

SPECIAL VALUES AND INTEGRAL REPRESENTATIONS FOR THE HURWITZ-TYPE EULER ZETA FUNCTIONS

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ABSTRACT. The Hurwitz-type Euler zeta function is defined as a deformation of the Hurwitz zeta function:

$$\zeta_E(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}.$$

In this paper, by using the method of Fourier expansions, we shall evaluate several integrals with integrands involving Hurwitz-type Euler zeta functions $\zeta_E(s, x)$. Furthermore, the relations between the values of a class of the Hurwitz-type (or Lerch-type) Euler zeta functions at rational arguments have also been given.

1. Introduction

The Hurwitz zeta function is defined by

$$(1.1) \quad \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$$

for $s \in \mathbb{C}$ and $x \neq 0, -1, -2, \dots$, and the series converges for $\operatorname{Re}(s) > 1$, so that $\zeta(s, x)$ is an analytic function of s in this region. Setting $x = 1$ in (1.1), it reduces to the Riemann zeta function

$$(1.2) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

From the well-known identity $\zeta(s, \frac{1}{2}) = (2^s - 1)\zeta(s)$, we see that the only real zeroes of $\zeta(s, x)$ with $x = \frac{1}{2}$ are $s = 0, -2, -4, \dots$.

Its special values at nonpositive integers are Bernoulli polynomials, that is,

$$(1.3) \quad \zeta(1-m, x) = -\frac{1}{m} B_m(x), \quad m \in \mathbb{N}$$

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(see [15, p. 162, (1.19)]). Here the Bernoulli polynomials $B_m(x)$ are defined by their generating function

$$(1.4) \quad \frac{ze^{xz}}{e^z - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{z^m}{m!}, \quad |z| < \pi.$$

Bernoulli polynomials have many interesting properties and arise in various areas of mathematics (see [9,34,36]). Setting $x = 0$ in (1.4), we get the Bernoulli numbers

$$B_m = B_m(0), \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

From an easy manipulations of the generating function (1.4), we see that $B_m(1) = B_m$ ($m \neq 1$) and $B_1(1) = -B_1$, where $m \in \mathbb{N}_0$ (see [13, p. 1480, (2.3)] and [34, p. 529, (17)]).

The Hurwitz zeta function is a fundamental tool for studying Stark's conjectures of algebraic number fields, since it represents the partial zeta function of cyclotomic field (see e.g., [18, p. 993, (4.2)] or [23, p. 4248]). It also plays a role in the evaluation of fundamental determinants that appears in mathematical physics (see [14] or [15, p. 161, (1.14) and (1.15)]).

In this paper, we consider the Hurwitz-type Euler zeta function, which is defined as a deformation of the Hurwitz zeta function

$$(1.5) \quad \zeta_E(s, x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}$$

for $s \in \mathbb{C}$ and $x \neq 0, -1, -2, \dots$. This series converges for $\operatorname{Re}(s) > 0$, and it can be analytically continued to the complex plane without any pole. It also represents a partial zeta function of cyclotomic fields in one version of Stark's conjectures in algebraic number theory (see [23, p. 4249, (6.13)]), and its special case, the alternative series,

$$(1.6) \quad \zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},$$

is also a particular case of Witten's zeta functions in mathematical physics. (See [31, p. 248, (3.14)]). We here mention that several properties of $\zeta_E(s)$ can be found in [1, 6, 10]. For example, in the form on [1, p. 811], the left hand side is the special values of the Riemann zeta functions at positive integers, and the right hand side is the special values of Euler zeta functions at positive integers.

Using the Fourier expansion formula

$$(1.7) \quad \zeta(s, x) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \sum_{n=1}^{\infty} \frac{\sin(2n\pi x + \frac{\pi s}{2})}{n^{1-s}},$$

Espinosa and Moll [15] evaluated several fundamental integrals with integrands involving Hurwitz zeta functions, where Γ is the Euler gamma-function (see, e.g., [30] or [36, Chapter 43]), who studied "the Hurwitz transform" meaning an integral over $[0, 1]$ of a Fourier series multiplied by the Hurwitz zeta function

$\zeta(s, x)$, and obtained numerous results for those which arises from the Hurwitz formula. Mező [29] determined the Fourier series expansion of the log-Barnes function, which is the analogue of the classical result of Kummer and Malmsten. Applying this expansion, he also got some integrals similar to the Espinosa–Moll log–Gamma integrals with respect to $\log G$ in [15, 16], where G is Barnes G function.

Recently, Shpot, Chaudhary and Paris [35] evaluated two integrals over $x \in [0, 1]$ involving products of the function $\zeta_1(s, x) = \zeta(s, x) - x^{-a}$ for $\operatorname{Re}(s) > 1$. As an application, they also calculated the $O(g)$ weak-coupling expansion coefficient $c_1(\epsilon)$ of the Casimir energy for a film with Dirichlet–Neumann boundary conditions, first stated by Symanzik [38] under the framework of $g\phi_{4-\epsilon}^4$ theory in quantum physics.

Here we will follow the approach of Espinosa and Moll in [15] to evaluate several integrals with integrands involving Hurwitz-type Euler zeta functions $\zeta_E(s, x)$.

Our main tool is the following Fourier expansion of $\zeta_E(s, x)$

$$(1.8) \quad \zeta_E(s, x) = \frac{2\Gamma(1-s)}{\pi^{1-s}} \sum_{n=0}^{\infty} \frac{\sin\left((2n+1)\pi x + \frac{\pi s}{2}\right)}{(2n+1)^{1-s}}$$

(comparing with (1.7) above). This expression, valid for $\operatorname{Re}(s) < 1$ and $0 < x \leq 1$, is derived by Williams and Zhang [40, p. 36, (1.7)].

The paper is organized as follows.

In Section 3, we determine the Fourier coefficients of $\zeta_E(s, x)$.

In Section 4, we evaluate integrals with integrands consisting of products of two Hurwitz-type Euler zeta functions. These will imply some classical relations for the integral of products of two Euler polynomials (see Propositions 4.8 and 4.10). We also evaluate the moments of the Hurwitz-type Euler zeta functions and the Euler polynomials (see Propositions 4.9 and 4.10).

In Section 5, we prove several further consequences of integrals with integrands consisting of products of the Hurwitz-type Euler zeta functions, the exponential function, the power of sine and cosine.

In Section 6, we obtain an Euler-type formula for the Dirichlet beta function $\beta(2m)$ and consider the Catalan’s constant G compiled by Adamchik in [2, Entry 19]. From this, we get a finite closed-form expression for any $\beta(2m)$ in terms of the elementary functions (see [24]).

In Section 7, we prove the relations between values of a class of the Hurwitz-type (or Lerch-type) Euler zeta functions at rational arguments.

2. Preliminaries

The methods employed in this section may also be used to derive several results of the Hurwitz-type Euler zeta functions, which are analogous of the known results for Hurwitz zeta functions.

The series expansion (1.5) for $\zeta_E(s, x)$ has a meaning if $\text{Re}(s) > 0$. In the following, we recall some basic results of this functions given in [40].

From (1.5), we can easily derive the following basic properties for $\zeta_E(s, x)$

$$(2.1) \quad \zeta_E(s, x) + \zeta_E(s, x + 1) = x^{-s},$$

$$(2.2) \quad \zeta_E(s, kx) = k^{-s} \sum_{n=0}^{k-1} (-1)^n \zeta_E\left(s, \frac{n}{k} + x\right) \quad \text{for } k \text{ odd,}$$

$$(2.3) \quad \zeta_E(s, x) - x^{-s} = - \sum_{n=0}^{\infty} \binom{-s}{n} \zeta_E(s + n) x^n$$

(see [4, (5), (6) and (7)]). For fixed $s \neq 1$, the series in (2.3) converges absolutely for $|x| < 1$. For $\text{Re}(s) > 1$, (1.2) and (1.6) are connected by the following equation

$$(2.4) \quad \zeta_E(s) = (1 - 2^{1-s})\zeta(s).$$

In [40, §3], Williams and Zhang established the following integral representation

$$(2.5) \quad \zeta_E(s, x) = \frac{e^{-\pi i s} \Gamma(1 - s)}{2\pi i} \int_C \frac{e^{(1-x)z} z^{s-1}}{e^z + 1} dz,$$

where C is the contour consisting of the real axis from $+\infty$ to ϵ , the circle $|z| = \epsilon$ and the real axis from ϵ to $+\infty$. By (2.5) and the generating function of the Euler polynomials

$$(2.6) \quad \frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi,$$

we have

$$\zeta_E(-m, x) = \frac{(-1)^m m!}{4\pi i} \int_{|z|=\epsilon} \sum_{n=0}^{\infty} \frac{E_n(1-x)}{n!} \frac{z^n}{z^{m+1}} dz,$$

that is,

$$(2.7) \quad \zeta_E(-m, x) = \frac{(-1)^m}{2} E_m(1-x) = \frac{1}{2} E_m(x), \quad m \in \mathbb{N}_0$$

(comparing with (1.3) above). Setting $s = -m$ in (2.4), from (1.3) and (2.7) it is easy to see that

$$(2.8) \quad E_m(0) = -E_m(1) = \frac{2}{m+1} (1 - 2^{m+1}) B_{m+1}$$

(see [34, p. 529, (16)]).

The above expansion (2.7) has also been derived in [40, p. 41, (3.8)].

The integers

$$E_m = 2^m E_m\left(\frac{1}{2}\right), \quad m \in \mathbb{N}_0,$$

are called m -th Euler numbers. For example, $E_0 = 1, E_2 = -1, E_4 = 5,$ and $E_6 = -61$. Notice that the Euler numbers with odd subscripts vanish, that is, $E_{2m+1} = 0$ for all $m \in \mathbb{N}_0$. The Euler polynomials can be expressed in terms of the Euler numbers in the following way:

$$(2.9) \quad E_m(x) = \sum_{i=0}^m \binom{m}{i} \frac{E_i}{2^i} \left(x - \frac{1}{2}\right)^{m-i}$$

(see [34, p. 531, (29)]).

Some properties of Euler polynomials can be easily derived from their generating functions, for example, from (2.6), we have

$$(2.10) \quad E_m(x+1) + E_m(x) = 2x^m, \quad m \in \mathbb{N}_0,$$

which have numerous applications (see [13, p. 1489, (5.4)] and [34, p. 530, (23) and (24)]).

For further properties of the Euler polynomials and numbers including their applications, we refer to [1, 7, 17, 22, 34]. It may be interesting to point out that there is also a connection between the generalized Euler numbers and the ideal class group of the p^{n+1} -th cyclotomic field when p is a prime number. For details, we refer to a recent paper [19], especially [19, Proposition 3.4].

The result

$$(2.11) \quad \int_0^1 \zeta_E(s, x) dx = \frac{4\Gamma(1-s)}{\pi^{2-s}} \cos\left(\frac{\pi s}{2}\right) \lambda(2-s),$$

valid for $\text{Re}(s) < 1$, follows directly from the representation (1.8). Here $\lambda(s)$ is the Dirichlet lambda function

$$(2.12) \quad \lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = (1-2^{-s})\zeta(s), \quad \text{Re}(s) > 1,$$

where $\zeta(s)$ is the Riemann zeta function (see [1, p. 807, 23.2.20] and [36, p. 32, Eq. 3:6:2]). If we put $s = -2m$ with $m \in \mathbb{N}_0$ in (2.11) and use (2.7), then we obtain

$$(2.13) \quad \int_0^1 E_{2m}(x) dx = \frac{8(2m)!(-1)^m}{\pi^{2m+2}} \lambda(2m+2).$$

If we replace m by $2m$ and let $n = 0$ in (4.15) below, then the left hand side of (2.13) becomes

$$(2.14) \quad \int_0^1 E_{2m}(x) dx = -\frac{2(2m)!}{(2m+1)!} E_{2m+1}(0),$$

where $m \in \mathbb{N}_0$. From (2.13) and (2.14), the values of the Dirichlet lambda function at even positive integers is given as

$$(2.15) \quad \lambda(2m) = (-1)^m \frac{\pi^{2m}}{4(2m-1)!} E_{2m-1}(0),$$

where $m \in \mathbb{N}$. In particular, if we put $m = 1, m = 2$ and $m = 3$ in (2.15), then we have the first few values $\lambda(2) = \frac{\pi^2}{8}, \lambda(4) = \frac{\pi^4}{96}$ and $\lambda(6) = \frac{\pi^6}{960}$ (see [1, p. 808]).

3. The Fourier expansion of $\zeta_E(s, x)$

In this section, by using the Fourier expansion (1.8) for $\zeta_E(s, x)$, we shall evaluate definite integrals of the Hurwitz-type transform

$$\mathfrak{H}(f, s) = \int_0^1 f(x)\zeta_E(s, x)dx.$$

The expansion is valid for $\operatorname{Re}(s) < 1$. For the basics we refer the reader to definite integrals of the Hurwitz transform, e.g., [15] and [25].

The following is the Fourier coefficients of $\zeta_E(s, x)$, as a direct consequence of (1.8) (see e.g., [15, p. 164, Proposition 2.1]).

Proposition 3.1. *The Fourier coefficients of $\zeta_E(s, x)$ are given by*

$$(3.1) \quad \int_0^1 \sin((2k+1)\pi x)\zeta_E(s, x)dx = \frac{\pi^s(2k+1)^{s-1}}{2\Gamma(s)} \operatorname{csc}\left(\frac{\pi s}{2}\right)$$

and

$$(3.2) \quad \int_0^1 \cos((2k+1)\pi x)\zeta_E(s, x)dx = \frac{\pi^s(2k+1)^{s-1}}{2\Gamma(s)} \operatorname{sec}\left(\frac{\pi s}{2}\right).$$

Remark 3.2. Recently, Luo [27], Bayad [8], Navas, Francisco and Varona [32] investigated Fourier expansions of the Apostol-Bernoulli polynomials, which are special values of Hurwitz-Lerch zeta function at non-positive integers [21, Lemma 2.1].

Proof of Proposition 3.1. The proof follows from the similar argument with [15, Proposition 2.1], by applying the orthogonality of the trigonometric functions and the expansion (1.8) above. \square

In the following proposition, we shall compute the Hurwitz-type transform of powers of sine and cosine, including some related integrals (see [15, §11]).

Proposition 3.3. *Let $s \in \mathbb{R}_0^- = (-\infty, 0]$ and $n \in \mathbb{N}$. Then we have*

$$(3.3) \quad \int_0^1 \sin^{2n-1}(\pi x)\zeta_E(s, x)dx = \frac{\Gamma(1-s)}{\pi^{1-s}2^{2n-2}} \cos\left(\frac{\pi s}{2}\right) \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^{1-s}} \binom{2n-1}{n-k-1}$$

and

$$(3.4) \quad \int_0^1 \cos^{2n-1}(\pi x)\zeta_E(s, x)dx = \frac{\Gamma(1-s)}{\pi^{1-s}2^{2n-2}} \sin\left(\frac{\pi s}{2}\right) \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{1-s}} \binom{2n-1}{n-k-1}.$$

Proof. The proofs follow from the similar arguments with [15, Examples 11.1 and 11.2], which is based on (3.1) and (3.2), and the integral representation

$$(3.5) \quad \sin^{2n-1}(\pi x) = \frac{1}{2^{2n-2}} \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{n-k-1} \sin((2k+1)\pi x)$$

by Kogan in [17, p. 31, 1.320]. \square

As in Espinosa and Moll [15], we may evaluate the integrals using the Fourier expansions of functions. The following result about this idea will become the basic tool for the proofs of the results in this paper.

Theorem 3.4. *Let $f(w, x)$ be defined for $x \in [0, 1]$ and a parameter w . Let*

$$(3.6) \quad f(w, x) = \sum_{n=0}^{\infty} \left(a_n(w) \cos((2n+1)\pi x) + b_n(w) \sin((2n+1)\pi x) \right)$$

be its Fourier expansion, so that

$$(3.7) \quad a_n(w) = 2 \int_0^1 f(w, x) \cos((2n+1)\pi x) dx,$$

$$(3.8) \quad b_n(w) = 2 \int_0^1 f(w, x) \sin((2n+1)\pi x) dx.$$

Then, for $s \in \mathbb{R}_0^-$, we have

$$(3.9) \quad \int_0^1 f(w, x) \zeta_E(s, x) dx = \frac{\Gamma(1-s)}{\pi^{1-s}} \left(\sin\left(\frac{\pi s}{2}\right) C(s, w) + \cos\left(\frac{\pi s}{2}\right) S(s, w) \right)$$

and

$$(3.10) \quad \int_0^1 f(w, x) \zeta_E(s, 1-x) dx = \frac{\Gamma(1-s)}{\pi^{1-s}} \left(\cos\left(\frac{\pi s}{2}\right) S(s, w) - \sin\left(\frac{\pi s}{2}\right) C(s, w) \right),$$

where

$$C(s, w) = \sum_{n=0}^{\infty} \frac{a_n(w)}{(2n+1)^{1-s}} \quad \text{and} \quad S(s, w) = \sum_{n=0}^{\infty} \frac{b_n(w)}{(2n+1)^{1-s}}.$$

Proof. The proof follows from the similar argument with [15, Theorem 2.1], by multiplying the formula (3.6) by $\zeta_E(s, x)$ and integrating both sides from 0 to 1, then applying (3.1) and (3.2). \square

4. Integral representations of Hurwitz-type Euler zeta functions

In this section, we shall evaluate integrals with integrands consisting of products of Hurwitz-type Euler zeta functions. These will imply some integral representations for the function and classical relations for the integral of products of two Euler polynomials.

Theorem 4.1. *Let $s, s' \in \mathbb{R}_0^-$. Then we have*

$$(4.1) \quad \int_0^1 \zeta_E(s', x) \zeta_E(s, x) dx = \frac{2\Gamma(1-s)\Gamma(1-s')}{\pi^{2-s-s'}} \lambda(2-s-s') \cos\left(\frac{\pi(s-s')}{2}\right).$$

Similarly, we have

$$(4.2) \quad \int_0^1 \zeta_E(s', x) \zeta_E(s, 1-x) dx = \frac{2\Gamma(1-s)\Gamma(1-s')}{\pi^{2-s-s'}} \lambda(2-s-s') \cos\left(\frac{\pi(s+s')}{2}\right).$$

Proof. In view of (3.4), (3.7) and (3.8), the expansion (1.8) shows that the coefficients of $\zeta_E(s', x)$ are given by

$$(4.3) \quad \begin{aligned} a_n(s') &= \frac{2\Gamma(1-s') \sin\left(\frac{\pi s'}{2}\right)}{\pi^{1-s'}} \frac{1}{(2n+1)^{1-s'}}, \\ b_n(s') &= \frac{2\Gamma(1-s') \cos\left(\frac{\pi s'}{2}\right)}{\pi^{1-s'}} \frac{1}{(2n+1)^{1-s'}}. \end{aligned}$$

From (2.12), (3.9) and (4.3), we obtain

$$(4.4) \quad \begin{aligned} \int_0^1 \zeta_E(s', x) \zeta_E(s, x) dx &= \frac{2\Gamma(1-s)\Gamma(1-s')}{\pi^{2-s-s'}} \left(\sin\left(\frac{\pi s}{2}\right) \sin\left(\frac{\pi s'}{2}\right) \right. \\ &\quad \left. + \cos\left(\frac{\pi s}{2}\right) \cos\left(\frac{\pi s'}{2}\right) \right) \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2-s-s'}} \\ &= \frac{2\Gamma(1-s)\Gamma(1-s')}{\pi^{2-s-s'}} \cos\left(\frac{\pi(s-s')}{2}\right) \lambda(2-s-s'), \end{aligned}$$

which gives (4.1). The proof of (4.2) is similar. \square

Corollary 4.2. *Let $s, s' \in \mathbb{R}_0^-$ and*

$$\delta_2(s) = \frac{1-2^{s+1}}{1-2^s}, \quad s \neq 0.$$

Then we have

$$(4.5) \quad \begin{aligned} \int_0^1 \zeta_E(s', x) \zeta_E(s, x) dx &= \delta_2(1-s-s') \frac{\cos\left(\frac{\pi(s-s')}{2}\right)}{\cos\left(\frac{\pi(s+s')}{2}\right)} B(1-s, 1-s') \\ &\quad \times \lambda(s+s'-1). \end{aligned}$$

Similarly, we have

$$(4.6) \quad \begin{aligned} \int_0^1 \zeta_E(s', x) \zeta_E(s, 1-x) dx &= \delta_2(1-s-s') B(1-s, 1-s') \\ &\quad \times \lambda(s+s'-1). \end{aligned}$$

Here $B(s, s')$ is the beta function,

$$(4.7) \quad B(s, s') = \frac{\Gamma(s)\Gamma(s')}{\Gamma(s+s')},$$

defined for $s, s' \in \mathbb{C}$ with $\operatorname{Re}(s), \operatorname{Re}(s') > 1$, see [40, (1.36)].

Proof. The functional equation of Riemann's zeta function is well-known, and it is given by (see [15, (1.35)])

$$(4.8) \quad \zeta(1-s) = 2 \cos\left(\frac{\pi s}{2}\right) \frac{\Gamma(s)}{(2\pi)^s} \zeta(s).$$

From (2.12) and (4.8), we have

$$(4.9) \quad \lambda(2-s-s') = -\frac{\delta_2(1-s-s')\pi^{2-s-s'}}{2\Gamma(2-s-s')\cos\left(\frac{\pi(2-s-s')}{2}\right)} \lambda(s+s'-1).$$

Finally, using (4.1) and (4.9), we obtain (4.5). The proof of (4.6) is similar. \square

Corollary 4.3. *Let $s \in \mathbb{R}_0^-$. Then we have*

$$(4.10) \quad \int_0^1 \zeta_E^2(s, x) dx = 2\Gamma^2(1-s)\pi^{2s-2}\lambda(2-2s)$$

and

$$(4.11) \quad \int_0^1 \zeta_E(s, x)\zeta_E(s, 1-x) dx = 2\Gamma^2(1-s)\pi^{2s-2}\lambda(2-2s)\cos(\pi s).$$

Proof. Put $s = s'$ in (4.1) and (4.2), we get our results. \square

Corollary 4.4. *Let $m \in \mathbb{N}$. Then we have*

$$(4.12) \quad \int_0^1 \zeta_E^2\left(-m + \frac{1}{2}, x\right) dx = 2 \left(\frac{(2m)!}{2^{2m}m!}\right)^2 \frac{\lambda(2m+1)}{\pi^{2m}}.$$

Proof. Setting $s = -m + \frac{1}{2}$ in (4.10), then by

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2m)!}{2^{2m}m!} \quad (\text{see [15, p. 167]}),$$

we get our result. \square

Corollary 4.5. *Let $s \in \mathbb{R}_0^-$ and $m \in \mathbb{N}$. Then we have*

$$(4.13) \quad \int_0^1 E_{m-1}(x)\zeta_E(s, x) dx = (-1)^{m+1}2\delta_2(m-s)\frac{(m-1)!\lambda(s-m)}{(1-s)_m},$$

where $(s)_k = s(s+1)\cdots(s+k-1)$ is the Pochhammer symbol.

Remark 4.6. The case $m = 1$ in (4.13) implies (2.11) from (2.12) and (4.8).

Proof of Corollary 4.5. Let $s' = 1 - m$ in (4.5), we have

$$\int_0^1 \zeta_E(1 - m, x) \zeta_E(s, x) dx = \delta_2(m - s) \frac{\cos\left(\frac{\pi(s-1+m)}{2}\right)}{\cos\left(\frac{\pi(s+1-m)}{2}\right)} B(1 - s, m) \\ \times \lambda(s - m).$$

The result then follows from (2.7), since

$$\frac{\cos\left(\frac{\pi(s-1+m)}{2}\right)}{\cos\left(\frac{\pi(s+1-m)}{2}\right)} = (-1)^{m+1}$$

and $B(1 - s, m) = (m - 1)! / (1 - s)_m$. \square

Lemma 4.7. For $m \in \mathbb{N}$, we have

$$(4.14) \quad \lambda(1 - m) = (-1)^{m+1} \frac{1}{2\delta_2(m - 1)} E_{m-1}(0).$$

Proof. This follows from (1.3) with $x = 1$, (2.8) and (2.12). \square

The following formula on the integral of two Euler polynomials has a long history, it has already appeared in a book by Nörlund [33, p. 36]. See e.g. [34, p. 530, (25)] and [37, p. 64, (52)].

Corollary 4.8. Let $m, n \in \mathbb{N}_0$. Then we have

$$(4.15) \quad \int_0^1 E_m(x) E_n(x) dx = 2(-1)^{n+1} \frac{m!n!}{(m+n+1)!} E_{m+n+1}(0).$$

Proof of Corollary 4.8. Set $s = 1 - n$, where $n \in \mathbb{N}$ in (4.13), then by (2.7) and (4.14), we obtain

$$\int_0^1 E_{m-1}(x) E_{n-1}(x) dx = 4(-1)^{m+1} \delta_2(m+n-1) \frac{(m-1)! \lambda(1 - (m+n))}{(n)_m} \\ = 2(-1)^n \frac{(m-1)!}{(n)_m} E_{m+n-1}(0) \\ = 2(-1)^n \frac{(m-1)!(n-1)!}{(m+n-1)!} E_{m+n-1}(0),$$

where $m, n \in \mathbb{N}$. This completes our proof. \square

Corollary 4.9. For $n \in \mathbb{N}_0$, the moments of the Hurwitz-type Euler zeta function are given by

$$(4.16) \quad \int_0^1 x^n \zeta_E(s, x) dx = \sum_{j=0}^n \binom{n}{j} (-1)^j \delta_2(j - s + 1) \frac{j! \lambda(s - j - 1)}{(1 - s)_{j+1}} \\ + (-1)^n \delta_2(n - s + 1) \frac{n! \lambda(s - n - 1)}{(1 - s)_{n+1}}.$$

Proof. The proof of (4.16) is obtained by using the expansion of x^n in terms of Euler polynomials (2.10) and the evaluation (4.13). \square

Corollary 4.10. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$(4.17) \quad \int_0^1 x^n E_{m-1}(x) dx = \frac{(-1)^m}{m} \left(\sum_{j=0}^n \frac{\binom{n}{j}}{\binom{m+j}{j}} E_{m+j}(0) + \frac{E_{m+n}(0)}{\binom{m+n}{n}} \right).$$

Proof. By (4.14), we have

$$(4.18) \quad \lambda(1 - (j + m + 1)) = (-1)^{j+m} \frac{1}{2\delta_2(j + m)} E_{j+m}(0).$$

Letting $s = 1 - m$ in (4.16), then using (2.7) and (4.18), we easily deduce that

$$\int_0^1 x^n E_{m-1}(x) dx = (-1)^m \left(\sum_{j=0}^n \binom{n}{j} \frac{j!}{(m)_{j+1}} E_{m+j}(0) + \frac{n!}{(m)_{n+1}} E_{m+n}(0) \right)$$

and this implies our result. \square

Remark 4.11. Corollary 4.10 gives some identities of $E_m(0)$ for each value of $m \in \mathbb{N}$. For instance, when $m = 1$ and 2, we have

$$\sum_{j=0}^n \binom{n}{j} \frac{E_{j+1}(0)}{j+1} = -\frac{1}{n+1} (E_{n+1}(0) + 1),$$

$$\sum_{j=0}^n \binom{n}{j} \frac{E_{j+2}(0)}{(j+1)(j+2)} = -\frac{1}{2(n+1)(n+2)} (2E_{n+2}(0) - n),$$

where $n \in \mathbb{N}_0$.

5. The exponential function

In this section, we shall evaluate the Hurwitz-type transform of the exponential function (see [15, §4]). As [15, §4], the result is expressed in terms of the transcendental function

$$(5.1) \quad F(x, s) = \sum_{n=0}^{\infty} \lambda(n + 2 - s) x^n \quad \text{for } |x| < 1,$$

where $\lambda(s)$ denotes the Dirichlet lambda function (define $0^0 = 1$ if $x = 0$).

Theorem 5.1. *Let $s \in \mathbb{R}_0^-$ and $|t| < 1$. Then we have*

$$(5.2) \quad \int_0^1 e^{2\pi tx} \zeta_E(s, x) dx = \frac{2(e^{2\pi t} + 1)\Gamma(1 - s)}{\pi^{2-s}} \operatorname{Re} \left(e^{\pi is/2} F(2it, s) \right),$$

where $F(x, s)$ is given in (5.1).

Remark 5.2. Setting $t = 0$ in (5.2), we get (2.11).

Proof of Theorem 5.1. The generating function for the Euler polynomials (2.6) gives

$$e^{xt} = \frac{e^t + 1}{2} \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

so that

$$(5.3) \quad \int_0^1 e^{xt} \zeta_E(s, x) dx = \frac{e^t + 1}{2} \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 E_n(x) \zeta_E(s, x) dx.$$

By (4.7), we have

$$(5.4) \quad \frac{n!}{(1-s)_{n+1}} = B(1-s, n+1) = \frac{\Gamma(1-s)\Gamma(n+1)}{\Gamma(n-s+2)}.$$

Therefore, from (4.13) and (5.4), (5.3) gives

$$(5.5) \quad \int_0^1 e^{xt} \zeta_E(s, x) dx = (e^t + 1) \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \delta_2(n-s+1) B(1-s, n+1) \\ \times \lambda(s-n-1).$$

By (4.9), we have

$$(5.6) \quad \lambda(s-n-1) = -\frac{2}{\delta_2(n-s+1)} \frac{\Gamma(n-s+2) \cos\left(\frac{\pi(n-s+2)}{2}\right)}{\pi^{n-s+2}} \lambda(n-s+2),$$

thus in viewing of (5.4) and (5.6), we may also write (5.5) in the form

$$(5.7) \quad \int_0^1 e^{xt} \zeta_E(s, x) dx = (e^t + 1) \Gamma(1-s) \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \delta_2(n-s+1) \frac{\Gamma(n+1)}{\Gamma(n-s+2)} \\ \times \lambda(s-n-1) \\ = \frac{2(e^t + 1) \Gamma(1-s)}{\pi^{2-s}} \sum_{n=0}^{\infty} \left(\frac{t}{\pi}\right)^n (-1)^n \cos\left(\frac{\pi(n-s)}{2}\right) \\ \times \lambda(n-s+2).$$

Finally, by replacing t with $2\pi t$ and using the identity

$$\cos \frac{\pi(n-s)}{2} = \operatorname{Re}(e^{i\pi(s-n)/2}) = \operatorname{Re}((-i)^n e^{i\pi s/2}),$$

we get

$$\int_0^1 e^{2\pi t x} \zeta_E(s, x) dx \\ = \frac{2(e^{2\pi t} + 1) \Gamma(1-s)}{\pi^{2-s}} \sum_{n=0}^{\infty} (-2t)^n \lambda(n-s+2) \cos\left(\frac{\pi(n-s)}{2}\right) \\ = \frac{2(e^{2\pi t} + 1) \Gamma(1-s)}{\pi^{2-s}} \sum_{n=0}^{\infty} (-2t)^n \lambda(n-s+2) \operatorname{Re}((-i)^n e^{i\pi s/2})$$

$$\begin{aligned}
&= \frac{2(e^{2\pi t} + 1)\Gamma(1-s)}{\pi^{2-s}} \operatorname{Re} \left(e^{\pi i s/2} \sum_{n=0}^{\infty} \lambda(n-s+2)(2it)^n \right) \\
&= \frac{2(e^{2\pi t} + 1)\Gamma(1-s)}{\pi^{2-s}} \operatorname{Re} \left(e^{\pi i s/2} F(2it, s) \right).
\end{aligned}$$

This completes our proof. \square

Corollary 5.3. *Let $m \in \mathbb{N}_0$ and $|t| < 1, t \neq 0$. Then we have*

$$\begin{aligned}
(5.8) \quad \int_0^1 e^{2\pi t x} E_m(x) dx &= \frac{(-1)^m 4(e^{2\pi t} + 1)m!}{(2\pi t)^{m+2}} \\
&\quad \times \left(\frac{\pi t}{2} \tanh(\pi t) - \sum_{r=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^r \lambda(2r+2)(2t)^{2r+2} \right),
\end{aligned}$$

where $\lfloor x \rfloor$ is the floor (greatest integer) function.

Proof. Suppose that $k \in \mathbb{N}_0$. Let $s = -2k$ in (5.2). From (5.1), we have

$$\begin{aligned}
(5.9) \quad \operatorname{Re} \left(e^{\pi i s/2} F(2it, s) \right) &= \operatorname{Re} \left(e^{\pi i(-2k)/2} F(2it, -2k) \right) \\
&= (-1)^k \operatorname{Re} \left(\sum_{r=0}^{\infty} \lambda(r+2+2k)(2it)^r \right) \\
&= (-1)^k \sum_{r=0}^{\infty} \lambda(2r+2+2k)(-1)^r (2t)^{2r}.
\end{aligned}$$

From [36, p. 37, Eq. 3:14:7], we obtain

$$\tanh \left(\frac{\pi x}{2} \right) = -\frac{4}{\pi x} \sum_{r=1}^{\infty} (-1)^r \lambda(2r) x^{2r},$$

so

$$\begin{aligned}
(5.10) \quad \frac{\pi x}{4} \tanh \left(\frac{\pi x}{2} \right) &= \sum_{r=0}^{\infty} (-1)^r \lambda(2r+2) x^{2r+2} \\
&= \sum_{r=k}^{\infty} (-1)^r \lambda(2r+2) x^{2r+2} + \sum_{r=0}^{k-1} (-1)^r \lambda(2r+2) x^{2r+2}.
\end{aligned}$$

Replacing x by $2t$ in (5.10), we have

$$(5.11) \quad \sum_{r=k}^{\infty} (-1)^r \lambda(2r+2)(2t)^{2r+2} = \frac{\pi t}{2} \tanh(\pi t) - \sum_{r=0}^{k-1} (-1)^r \lambda(2r+2)(2t)^{2r+2}.$$

On the other hand, by (5.9), we have

$$\begin{aligned}
 & (2t)^{-2k-2} \sum_{r=k}^{\infty} (-1)^r \lambda(2r+2)(2t)^{2r+2} \\
 (5.12) \quad &= (-1)^k \sum_{r=0}^{\infty} \lambda(2r+2+2k)(-1)^r (2t)^{2r} \\
 &= \operatorname{Re} \left(e^{\pi i(-2k)/2} F(2it, -2k) \right)
 \end{aligned}$$

for $|t| < 1, t \neq 0$. By (5.11) and (5.12), we get

$$\begin{aligned}
 & \operatorname{Re} \left(e^{\pi i(-2k)/2} F(2it, -2k) \right) \\
 (5.13) \quad &= \frac{(-1)^{2k}}{(2t)^{2k+2}} \left(\frac{\pi t}{2} \tanh(\pi t) - \sum_{r=0}^{k-1} (-1)^r \lambda(2r+2)(2t)^{2r+2} \right).
 \end{aligned}$$

Let $s = -2k - 1$ in (5.2), similar with the above, we obtain

$$\begin{aligned}
 & \operatorname{Re} \left(e^{\pi i s/2} F(2it, s) \right) \\
 (5.14) \quad &= \operatorname{Re} \left(e^{\pi i(-2k-1)/2} F(2it, -2k-1) \right) \\
 &= \frac{(-1)^{2k+1}}{(2t)^{2k+3}} \left(\frac{\pi t}{2} \tanh(\pi t) - \sum_{r=0}^k (-1)^r \lambda(2r+2)(2t)^{2r+2} \right).
 \end{aligned}$$

The proof now follows directly from (2.7), (5.2), (5.13) and (5.14). \square

6. An expression for Catalan's constant and an Euler-type formula for the Dirichlet beta function

Following Berndt [9, p. 200, Entry 17(v)], Espinosa and Moll [15, p. 177, (8.1)], for $x > 0$, we define

$$(6.1) \quad G_E(s, x) = \zeta_E(s, x) - \zeta_E(s, 1-x).$$

From (1.8), the Fourier expansion for $\zeta_E(s, x)$, we have the following Fourier expansion of $G_E(s, x)$

$$(6.2) \quad G_E(s, x) = \frac{4\Gamma(1-s)}{\pi^{1-s}} \sin\left(\frac{\pi s}{2}\right) \sum_{n=0}^{\infty} \frac{\cos((2n+1)\pi x)}{(2n+1)^{1-s}}.$$

We introduce the anti-symmetrized Hurwitz transform for $G_E(s, x)$ as follows (also see [15])

$$(6.3) \quad \mathfrak{H}_A(f) = \frac{1}{2} \int_0^1 f(w, x) G_E(s, x) dx,$$

where $f(w, x)$ is given in (3.6). For a function $f(w, x)$ with Fourier expansion as in (3.6), we have

$$(6.4) \quad \frac{1}{2} \int_0^1 f(w, x) G_E(s, x) dx = \frac{\Gamma(1-s)}{\pi^{1-s}} \sin\left(\frac{\pi s}{2}\right) \sum_{n=0}^{\infty} \frac{a_n(w)}{(2n+1)^{1-s}}.$$

From [17, p. 408, 3.612], we get

$$(6.5) \quad \int_0^1 \frac{\cos((2n+1)\pi x)}{\cos(\pi x)} dx = (-1)^n.$$

By (6.2) and (6.5), we see that

$$(6.6) \quad \frac{1}{2} \int_0^1 \frac{G_E(s, x)}{\cos(\pi x)} dx = \frac{2\Gamma(1-s)}{\pi^{1-s}} \sin\left(\frac{\pi s}{2}\right) \beta(1-s),$$

where

$$(6.7) \quad \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

is the Dirichlet beta function (see [1, p. 807, 23.2.21]). It is necessary to remark that by the procedure of analytic continuation, the function $\beta(s)$ can extend to all points in the complex plane, without any singularities. In particular, when $s = -1$, (6.6) becomes

$$(6.8) \quad \frac{1}{2} \int_0^1 \frac{G_E(-1, x)}{\cos(\pi x)} dx = -\frac{2}{\pi^2} \beta(2).$$

Espinosa and Moll [15, p. 178, (8.8)] established the integral representation of the Catalan constant G (see [2]) as

$$(6.9) \quad \int_0^1 \frac{\frac{1}{2} - x}{\cos(\pi x)} dx = \frac{4G}{\pi^2}.$$

Using (2.7), (6.1) with $s = -1$ and (6.9), the left hand side of (6.8) becomes

$$(6.10) \quad \begin{aligned} \frac{1}{2} \int_0^1 \frac{G_E(-1, x)}{\cos(\pi x)} dx &= \frac{1}{2} \int_0^1 \frac{x - \frac{1}{2}}{\cos(\pi x)} dx \\ &= -\frac{2G}{\pi^2}. \end{aligned}$$

From (6.8) and (6.10), we then have

$$(6.11) \quad G = \beta(2) = -\frac{\pi^2}{4} \int_0^1 \frac{G_E(-1, x)}{\cos(\pi x)} dx.$$

This form of Catalan's constant is Entry 14 and 19 in the list of expressions for G compiled by Adamchik in [2].

From (6.6), we have the following lemma, which were given in [1, p. 807, 23.2.23], [36, p. 36, Eq. 3:13:4] and [40, (4.13)].

Lemma 6.1. *Let $m \in \mathbb{N}$. Then we have*

$$(6.12) \quad \int_0^1 \sec(\pi x) E_{2m-1}(x) dx = (-1)^m \frac{4(2m-1)!}{\pi^{2m}} \beta(2m).$$

Proof. For $m \in \mathbb{N}$, letting $s = -(2m-1)$ in (6.6), we have

$$(6.13) \quad \frac{1}{2} \int_0^1 \frac{G_E(-(2m-1), x)}{\cos(\pi x)} dx = \frac{2(2m-1)!}{\pi^{2m}} (-1)^m \beta(2m).$$

It is easy to see that by (2.6),

$$E_m(1-x) = (-1)^m E_m(x), \quad m \in \mathbb{N}_0.$$

Hence, by (2.7) and (6.1), the left hand side of (6.13) becomes

$$(6.14) \quad \begin{aligned} \frac{1}{2} \int_0^1 \frac{G_E(-(2m-1), x)}{\cos(\pi x)} dx &= \frac{1}{4} \int_0^1 \frac{E_{2m-1}(x) - E_{2m-1}(1-x)}{\cos(\pi x)} dx \\ &= \frac{1}{2} \int_0^1 \frac{E_{2m-1}(x)}{\cos(\pi x)} dx \end{aligned}$$

for $m \in \mathbb{N}$. The proof now follows directly from (6.13) and (6.14). \square

We now evaluate $\beta(2m)$ using (4.17). This new formula can be regarded as an analogue of Theorem 1 of [24].

Theorem 6.2 (Euler-type formula for $\beta(2m)$). *Let $m \in \mathbb{N}$. Then we have*

$$(6.15) \quad \beta(2m) = \sum_{n=1}^{\infty} \frac{(-1)^{n+m} \pi^{2m+2n} E_{2n}}{4} \left(\sum_{j=1}^n \frac{E_{2m+2j-1}(0)}{(2n-2j+1)!(2m+2j-1)!} \right).$$

Proof. From the Taylor expansion for $\sec(\pi x)$, namely

$$\sec(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} (\pi x)^{2n},$$

it is clear that

$$\int_0^1 \sec(\pi x) E_{2m-1}(x) dx = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} E_{2n}}{(2n)!} \int_0^1 x^{2n} E_{2m-1}(x) dx,$$

where $m \in \mathbb{N}$. Note that $E_{2m}(0) = 0, m \in \mathbb{N}$. Therefore, by (4.17) and (6.12), we get our result. \square

For instance, if we set $m = 1$ in (6.15), then we obtain:

Corollary 6.3.

$$\beta(2) = G = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \pi^{2n+2} E_{2n}}{4(2n+2)!}.$$

Remark 6.4. For $m \in \mathbb{N}_0$ we write $s = -2m$ in (1.8) and let $x = 1/2$. We obtain easily

$$(6.16) \quad \zeta_E \left(-2m, \frac{1}{2} \right) = \frac{(-1)^m 2(2m)!}{\pi^{2m+1}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2m+1}},$$

so (6.16) becomes, after using (2.7),

$$(6.17) \quad \frac{1}{2} E_{2m} \left(\frac{1}{2} \right) = \frac{(-1)^m 2(2m)!}{\pi^{2m+1}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2m+1}}.$$

By using $E_{2m} = 2^{2m} E_{2m}(1/2)$, $m \in \mathbb{N}_0$, we are easy to show that, for the Dirichlet beta function (6.7),

$$(6.18) \quad \beta(2m+1) = (-1)^m \frac{E_{2m}}{2^{2m+2}(2m)!} \pi^{2m+1},$$

where E_m are the Euler number, that is, the integers obtained as the coefficients of $z^m/m!$ in the Taylor expansion of $1/\cosh z$, $|z| < \pi/2$ (see [24, (3)] and [40, (4.10)]). The formula (6.12) for $\beta(2m)$ gives us

$$(6.19) \quad \beta(2m) = (-1)^m \frac{\pi^{2m}}{4(2m-1)!} \int_0^1 \sec(\pi x) E_{2m-1}(x) dx,$$

which is also Equation 23.2.23 of [1], Equation 3:13:4 of [36] and (4.13) of [40]. (6.19) may give some hint for the question “why it is so difficult to find closed-form expressions for even beta values?” However, Lima [24] has derived the exact closed-form expression for a certain class of zeta series related to $\beta(2m)$ and a finite number of odd zeta values.

7. Integral representations of Lerch-type Euler zeta functions and special values for Lerch-type Euler zeta functions at rational arguments

The counterpart of the Hurwitz-type Euler zeta function, the Lerch-type Euler zeta function $\ell_{E,s}(x)$ of the complex exponential argument is defined by

$$(7.1) \quad \ell_{E,s}(x) = \sum_{n=0}^{\infty} \frac{e^{(2n+1)\pi i x}}{(2n+1)^s}$$

for $\operatorname{Re}(s) > 1$ and $x \in \mathbb{R}$ (see [12, p. 1530, (7)]). The Hurwitz-type Euler zeta function and the Lerch-type Euler zeta function are related by the functional equation

$$(7.2) \quad \zeta_E(1-s, x) = \frac{\Gamma(s)}{\pi^s} \left(e^{-\frac{\pi i s}{2}} \ell_{E,s}(x) - e^{\frac{\pi i s}{2}} \ell_{E,s}(1-x) \right),$$

the inverse is

$$(7.3) \quad \ell_{E,s}(x) = \frac{\Gamma(1-s)}{2\pi^{1-s}} \left(e^{\frac{\pi i(1-s)}{2}} \zeta_E(1-s, x) - e^{-\frac{\pi i(1-s)}{2}} \zeta_E(1-s, 1-x) \right).$$

The limiting case $x \rightarrow 1^-$ of (7.2) leads to the asymmetric form of the functional equation for $\zeta_E(s)$ and $\lambda(s)$

$$(7.4) \quad \zeta_E(1-s) = -\frac{2\Gamma(s)}{\pi^s} \cos\left(\frac{\pi s}{2}\right) \lambda(s),$$

where $\lambda(s)$ is the Dirichlet lambda function (2.12). Similarly, in (7.3), with $x = \frac{1}{2}$, we have

$$(7.5) \quad \ell_{E,s}\left(\frac{1}{2}\right) = \frac{i\Gamma(1-s)}{\pi^{1-s}} \sin\left(\frac{\pi(1-s)}{2}\right) \zeta_E\left(1-s, \frac{1}{2}\right).$$

Theorem 7.1. *Let $s, s' \in \mathbb{R}_0^-$. Then we have*

$$(7.6) \quad \int_0^1 \zeta_E(s', x) \ell_{E,s}(x) dx = \pi^{s'-1} \Gamma(1-s') e^{\frac{\pi i}{2}(1-s')} \lambda(1+s-s').$$

Proof. We have the functional equation for the Hurwitz-type Euler zeta function (see (7.3)):

$$\ell_{E,s}(x) = \frac{i\Gamma(1-s)}{2\pi^{1-s}} \left(e^{-\frac{\pi i s}{2}} \zeta_E(1-s, x) + e^{\frac{\pi i s}{2}} \zeta_E(1-s, 1-x) \right).$$

From which, we obtain

$$(7.7) \quad \int_0^1 \zeta_E(s', x) \ell_{E,s}(x) dx = \frac{i\Gamma(1-s)}{2\pi^{1-s}} \left[e^{-\frac{\pi i s}{2}} \int_0^1 \zeta_E(s', x) \zeta_E(1-s, x) dx + e^{\frac{\pi i s}{2}} \int_0^1 \zeta_E(s', x) \zeta_E(1-s, 1-x) dx \right].$$

Substituting (4.1) and (4.2) in the integral (7.7), we deduce that

$$(7.8) \quad \int_0^1 \zeta_E(s', x) \ell_{E,s}(x) dx = \frac{i\Gamma(1-s)\Gamma(s)\Gamma(1-s')}{\pi^{2-s'}} \lambda(1+s-s') \times \left[e^{-\frac{\pi i s}{2}} \cos\left(\frac{\pi(1-s-s')}{2}\right) + e^{\frac{\pi i s}{2}} \cos\left(\frac{\pi(1-s+s')}{2}\right) \right].$$

Now the factor on the right hand side of (7.8) is

$$\left[e^{-\frac{\pi i s}{2}} \cos\left(\frac{\pi(1-s-s')}{2}\right) + e^{\frac{\pi i s}{2}} \cos\left(\frac{\pi(1-s+s')}{2}\right) \right] = e^{-\frac{\pi i s'}{2}} \sin(\pi s).$$

Therefore, from the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)},$$

we have

$$\int_0^1 \zeta_E(s', x) \ell_{E,s}(x) dx = \frac{i\Gamma(1-s')}{\pi^{1-s'}} e^{-\frac{\pi i s'}{2}} \lambda(1+s-s').$$

This completes the proof. \square

Let $\phi(x, a, s)$ be the power Dirichlet series defined for $\operatorname{Re}(s) > 1$, x real, $a \neq$ negative integer or zero, by the series

$$(7.9) \quad \phi(x, a, s) = \sum_{n=0}^{\infty} \frac{e^{2n\pi ix}}{(n+a)^s}$$

(see [3, p. 161, (1.1)]). It is known that $\phi(x, a, s)$ can be extended to whole s -plane by means of the contour integral. The value of $\phi(x, a, s)$ when s is 0 or a negative integer can be calculated by applying Cauchy's residue theorem. For $m \in \mathbb{N}_0$, the Apostol's formula (see [3, p. 164]) gives

$$(7.10) \quad \phi(x, a, -m) = -\frac{B_{m+1}(a, e^{2\pi ix})}{m+1},$$

where $B_m(a, \alpha)$ are the Apostol-Bernoulli polynomials defined by

$$(7.11) \quad \frac{ze^{az}}{\alpha e^z - 1} = \sum_{m=0}^{\infty} B_m(a, \alpha) \frac{z^m}{m!}.$$

We refer to [3, 8, 20, 27, 32] for an account of the properties of Apostol-Bernoulli polynomials $B_m(a, \alpha)$. For a later purpose, we observe that, by (7.1) and (7.9), the following equality is established,

$$(7.12) \quad \ell_{E,s}(x) = 2^{-s} e^{\pi ix} \phi\left(x, \frac{1}{2}, s\right),$$

so that by (7.10) and (7.12),

$$(7.13) \quad \ell_{E,-m}(x) = -2^m e^{\pi ix} \frac{B_{m+1}\left(\frac{1}{2}, e^{2\pi ix}\right)}{m+1},$$

where $m \in \mathbb{N}_0$.

We observe that the function $\zeta_E(s, x)$ in (1.5) is a linear combination of the Hurwitz zeta functions $\zeta(s, x)$ in (1.1) when x is a rational number.

Theorem 7.2. *Let p, q be integers, $q \in \mathbb{N}$ and $1 \leq p \leq q$. Then for all $s \in \mathbb{C}$, we have*

$$(7.14) \quad \zeta_E\left(1-s, \frac{p}{q}\right) = \frac{2\Gamma(s)}{(2q\pi)^s} \sum_{r=0}^{q-1} \cos\left(\frac{\pi s}{2} - \frac{(2r+1)\pi p}{q}\right) \zeta\left(s, \frac{2r+1}{2q}\right).$$

Remark 7.3. This is an analogue of [5, p. 261, Theorem 12.8], where the Hurwitz zeta function $\zeta(s, x)$ instead of $\zeta_E(s, x)$.

Proof of Theorem 7.2. Following an idea in [37, p. 337, (8)], we have

$$(7.15) \quad \sum_{n=1}^{\infty} f(n) = \sum_{r=1}^q \sum_{k=0}^{\infty} f(kq+r)$$

if the involved series is absolutely convergent. Applying (7.15) to the series (7.1), we have (see [12, p. 1530, (8b)] and [11, p. 1487, (3.4)])

$$(7.16) \quad \ell_{E,s} \left(\frac{p}{q} \right) = \frac{1}{(2q)^s} \sum_{r=1}^q e^{(2r-1)\pi i \frac{p}{q}} \zeta \left(s, \frac{2r-1}{2q} \right).$$

Therefore, if we take $x = \frac{p}{q}$ in the formula (7.2) to (7.16), in considerations of $\ell_{E,s}(1-x) = -\ell_{E,s}(x)$, we have the following equality

$$\begin{aligned} \zeta_E \left(1-s, \frac{p}{q} \right) &= \frac{\Gamma(s)}{\pi^s} \left(e^{-\frac{\pi i s}{2}} \ell_{E,s} \left(\frac{p}{q} \right) + e^{\frac{\pi i s}{2}} \ell_{E,s} \left(-\frac{p}{q} \right) \right) \\ &= \frac{2\Gamma(s)}{(2q\pi)^s} \sum_{r=1}^q \cos \left(\frac{\pi s}{2} - \frac{(2r-1)\pi p}{q} \right) \zeta \left(s, \frac{2r-1}{2q} \right), \end{aligned}$$

which holds true, by the principle of analytic continuation, for all admissible values of $s \in \mathbb{C}$. \square

Corollary 7.4 ([12, p. 1529, Theorem B]). *Let p, q be integers, $q \in \mathbb{N}$ and $1 \leq p \leq q$. For $m \in \mathbb{N}$, the Euler polynomials $E_m(x)$ at rational arguments are given by*

$$(7.17) \quad E_m \left(\frac{p}{q} \right) = \frac{4m!}{(2q\pi)^{m+1}} \sum_{r=1}^q \sin \left(\frac{(2r-1)\pi p}{q} - \frac{m\pi}{2} \right) \zeta \left(m+1, \frac{2r-1}{2q} \right).$$

Proof. By setting $s = m+1$ ($m \in \mathbb{N}_0$) in (7.14) and using (2.7), we obtain the formula in (7.17), which completes the proof. \square

Corollary 7.5. *Let p, q be integers, $q \in \mathbb{N}$ and $1 \leq p < q$. Then we have*

$$(7.18) \quad B_{m+1} \left(\frac{1}{2}, e^{2\pi i \frac{p}{q}} \right) = q^m \sum_{r=1}^q e^{2(r-1)\pi i \frac{p}{q}} B_{m+1} \left(\frac{2r-1}{2q} \right), \quad m \in \mathbb{N}_0.$$

Proof. Letting $m \in \mathbb{N}_0$, then replacing s by $-m$ in (7.16), using (1.3) and (7.13), we get our result. \square

Remark 7.6. This identity is also a special case of the multiplication theorem for Apostol-Bernoulli polynomials (see eg. [28, Eq. (17)]).

Since $\ell_{E,s}(x)$ is a relative of the Hurwitz-type Euler zeta function $\zeta_E(s, x)$ defined (1.5) (see (7.2) and (7.3) above), it is natural to expect a relation involving $\zeta(s, x)$:

Lemma 7.7 (Generalized Eisenstein formula). *Let p, q be integers, $q \in \mathbb{N}$ and $1 \leq p < q$. Then we have*

$$(7.19) \quad \zeta \left(s, \frac{2p-1}{2q} \right) = \frac{1}{q} \sum_{r=1}^q (2q)^s e^{-(2p-1)\pi i \frac{r}{q}} \ell_{E,s} \left(\frac{r}{q} \right).$$

Remark 7.8. The equation (7.19) was announced without a proof in [11, p. 1487, (3.3)].

Proof of Lemma 7.7. The formula (1.1) could be written in the following form

$$\begin{aligned}
 \zeta\left(s, \frac{2p+1}{2q}\right) &= (2q)^s \sum_{k=0}^{\infty} \frac{1}{(2(qk+p)+1)^s} \\
 (7.20) \qquad \qquad \qquad &= (2q)^s \sum_{\substack{n=0 \\ n \equiv p \pmod{q}}}^{\infty} \frac{1}{(2n+1)^s}.
 \end{aligned}$$

Note that in fact (see [5, p. 158, Theorem 8.1])

$$\frac{1}{q} \sum_{r=1}^q e^{2\pi i(n-p)\frac{r}{q}} = \begin{cases} 1 & \text{if } n \equiv p \pmod{q} \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\begin{aligned}
 \zeta\left(s, \frac{2p+1}{2q}\right) &= (2q)^s \sum_{n=0}^{\infty} \frac{\frac{1}{q} \sum_{r=1}^q e^{2\pi i(n-p)\frac{r}{q}}}{(2n+1)^s} \\
 (7.21) \qquad \qquad \qquad &= \frac{1}{q} \sum_{r=1}^q (2q)^s \sum_{n=0}^{\infty} \frac{e^{(2n+1)\pi i\frac{r}{q}}}{(2n+1)^s} e^{-(2p+1)\pi i\frac{r}{q}}
 \end{aligned}$$

for $1 \leq p < q$, the conclusion is immediate. □

Corollary 7.9. *Let p, q be integers, $q \in \mathbb{N}$ and $1 \leq p < q$. Then we have*

$$(7.22) \quad B_{m+1}\left(\frac{2p-1}{2q}\right) = \frac{1}{q^{m+1}} \sum_{r=1}^q e^{-2(p-1)\pi i\frac{r}{q}} B_{m+1}\left(\frac{1}{2}, e^{2\pi i\frac{r}{q}}\right), \quad m \in \mathbb{N}_0.$$

Here, the term $B_{m+1}\left(\frac{1}{2}, 1\right)$ is understood as $B_{m+1}\left(\frac{1}{2}\right)$ (the value of the Bernoulli polynomials $B_{m+1}(x)$ at $x = \frac{1}{2}$).

Proof. Let $m \in \mathbb{N}_0$, and replace s by $-m$ in (7.19). Then, using (1.3) and (7.13), we get our result. □

Theorem 7.10. *Let p, q be integers, $q \in \mathbb{N}$ and $1 \leq p \leq q$. Then we have the identities*

$$\begin{aligned}
 &\frac{1}{q} \sum_{r=1}^q (2q)^s e^{-(2p+1)\pi i\frac{r}{q}} \ell_{E,s}\left(\frac{r}{q}\right) \\
 (7.23) \qquad \qquad \qquad &= \frac{\Gamma(1-s)}{\pi^{1-s}} \left\{ e^{-\frac{\pi i(1-s)}{2}} \ell_{E,1-s}(x) - e^{\frac{\pi i(1-s)}{2}} \ell_{E,1-s}(1-x) \right\}
 \end{aligned}$$

and

$$(7.24) \quad \begin{aligned} & \frac{1}{(2q)^s} \sum_{r=1}^q e^{(2r-1)\pi i \frac{p}{q}} \zeta \left(s, \frac{2r-1}{2q} \right) \\ &= \frac{\Gamma(1-s)}{2\pi^{1-s}} \left\{ e^{\frac{\pi i(1-s)}{2}} \zeta_E \left(1-s, \frac{p}{q} \right) - e^{-\frac{\pi i(1-s)}{2}} \zeta_E \left(1-s, 1 - \frac{p}{q} \right) \right\}. \end{aligned}$$

Proof. From (7.2), (7.3), (7.16) and (7.19), we prove the theorem. \square

We will use $[x]$ to denote the integer part of $x \in \mathbb{R}$ (i.e., $[x]$ is the largest integer $\leq x$); then, the fractional part of x will be $\{x\} = x - [x]$.

Lemma 7.11. *For $n \in \mathbb{N}$ we have*

$$(7.25) \quad E_n(\{x\}) = 2(-i)^{n-1} n! \sum_{k=0}^{\infty} \frac{(-1)^{n-1} e^{(2k+1)\pi i x} + e^{-(2k+1)\pi i x}}{((2k+1)\pi)^{n+1}}.$$

Proof. Let $\{x\} = x - [x]$. The function $E_n(\{x\})$ is a quasi-periodic function of period 2 (see [20]). The Fourier series of $E_n(\{x\})$ are given as follows ([1, p. 805, 23.1.16]):

$$(7.26) \quad E_n(\{x\}) = \frac{4n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x - \frac{1}{2}\pi n)}{(2k+1)^{n+1}},$$

where $0 \leq x < 1$ if $n \in \mathbb{N}$ and $0 < x < 1$ if $n = 0$, immediately yields

$$(7.27) \quad E_{2m}(\{x\}) = 4(-1)^m (2m)! \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{((2k+1)\pi)^{2m+1}},$$

and

$$(7.28) \quad E_{2m+1}(\{x\}) = 4(-1)^{m-1} (2m+1)! \sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi x)}{((2k+1)\pi)^{2m+2}}.$$

Thus (7.25) follows immediately from (7.27) and (7.28). \square

Theorem 7.12. *Let $m > 1$ be a fixed positive odd integer and let α be an integer such that $\alpha \not\equiv 0 \pmod{m}$. Then, for $n \in \mathbb{N}$, we have*

$$(7.29) \quad \begin{aligned} & \sum_{r=1}^{m-1} (-1)^r e^{-2\pi i r \frac{\alpha}{m}} \ell_{E,1-n} \left(\frac{r}{m} \right) \\ &= \frac{(-1)^{n-1}}{4} \left(m^n E_{n-1} \left(\frac{2\alpha}{m} - \left[\frac{2\alpha}{m} \right] \right) + E_{n-1}(0) \right) \\ & \quad - \frac{1}{2n} \left(m^n B_n \left(\frac{2\alpha}{m} - \left[\frac{2\alpha}{m} \right] \right) + B_n(0) \right). \end{aligned}$$

Proof. It follows, from (7.3), that for odd $m > 1$,

$$\begin{aligned}
& \sum_{r=1}^{m-1} (-1)^r e^{-2\pi i r \frac{\alpha}{m}} \ell_{E,1-n} \left(\frac{r}{m} \right) \\
&= \frac{(n-1)! i^n}{2\pi^n} \left(\sum_{r=1}^{m-1} (-1)^r e^{-2\pi i r \frac{\alpha}{m}} \zeta_E \left(n, \frac{r}{m} \right) \right. \\
&\quad \left. - (-1)^n \sum_{r=1}^{m-1} (-1)^r e^{-2\pi i r \frac{\alpha}{m}} \zeta_E \left(n, \frac{m-r}{m} \right) \right) \\
&= \frac{(n-1)! i^n}{2\pi^n} \left(\sum_{r=1}^{m-1} (-1)^r e^{-2\pi i r \frac{\alpha}{m}} \zeta_E \left(n, \frac{r}{m} \right) \right. \\
&\quad \left. + (-1)^n \sum_{r'=1}^{m-1} (-1)^{r'} e^{-2\pi i r' \frac{\alpha}{m}} \zeta_E \left(n, \frac{r'}{m} \right) \right) \\
(7.30) \quad &= \frac{(n-1)! i^n}{2\pi^n} \sum_{r=1}^{m-1} (-1)^r \sum_{k=0}^{\infty} \frac{(-1)^k [e^{-2\pi i r \frac{\alpha}{m}} + (-1)^n e^{2\pi i r \frac{\alpha}{m}}]}{(k + \frac{\alpha}{m})^n} \\
&= \frac{(n-1)! i^n m^n}{2\pi^n} \sum_{r=1}^{m-1} \sum_{k=0}^{\infty} \frac{(-1)^{mk+r} [e^{-2\pi i \alpha \frac{(mk+r)}{m}} + (-1)^n e^{2\pi i \alpha \frac{(mk+r)}{m}}]}{(mk+r)^n} \\
&= \frac{(n-1)! i^n m^n}{2\pi^n} \sum_{\substack{h=1 \\ h \not\equiv 0 \pmod{m}}}^{\infty} \sum_{h=1}^{\infty} \frac{(-1)^h [e^{-2\pi i \alpha \frac{h}{m}} + (-1)^n e^{2\pi i \alpha \frac{h}{m}}]}{h^n} \\
&= \frac{(n-1)! i^n m^n}{2\pi^n} \left(\sum_{h=1}^{\infty} \frac{(-1)^h [e^{-2\pi i \alpha \frac{h}{m}} + (-1)^n e^{2\pi i \alpha \frac{h}{m}}]}{h^n} \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \frac{(-1)^{mk} [e^{-2\pi i k \alpha} + (-1)^n e^{2\pi i k \alpha}]}{(km)^n} \right).
\end{aligned}$$

In the following, we will use the identities below:

$$(7.31) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} + \sum_{k=1}^{\infty} \frac{1}{(2k)^s}$$

and

$$(7.32) \quad B_n(\{x\}) = -(-i)^n n! \sum_{k=1}^{\infty} \frac{e^{2\pi i k x} + (-1)^n e^{-2\pi i k x}}{(2\pi k)^n}$$

for $n \in \mathbb{N}$ (see [39, p. 13, §3] or [26]). Note that, by (7.25) and $E'_n(\{x\}) = nE_{n-1}(\{x\})$, we have

$$(7.33) \quad E'_n(\{x\}) = 2(-i)^n n! \sum_{k=0}^{\infty} \frac{(-1)^n e^{(2k+1)\pi ix} + e^{-(2k+1)\pi ix}}{\pi^n (2k+1)^n}.$$

Therefore we have

$$(7.34) \quad \begin{aligned} & \sum_{r=1}^{m-1} (-1)^r e^{-2\pi i \frac{r}{m}} \ell_{E,1-n} \left(\frac{r}{m} \right) \\ &= \frac{(n-1)! i^n m^n}{2\pi^n} \left(\sum_{k=1}^{\infty} \frac{e^{-2\pi i (2k) \frac{\alpha}{m}} + (-1)^n e^{2\pi i (2k) \frac{\alpha}{m}}}{(2k)^n} \right. \\ & \quad - \sum_{k=1}^{\infty} \frac{e^{-2\pi i (2k-1) \frac{\alpha}{m}} + (-1)^n e^{2\pi i (2k-1) \frac{\alpha}{m}}}{(2k-1)^n} \\ & \quad + m^{-n} \sum_{k=1}^{\infty} \frac{e^{-2\pi i (2k)\alpha} + (-1)^n e^{2\pi i (2k)\alpha}}{(2k)^n} \\ & \quad \left. - m^{-n} \sum_{k=1}^{\infty} \frac{e^{-2\pi i (2k-1)\alpha} + (-1)^n e^{2\pi i (2k-1)\alpha}}{(2k-1)^n} \right) \\ &= \frac{(-1)^{n-1}}{4n} \left(m^n E'_n \left(\frac{2\alpha}{m} - \left[\frac{2\alpha}{m} \right] \right) + E'_n(2\alpha - [2\alpha]) \right) \\ & \quad - \frac{1}{2n} \left(m^n B_n \left(\frac{2\alpha}{m} - \left[\frac{2\alpha}{m} \right] \right) + B_n(2\alpha - [2\alpha]) \right) \\ &= \frac{(-1)^{n-1}}{4} \left(m^n E_{n-1} \left(\frac{2\alpha}{m} - \left[\frac{2\alpha}{m} \right] \right) + E_{n-1}(0) \right) \\ & \quad - \frac{1}{2n} \left(m^n B_n \left(\frac{2\alpha}{m} - \left[\frac{2\alpha}{m} \right] \right) + B_n(0) \right). \end{aligned}$$

This completes our proof. \square

Remark 7.13. Equation (7.29) is an analogue of Theorem D in [39], which is an extension of the classical Eisenstein formula

$$\frac{\alpha}{m} - \left[\frac{\alpha}{m} \right] - \frac{1}{2} = -\frac{1}{2m} \sum_{\gamma=1}^{m-1} \sin \left(\frac{2\pi\gamma\alpha}{m} \right) \cot \left(\frac{\pi\gamma}{n} \right),$$

where α is an integer with $\alpha \not\equiv 0 \pmod{m}$ and $m > 2$ is a fixed positive integer.

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