (6) is as follows:

(1) If $1 < p < 2$, then, up to a harmonic function,

$$\widetilde{u} = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)}{|x-y|^2}\gamma(y)dy + \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^{\perp}}{|x-y|^2}\omega(y)dy. \tag{7}$$

In this case, by the Sobolev embedding and the property of multiplier operator for homogeneous Besov spaces, it yields

$$\widetilde{u} \in L^{\frac{2p}{2-p}} \cap \dot{B}^{s+1}_{p,q}. \tag{8}$$

(2) If $2 \le p < \infty$, then we define

$$\begin{aligned}\widetilde{u} = \widetilde{u}(0) &- \frac{1}{2\pi}\int_{\mathbb{R}^2}\left\{\frac{(x-y)}{|x-y|^2} + \frac{y}{|y|^2}\right\}\gamma(y)dy \\ &+\frac{1}{2\pi}\int_{\mathbb{R}^2}\left\{\frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2}\right\}\omega(y)dy.\end{aligned} \tag{9}$$

Note that the above integrals are well defined since the contribution near infinity cancels out by the mean value theorem. And by the Sobolev embeddings, we have

$$\widetilde{u} \in \begin{cases} BMO, & \text{if } p = 2, \\ C^{0,1-\frac{2}{p}} = B^{1-\frac{2}{p}}_{\infty,\infty}, & \text{if } 2 < p < \infty, \end{cases} \tag{10}$$

where BMO is the space of functions of bounded mean oscillations and $C^{0,1-\frac{2}{p}}$ is the space of Hölder continuous functions with the exponent $1 - \frac{2}{p}$.

Then, we can reformulate (4) as the following closed system:

$$\begin{aligned}\frac{\partial \omega}{\partial t} + \widetilde{u}\cdot\nabla\omega - \gamma\omega = 0,\\ \frac{\partial \gamma}{\partial t} + \widetilde{u}\cdot\nabla\gamma + \gamma^2 = 0,\\ \frac{\partial \varphi}{\partial t} + \widetilde{u}\cdot\nabla\varphi + \gamma\varphi = 0,\end{aligned} \tag{11}$$

with $\widetilde{u}$ given by (7) and (9) and with the initial data $(\omega(0,x), \gamma(0,x), \varphi(0,x)) = (\omega_0, \gamma_0, \varphi_0)$. From now on, we will interpret the formula (6) by (7) and (9) in the subsequent sections.

Now we are ready to give the main results of this paper.

Theorem 1. *Let* $p \in (1,\infty)$, $q \in [1,\infty]$, $s > 0$ *and* (s,p,q) *satisfy*

$$\begin{aligned} s > \frac{2}{p},\ \text{if } q < \infty, \\ s \ge \frac{2}{p},\ \text{if } q = 1. \end{aligned} \tag{12}$$

If $\omega_0, \gamma_0, \varphi_0 \in B^s_{p,q}$, *there exists a time* T *such that* (11) *has a distributional solution* ω, γ, φ *in* $L^\infty([0,T); B^s_{p,q})$. *Moreover, if*

$$\begin{aligned} s > 1 + \frac{2}{p},\ \text{if } q < \infty, \\ s \ge 1 + \frac{2}{p},\ \text{if } q = 1, \end{aligned} \tag{13}$$

then the solution is unique and $\omega, \gamma, \varphi \in L^\infty([0,T); B^s_{p,q}) \cap Lip([0,T); B^{s-1}_{p,q})$.

Remark 1. For any $1 < p < \infty$, the condition $\omega, \gamma \in B^s_{p,q}$ with (12) imply $\widetilde{u} \in C^{0,\sigma}$ for some $\sigma \in (0,1)$. Indeed, since $\omega, \gamma \in L^\infty \cap L^p$ by $B^s_{p,q} \hookrightarrow L^p \cap L^\infty$, we have $\omega, \gamma \in L^{p'}$ for some $2 < p' < \infty$. Then, by the Calderon-Zygmund inequality and the Sobolev inequality,

$$\widetilde{u} \in C^{0,\sigma}, \text{ with } \sigma = 1 - \frac{2}{p'}. \tag{14}$$

We will use this information to estimate the convective term.

Theorem 2. *Let* $\omega_0, \gamma_0, \varphi_0 \in B^s_{p,q}$ *and* $s > \frac{2}{p}$. *If* $\gamma_0 \geq 0$, *then there exists a global solution* ω, γ, φ *in* $L^\infty([0,\infty); B^s_{p,q})$. *Furthermore, it is unique if* (13) *is satisfied.*

In the sequel, for the sake of convenience, we use the notations $\|f\|_p = \|f\|_{L^p(\mathbb{R}^2)}$, $\theta = (\omega, \gamma, \varphi)$ and define

$$\|\theta\|_p = \|\omega\|_p + \|\gamma\|_p + \|\varphi\|_p,$$
$$\|\theta\|_{B^s_{p,q}} = \|\omega\|_{B^s_{p,q}} + \|\gamma\|_{B^s_{p,q}} + \|\varphi\|_{B^s_{p,q}}.$$

Also, all the constants will be denoted by the same C.

The outline of our paper is as follows. In Section 2, we collect the basic tools and the preliminary information. In Section 3, we give the local existence result. In Section 4, we establish the global existence result based on the nonnegativeness of γ_0 and the logarithmic inequality for Besov spaces.

2. Preliminaries

In this section, we give the essential tools for the main results. For more detailed information for Besov spaces and their properties, we refer to [1], [2], [5] and [13].

Fix a smooth bump function χ with $supp(\chi) \subset \{\xi \in \mathbb{R}^2 : |\xi| < 2\}$ and $\chi \equiv 1$ on $\{\xi \in \mathbb{R}^2 : |\xi| < 1/2\}$ and set $\varphi(\xi) = \chi(\xi/2) - \chi(\xi)$ so that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \text{ and } \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \text{ if } \xi \neq 0.$$

The nonhomogeneous dyadic blocks Δ_j are defined as

$$\Delta_j f(x) := 0 \text{ if } j \leq -2,$$

$$\Delta_{-1} f(x) := \chi(D) f = \mathcal{F}^{-1}(\chi(\xi)\widehat{f}(\xi)),$$
$$\Delta_j f(x) := \varphi(2^{-j}D) f = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\widehat{f}(\xi)) \text{ if } j \geq 0,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

The low frequency cut-off operator S_j is defined as

$$S_j f(x) = \sum_{i \leq j-1} \Delta_i f(x).$$

The Bony decomposition for the product is

$$fg = T_f g + T_g f + R(f,g),$$

where the paraproduct $T_f g$ and remainder $R(f,g)$ are defined respectively by

$$T_f g = \sum_j S_{j-1} f \Delta_j g, \;\; R(f,g) = \sum_{|i-j|\le 1} \Delta_i f \Delta_j g.$$

Let $s \in \mathbb{R}$ and $p,q,r \in [1;\infty]$. For $f \in S'(\mathbb{R}\times\mathbb{R}^2)$, the nonhomogeneous Besov spaces $B^s_{p,q}$ and $L^r_t B^s_{p,q}$ are defined by the following norms:

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j\ge -1} 2^{jsq}\|\Delta_j f\|^q_{L^p}\Big)^{1/q},$$

$$\|f\|_{L^r_t B^s_{p,q}} = \Big\|\Big(\sum_{j\ge -1} 2^{jsq}\|\Delta_j f\|^q_{L^p}\Big)^{1/q}\Big\|_{L^r(0,t)},$$

with the usual convention when $q=\infty$.

Lemma 1 (Bernstein, [1], Lemma 2.1)**.** *Let $1\le p\le q\le\infty$. Then*

$$supp\hat{f} \subset \{|\xi|\le C2^j\} \Rightarrow \|\partial^\alpha f\|_q \le C 2^{j|\alpha|+2j(1/p-1/q)}\|f\|_p,$$
$$supp\hat{f} \subset \{2^j/C \le |\xi|\le C2^j\} \Rightarrow \|f\|_p \le C 2^{-j|\alpha|}\|\partial^\alpha f\|_p.$$

Next, we recall an L^p estimate for the solution of the following general scalar transport equation:

$$\frac{\partial f}{\partial t} + v\cdot\nabla f = g, \qquad f|_{t=0} = f_0. \tag{15}$$

Proposition 1 ([5])**.** *Let $\nabla v \in L^\infty([0,T]\times\mathbb{R}^2)$. For $r\in[1,\infty]$, the solution of* (15) *satisfies*:

$$\|f(t)\|_r \le e^{\frac{1}{r}\int_0^t \|\nabla\cdot v\|_\infty d\tau}\Big(\|f_0\|_r + \int_0^t e^{-\frac{1}{r}\int_0^\tau \|\nabla\cdot v\|_\infty d\tau'}\|g(\tau)\|_r d\tau\Big), \tag{16}$$

where it is understood that $\frac{1}{r}=0$ if $r=\infty$.

We give a version of the commutator estimate in the Besov spaces, which is not included in Lemma 2.100 in [1].

Proposition 2. *Let $s>0$ and $p,q\in[1,\infty]$. For the commutator*

$$R_j = [v\cdot\nabla, \Delta_j]f = v\cdot\nabla\Delta_j f - \Delta_j(v\cdot\nabla f), \tag{17}$$

the estimate

$$\|\left(2^{js}\|R_j\|_p\right)_j\|_{l^q} \le C\left(\|\nabla v\|_\infty \|f\|_{B^s_{p,q}} + \|f\|_\infty\|\nabla v\|_{B^s_{p,q}}\right) \tag{18}$$

is valid.

Proof. First, after splitting v into low and high frequencies by $v = S_0 v + \widetilde{v}$, we have

$$R_j = v\cdot\nabla\Delta_j f - \Delta_j(v\cdot\nabla f) = [\widetilde{v}^k, \Delta_j]\partial_k f + [S_0 v^k, \Delta_j]\partial_k f. \tag{19}$$

Now, if we write the Bony's decomposition for the term $[\widetilde{v}^k, \Delta_j]\partial_k f$, we get $R_j = \sum_{i=1}^{8} R_j^i$ with

$$\begin{array}{llllll} R_j^1 &=& [T_{\widetilde{v}^k}, \Delta_j]\partial_k f, & R_j^2 &=& T_{\partial_k \Delta_j f}\widetilde{v}^k, \\ R_j^3 &=& -\Delta_j T_{\partial_k f}\widetilde{v}^k, & R_j^4 &=& \partial_k R(\widetilde{v}^k, \Delta_j f), \\ R_j^5 &=& -R(\nabla\cdot\widetilde{v}, \Delta_j f), & R_j^6 &=& -\partial_k\Delta_j R(\widetilde{v}^k, f), \\ R_j^7 &=& \Delta_j R(\nabla\cdot\widetilde{v}, f), & R_j^8 &=& [S_0 v^k, \Delta_j]\partial_k f. \end{array} \tag{20}$$

The terms $R_j^1,\ R_j^2\ ,R_j^4,\ R_j^5,\ R_j^6,\ R_j^7$ and R_j^8 are estimated as in the proof of Lemma 2.100 in [1]:

$$\begin{aligned} 2^{js}\|R_j^1\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \\ 2^{js}\|R_j^2\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \\ 2^{js}\|R_j^4\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \\ 2^{js}\|R_j^5\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \\ 2^{js}\|R_j^6\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \\ 2^{js}\|R_j^7\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \\ 2^{js}\|R_j^8\|_p &\le Cc_j\|\nabla v\|_\infty\|f\|_{B^s_{p,q}}, \end{aligned} \tag{21}$$

where $(c_j)_{j\ge -1}$ is a sequence such that $\|(c_j)\|_{l^q} \le 1$.

For the term R_j^3, we have

$$R_j^3 = -\sum_{|j'-j|\le 4} \Delta_j(S_{j'-1}\partial_k f \Delta_{j'}\widetilde{v}^k). \tag{22}$$

By Bernsteins' lemma,

$$2^{js}\|R_j^3\|_p \le C\sum_{|j'-j|\le 4} \|S_{j'-1}f\|_\infty 2^{j's}\|\Delta_{j'}\nabla\widetilde{v}\|_p, \tag{23}$$

which gives

$$2^{js}\|R_j^3\|_p \le Cc_j\|f\|_\infty\|\nabla v\|_{B^s_{p,q}}. \tag{24}$$

Combining the inequalities (21) and (24) yields (18). □

Proposition 3 (Corollary 2.86 in [1])**.** *Let $s > 0$ and $f, g \in L^\infty \cap B^s_{p,q}$ for $p, q \in [1,\infty]$. There exists a constant C such that the following inequality is true.*

$$\|fg\|_{B^s_{p,q}} \le C(\|f\|_\infty\|g\|_{B^s_{p,q}} + \|g\|_\infty\|f\|_{B^s_{p,q}}). \tag{25}$$

We provide a product estimate involving a derivative and recall the logarithmic inequality in the Besov spaces.

Proposition 4. *For $s, \sigma > 0$, $f \in B^{\sigma}_{\infty,\infty} \cap B^{s+1}_{p,q}$, $\nabla f \in L^\infty$ and $g \in L^\infty \cap B^{s+1}_{p,q}$, the inequality*

$$\|f\nabla g\|_{B^s_{p,q}} \le C(\|f\|_{B^{\sigma}_{\infty,\infty}}\|\nabla g\|_{B^s_{p,q}} + \|\nabla f\|_\infty \|g\|_{B^s_{p,q}} + \|\nabla f\|_{B^s_{p,q}}\|g\|_\infty) \tag{26}$$

is valid.

Proof. Using Theorem 2.82 and Theorem 2.85 in [1] in conjunction with Bernstein's lemma, we obtain

$$\begin{aligned} \|T_{\partial_k g} f\|_{B^s_{p,q}} &\le C\|g\|_{L^\infty}\|\nabla f\|_{B^s_{p,q}}, \\ \|R(f, \partial_k g)\|_{B^s_{p,q}} &\le C\|\nabla f\|_{B^0_{\infty,\infty}}\|g\|_{B^s_{p,q}}. \end{aligned}$$

The remaining paraproduct is estimated as follows:

$$\begin{aligned} \|T_f \partial_k g\|_{B^s_{p,q}} &\le C\|2^{js}\|S_{j-1} f \Delta_j(\partial_k g)\|_p\|_{l^q} \\ &\le C \sup_{1\le j}\|S_{j-1} f\|_\infty \|\partial_k g\|_{B^s_{p,q}} \\ &\le C\|\partial_k g\|_{B^s_{p,q}} \sup_{1\le j} \sum_{i \le j-2} 2^{-\sigma i} 2^{\sigma i}\|\Delta_i f\|_\infty \\ &\le C\|\partial_k g\|_{B^s_{p,q}}\|f\|_{B^\sigma_{\infty,\infty}} \sup_{1\le j}\sum_{i\le j-2} 2^{-\sigma i} \\ &\le C\|\nabla g\|_{B^s_{p,q}}\|f\|_{B^\sigma_{\infty,\infty}}. \end{aligned}$$

Since $L^\infty \hookrightarrow B^0_{\infty,\infty}$, the desired inequality follows. ☐

Proposition 5 (Proposition 1.1 in [3])**.** *Let $s > \frac{2}{p}$, $p \in (1, \infty)$ and $q \in [1, \infty]$. There exists a constant C such that the following inequality is true.*

$$\|f\|_\infty \le C(1 + \|f\|_{\dot{B}^s_{\infty,\infty}}(\log^+ \|f\|_{B^s_{p,q}} + 1)). \tag{27}$$

3. The proof of Theorem 1.

We will prove the theorem by the Galerkin argument. Let us assume the condition (12) is satisfied.

We start by recalling the orthonormal wavelet basis and multiresolution approximation. We refer to [6], [9] and [15] for more details of the characterization of functions spaces and the projection kernel properties. For some integer $r > s$, there are real valued compactly supported functions

$$\psi_0(x) \in C^r(\mathbb{R}^2) \text{ and } \psi^l(x) \in C^r(\mathbb{R}^2),\ l = 1, 2, 3, \tag{28}$$

with

$$\int_{\mathbb{R}^2} x^\alpha \psi^l(x) dx = 0, |\alpha| \le r, \tag{29}$$

such that

$$\{2^k \psi^l_{km}(x) : k \in \mathbb{N}_0, 1 \le l \le 3, m \in \mathbb{Z}^2\}, \tag{30}$$

with

$$\psi^l_{km}(x) = \begin{cases} \psi_0(x-m), & \text{if } k=0, l=1, \\ \psi^l(2^{k-1}x-m), & \text{if } k\in\mathbb{N}, 1\le l\le 3, \end{cases} \tag{31}$$

is an orthonormal basis in $L^2(\mathbb{R}^2)$ (see [17]).

Let $\{V_k,\ k\in\mathbb{Z}\}$ be r-regular multiresolution approximation of $L^2(\mathbb{R}^2)$ and the space V_k is spanned by the orthonormal wavelet basis $\{2^k\psi^l_{km}(x),\ m\in\mathbb{Z}^2,\ 0\le l\le 3\}$, and $P_k: L^2(\mathbb{R}^2)\to V_k$ denote the orthogonal projection. The integral kernel of the operator P_k is given by

$$P_k(x,y) = 2^{2k}\sum_{m\in\mathbb{Z}^2, 0\le l\le 3}\psi^l_{km}(x)\psi^l_{km}(y). \tag{32}$$

Let us consider the following approximated equation in the space V_k:

$$\begin{aligned} &\frac{\partial\omega^k}{\partial t} + P_k(\widetilde{u}^k\cdot\nabla\omega^k) - P_k(\gamma^k\omega^k) = 0, \\ &\frac{\partial\gamma^k}{\partial t} + P_k(\widetilde{u}^k\cdot\nabla\gamma^k) + P_k(\gamma^k\gamma^k) = 0, \\ &\frac{\partial\varphi^k}{\partial t} + P_k(\widetilde{u}^k\cdot\nabla\varphi^k) + P_k(\gamma^k\varphi^k) = 0, \end{aligned} \tag{33}$$

where ω^k, γ^k and φ^k belongs to V_k and $\widetilde{u}^k$ is defined by (6).

The system (33) is the system of ODEs and it can be solved by the standard methods with the initial data $(\omega^k_0, \gamma^k_0, \varphi^k_0) = (P_k\omega_0, P_k\gamma_0, P_k\varphi_0)$.

Next, we prove the approximate solutions $(\omega^k, \gamma^k, \varphi^k)$ are uniformly bounded in the space $B^s_{p,q}$. To do so, we apply the spectral localization operator Δ_j to the first equation in (33) and obtain

$$\frac{\partial\Delta_j\omega^k}{\partial t} + \widetilde{u}^k\cdot\nabla\Delta_j\omega^k = \Delta_j P_k(\gamma^k\omega^k) + R_j, \tag{34}$$

with $R_j = \widetilde{u}^k\cdot\nabla\Delta_j\omega^k - \Delta_j P_k(\widetilde{u}^k\cdot\nabla\omega^k)$. By multiplying the both sides of (34) by $\operatorname{sgn}(\Delta_j\omega^k)|\Delta_j\omega^k|^{p-1}$, integrating over $[0,t]\times\mathbb{R}^2$ and using integration by parts, we get

$$\begin{aligned} \|\Delta_j\omega^k(t)\|_p &\le \|\Delta_j\omega^k_0\|_p + \int_0^t \|\Delta_j P_k(\gamma^k(\tau)\omega^k(\tau))\|_p d\tau \\ &\quad + \int_0^t\left(\|R_j(\tau)\|_p + \frac{1}{p}\|\nabla\cdot\widetilde{u}^k(\tau)\|_\infty\|\Delta_j\omega^k(\tau)\|_p\right)d\tau. \end{aligned} \tag{35}$$

Since we are using real wavelet bases, the spectral localization operator Δ_j and projection operator P_k commute:

$$\Delta_j P_k f = P_k\Delta_j f. \tag{36}$$

Moreover, from the property of the wavelet basis, we find that the operator P_k has the L^p boundedness (Calderon-Zygmund operator):

$$\|P_k f\|_p \le C_0\|f\|_p \text{ for } 1<p<\infty, \tag{37}$$

and the constant C_0 is independent of k. Now, multiplying (35) by 2^{js} and taking l^q norm, it yields

(38)
$$\begin{aligned}&\|\omega^k(t)\|_{B^s_{p,q}}\\ &\le \|\omega_0^k\|_{B^s_{p,q}} + C\int_0^t \Big(\|\gamma^k\omega^k\|_{B^s_{p,q}} + \|\nabla\widetilde{u}^k\|_\infty\|\omega^k\|_{B^s_{p,q}} + \|\nabla\widetilde{u}^k\|_{B^s_{p,q}}\|\omega^k\|_\infty\Big)\,d\tau,\end{aligned}$$

where we used the properties (36) and (37) for estimating R_j as in the proof of Proposition 2. Repeating the same argument to the second and third equation in (33), and using the product estimate and the embedding $B^s_{p,q} \hookrightarrow L^\infty$, in particular, we obtain

$$\|\theta^k(t)\|_{B^s_{p,q}} \le \|\theta_0^k\|_{B^s_{p,q}} + C\int_0^t \Big(\|\theta^k(\tau)\|_{B^s_{p,q}}\Big)^2\,d\tau. \tag{39}$$

Fix some T such that $T < \frac{1}{C\|\theta_0\|_{B^s_{p,q}}}$ and note that we have $\|\theta_0^k\|_{B^s_{p,q}} \le \|\theta_0\|_{B^s_{p,q}}$ from the theory of characterization of Besov spaces in terms of the wavelet expansions. By the Grönwall type inequality, from (39), we readily get

$$\|\theta^k(t)\|_{B^s_{p,q}} \le \frac{\|\theta_0\|_{B^s_{p,q}}}{1 - CT\|\theta_0\|_{B^s_{p,q}}} \quad \text{for all } t < T. \tag{40}$$

This means that our approximated solutions $(\omega^k, \gamma^k, \varphi^k)$ are uniformly bounded in the space $L^\infty([0;T); B^s_{p,q})$.

Now, by the Theorem 2.94 in [1], as the embedding $B^s_{p,q} \hookrightarrow B^{s-1}_{p,q}$ is locally compact, the Arzelà-Ascoli theorem and Cantor's diagonal process give us, up to a subsequence, the sequence $\{\theta^k\}$ tends to a limit θ in $L^\infty([0;T); (B^{s-1}_{p,q})_{loc})$. And, by Fatou's property, the limit θ belongs to $L^\infty([0;T); B^s_{p,q})$. By interpolation, the convergence of $\psi\theta^k$ to $\psi\theta$ holds in $L^\infty([0;T); B^{s'}_{p,q})$ for any $s' < s$ and $\psi \in C_0^\infty(\mathbb{R}^2)$. This tells us that $\theta = (\omega, \gamma, \varphi)$ solves the equation (11) in a distributional sense.

Next, we show that if we impose more regularity on the initial data, i.e., the condition (13) is satisfied, then the solution is unique and

$$\omega, \gamma, \varphi \in L^\infty([0,T); B^s_{p,q}) \cap Lip([0,T); B^{s-1}_{p,q})$$

for some T.

Suppose that there are two solutions $\theta = (\omega, \gamma, \varphi)$ and $\theta' = (\omega', \gamma', \varphi')$ with the same initial data and (13) is satisfied. Let us introduce the notations

$$\begin{aligned}\delta\omega &= \omega - \omega',\\ \delta\gamma &= \gamma - \gamma',\\ \delta\varphi &= \varphi - \varphi',\\ \delta\widetilde{u} &= \widetilde{u} - \widetilde{u}'.\end{aligned} \tag{41}$$

Subtracting the equations of θ' from the equations of θ, we have

$$\begin{aligned}
&\frac{\partial\delta\omega}{\partial t}+\widetilde{u}\cdot\nabla\delta\omega+\delta\widetilde{u}\cdot\nabla\omega'-\gamma\delta\omega-\omega'\delta\gamma=0,\\
&\frac{\partial\delta\gamma}{\partial t}+\widetilde{u}\cdot\nabla\delta\gamma+\delta\widetilde{u}\cdot\nabla\gamma'+\gamma\delta\gamma+\gamma'\delta\gamma=0,\\
&\frac{\partial\delta\varphi}{\partial t}+\widetilde{u}\cdot\nabla\delta\varphi+\delta\widetilde{u}\cdot\nabla\varphi'+\gamma\delta\varphi+\varphi'\delta\gamma=0.
\end{aligned}\tag{42}$$

Applying the similar procedure to (34), (34) and (38) to each equation in (42) and combining them, it yields

$$\begin{aligned}
\|\delta\theta\|_{L^\infty_T B^{s-1}_{p,q}} \le{}& C\int_0^T(\|\delta\widetilde{u}\nabla\theta'\|_{B^{s-1}_{p,q}}+\|\theta\delta\theta\|_{B^{s-1}_{p,q}}+\|\theta'\delta\theta\|_{B^{s-1}_{p,q}})d\tau\\
&+C\int_0^T(\|\nabla\widetilde{u}\|_\infty\|\delta\theta\|_{B^{s-1}_{p,q}}+\|\nabla\widetilde{u}\|_{B^{s-1}_{p,q}}\|\delta\theta\|_\infty)d\tau.
\end{aligned}\tag{43}$$

By Remark 1, $\delta\widetilde{u}\in C^{0,\sigma}$ for some $0<\sigma<1$. Then, by Proposition 4, we obtain

$$\begin{aligned}
&\|\delta\widetilde{u}\nabla\theta'\|_{B^{s-1}_{p,q}}\\
&\le C(\|\delta\widetilde{u}\|_{B^\sigma_{\infty,\infty}}\|\nabla\theta'\|_{B^{s-1}_{p,q}}+\|\nabla\delta\widetilde{u}\|_\infty\|\theta'\|_{B^{s-1}_{p,q}}+\|\nabla\delta\widetilde{u}\|_{B^{s-1}_{p,q}}\|\theta'\|_\infty)\\
&\le C(\|\delta\theta\|_{B^{s-1}_{p,q}}\|\nabla\theta'\|_{B^{s-1}_{p,q}}+\|\delta\theta\|_{B^{s-1}_{p,q}}\|\theta'\|_{B^{s-1}_{p,q}}+\|\delta\theta\|_{B^{s-1}_{p,q}}\|\theta'\|_{B^s_{p,q}})\\
&\le C\|\delta\theta\|_{B^{s-1}_{p,q}}\|\theta'\|_{B^s_{p,q}}.
\end{aligned}\tag{44}$$

By product estimate (25), we obtain

$$\|\theta\delta\theta\|_{B^{s-1}_{p,q}}\le C\|\theta\|_{B^s_{p,q}}\|\delta\theta\|_{B^{s-1}_{p,q}}\tag{45}$$

and

$$\|\theta'\delta\theta\|_{B^{s-1}_{p,q}}\le C\|\theta'\|_{B^s_{p,q}}\|\delta\theta\|_{B^{s-1}_{p,q}}.\tag{46}$$

Since θ and θ' are uniformly bounded in $L^\infty_T B^s_{p,q}$, from (43)-(46), we have

$$\|\delta\theta\|_{L^\infty_T B^{s-1}_{p,q}}\le CT\|\delta\theta\|_{L^\infty_T B^{s-1}_{p,q}}.\tag{47}$$

We can choose some smaller time $T>0$ such that

$$CT<\frac{1}{2}.\tag{48}$$

We find that

$$\|\theta-\theta'\|_{L^\infty_T B^{s-1}_{p,q}}\le\frac{1}{2}\|\theta-\theta'\|_{L^\infty_T B^{s-1}_{p,q}}.\tag{49}$$

This implies $\theta=\theta'$ and the solution is unique on $[0,T)$.

From the equation of ω in (11), we obtain

$$\begin{aligned}
\|\partial_t\omega\|_{L^\infty_T B^{s-1}_{p,q}}&\le\|\widetilde{u}\cdot\nabla\omega\|_{L^\infty_T B^{s-1}_{p,q}}+\|\gamma\omega\|_{L^\infty_T B^{s-1}_{p,q}}\\
&\le C\|\theta\|^2_{L^\infty_T B^s_{p,q}},
\end{aligned}\tag{50}$$

where we repeated the arguments in (44) and (45). Repeating (50) for the equations of γ and φ, we obtain $\partial_t\theta \in L^\infty([0,T); B^{s-1}_{p,q})$.

Therefore, we have $\theta \in L^\infty([0,T); B^s_{p,q}) \cap Lip([0,T); B^{s-1}_{p,q})$ and the proof of Theorem 1 is completed.

4. The proof of Theorem 2.

In this section, we prove Theorem 2, i.e., the local solution, which was found in Theorem 1, becomes global by obtaining a priori estimate in the corresponding space. The boundedness of γ, ω and φ for positive γ_0 is the key estimate for the global existence.

Lemma 2. *Let* $\gamma(t) \in B^s_{p,q},\ 0 < t < T$ *be a local solution of* (11) *with* (s,p,q) *satisfy* (12). *If* $\gamma_0 \geq 0$, *then we have*

$$0 \leq \gamma(x,t) \leq \|\gamma_0\|_\infty \tag{51}$$

for all $x \in \mathbb{R}^2$ *and* $0 < t < T$.

Proof. To prove (51), we use the maximum principle to the equation of γ in (11).

First, let us prove the right hand side of the inequality (51). Let $\gamma^\epsilon = \gamma - \epsilon t,\ \epsilon > 0$, then γ^ϵ satisfies

$$\frac{\partial \gamma^\epsilon}{\partial t} = -\widetilde{u} \cdot \nabla \gamma^\epsilon - \gamma^2 - \epsilon. \tag{52}$$

Suppose the right inequality in (51) is not satisfied for γ^ϵ for some $\epsilon > 0$.

We have $\gamma \to 0$ as $|x| \to \infty$, since $\gamma \in L^p(\mathbb{R}^2)$. It follows that $\gamma^\epsilon \to -\epsilon t$ as $|x| \to \infty$. Therefore, γ^ϵ attains its maximum at some point (x^*, t^*) in $\mathbb{R}^2 \times (0,T]$. At the point (x^*, t^*), we have

$$\frac{\partial \gamma^\epsilon}{\partial t} \geq 0,\ \nabla \gamma^\epsilon = 0. \tag{53}$$

This contradicts (52), since $-\gamma^2 - \epsilon < 0$. Sending ϵ to 0, we obtain the result.

In order to prove the left hand side of inequality (51), denote $\gamma^\epsilon = \gamma + \epsilon + \epsilon t$ for $\epsilon > 0$. Then γ^ϵ satisfies

$$\frac{\partial \gamma^\epsilon}{\partial t} = -\widetilde{u} \cdot \nabla \gamma^\epsilon - \gamma^2 + \epsilon. \tag{54}$$

We will prove the inequality for γ^ϵ. Suppose the inequality is not satisfied for γ^ϵ for some small enough $\epsilon > 0$. We know, initially, $\gamma^\epsilon(0) = \gamma_0 + \epsilon > 0$. Therefore, there exists the first time t^* at which γ^ϵ hits 0, that is $\gamma(x) = -\epsilon - \epsilon t^*$ for some $x \in \mathbb{R}^2$.

Since this point is minimum for γ^ϵ, it must be

$$0 \geq \frac{\partial \gamma^\epsilon}{\partial t} = -\gamma^2 + \epsilon, \tag{55}$$

which implies

$$(1+t^*)^2 \geq \frac{1}{\epsilon}. \tag{56}$$

The inequality (56) contradicts the condition $t^* \in (0,T]$ for small enough ϵ. Therefore, $\gamma^\epsilon \geq 0$ for small enough ϵ and $t \in [0,T]$. Since $\epsilon > 0$ is arbitrary, $\gamma \geq 0$ for $t \in [0,T]$. This proves (51). □

Now if we apply Proposition 1 to the equation of ω in (11), it yields

$$e^{-\frac{1}{r}\int_0^t \|\gamma\|_\infty d\tau}\|\omega(t)\|_r \leq (\|\omega_0\|_r + \int_0^t \|\gamma\|_\infty e^{-\frac{1}{r}\int_0^\tau \|\gamma\|_\infty d\tau'}\|\omega(\tau)\|_r d\tau) \tag{57}$$

for $r < \infty$.

Applying the Grönwall's inequality to (57) and using Lemma 2, we get

$$\|\omega(t)\|_r \leq \|\omega_0\|_r e^{\frac{2}{r}t\|\gamma_0\|_{L^\infty}},\ t \in [0,T]. \tag{58}$$

If we let $r \to \infty$ in (58), we obtain

$$\|\omega(t)\|_\infty \leq \|\omega_0\|_\infty,\ t \in [0,T]. \tag{59}$$

Repeating the above argument to the equation of φ, in particular, we have

$$\|\theta(t)\|_\infty \leq \|\theta_0\|_\infty,\ t \in [0,T]. \tag{60}$$

By analogy with (38),

$$\begin{aligned} &\|\omega(t)\|_{B^s_{p,q}} \\ &\leq \|\omega_0\|_{B^s_{p,q}} + C\int_0^t (\|\gamma\omega\|_{B^s_{p,q}} + \|\nabla\widetilde{u}\|_\infty\|\omega\|_{B^s_{p,q}} + \|\omega\|_\infty\|\nabla\widetilde{u}\|_{B^s_{p,q}})d\tau \\ &\quad + \frac{1}{p}\int_0^t \|\gamma\|_\infty\|\omega\|_{B^s_{p,q}}d\tau. \end{aligned} \tag{61}$$

If we repeat the same computation for the equation of γ and φ, then we get

$$\|\theta(t)\|_{B^s_{p,q}} \leq \|\theta_0\|_{B^s_{p,q}} + C\int_0^t (\|\theta\|_\infty + \|\nabla\widetilde{u}\|_\infty)\|\theta\|_{B^s_{p,q}}d\tau, \tag{62}$$

where the product estimate is used. According to Proposition 5, $\widetilde{u}$ satisfies

$$\|\nabla\widetilde{u}\|_\infty \leq C(1 + \|\nabla\widetilde{u}\|_{\dot{B}^0_{\infty,\infty}}(\log^+\|\nabla\widetilde{u}\|_{B^s_{p,q}} + 1)). \tag{63}$$

By the property of multiplier operators for the homogeneous Besov space and the embedding $L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$, we have

$$\|\nabla\widetilde{u}\|_{\dot{B}^0_{\infty,\infty}} \leq C\|\theta\|_\infty. \tag{64}$$

Thus, applying (60), (63) and (64) to (62), we obtain

$$\|\theta(t)\|_{B^s_{p,q}} \leq \|\theta_0\|_{B^s_{p,q}} + C\int_0^t (1 + \|\theta_0\|_{B^s_{p,q}})(\log^+\|\theta\|_{B^s_{p,q}} + 1)\|\theta\|_{B^s_{p,q}}d\tau. \tag{65}$$

Finally, Grönwall's inequality gives us

$$\|\theta(t)\|_{B^s_{p,q}} \le \|\theta_0\|_{B^s_{p,q}} \exp(C\exp(Ct(1+\|\theta_0\|_{B^s_{p,q}}))). \tag{66}$$

The inequality (66) implies the solution obtained in Theorem 1 does not blow up in a finite time, therefore, the solution becomes global by the standard continuation argument.

Acknowledgments. The authors express their sincere gratitude to the reviewer for her/his valuable suggestions and for pointing out the simple proof of the boundedness of γ, ω and φ using the particle trajectory method.

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