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ON THE COEFFICIENTS OF GAMMA-STARLIKE FUNCTIONS

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Abstract. We give several sharp estimates for some initial coefficients problems for the so-called gamma starlike functions f, analytic and univalent in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and normalized so that $f(0) = 0 = f'(0) - 1$, and satisfying $\operatorname{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\right] > 0$.

1. Introduction and definitions

Denote by $\mathcal{A}$ the class of functions f analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with Taylor series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$, consisting of univalent functions. A function $f \in \mathcal{A}$ is called starlike, if $f(\mathbb{D})$ is starlike (with respect to the origin), and convex if $f(\mathbb{D})$ is convex. Let $\mathcal{S}^*$ and $\mathcal{C}$ denote the classes of starlike and convex functions in $\mathcal{S}$ respectively. It is well-known that a function $f \in \mathcal{A}$ belongs to $\mathcal{S}^*$ if, and only, if $\operatorname{Re}(zf'(z)/f(z)) > 0$ for $z \in \mathbb{D}$. Similarly, a function $f \in \mathcal{A}$ belongs to $\mathcal{C}$ if and only if $\operatorname{Re}(1 + (zf''(z)/f'(z))) > 0$ for $z \in \mathbb{D}$. Thus it is easy to see that $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$.

For $\alpha \in \mathbb{R}$, the class $\mathcal{M}_\alpha$ of α-convex functions defined by

$$\operatorname{Re}\left[\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)}\right] > 0,$$

is well-known, and contains a great many interesting properties, the most basic being that for all $\alpha \in \mathbb{R}$, $\mathcal{M}_\alpha \subset \mathcal{S}^*$ [4, 7–9]. Thus $\mathcal{M}_\alpha$ is a natural subset of $\mathcal{S}$, with $\mathcal{M}_0 = \mathcal{S}^*$ and $\mathcal{M}_1 = \mathcal{C}$.

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In contrast, the corresponding class $\mathcal{M}^\gamma$ of so-called gamma-starlike functions, defined for $f \in \mathcal{A}$, and $\gamma \in \mathbb{R}$ by

$$\operatorname{Re}\left[\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\right] > 0, \tag{1.2}$$

has been less well studied. We note again that $\mathcal{M}^0 = \mathcal{S}^*$ and $\mathcal{M}^1 = \mathcal{C}$. The presence of powers in (1.2) obviously creates difficulties, and is probably the reason why relatively little appears to be known about $\mathcal{M}^\gamma$. However, as in the case of $\mathcal{M}_\alpha$, functions in $\mathcal{M}^\gamma$ are also contained in $\mathcal{S}^*$ [5], again providing a natural subset of $\mathcal{S}$.

It is the purpose of this paper to give a series of sharp inequalities involving the initial coefficients of functions in $\mathcal{M}^\gamma$, which complete and extend those given in [2], resulting in most of what is now known about these problems.

2. Preliminaries

We shall need the following lemmas concerning functions with positive real part, (see e.g. [1, 6]).

Denote by $\mathcal{P}$, the set of functions p analytic in $\mathbb{D}$ with Taylor expansion $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, and satisfying $\operatorname{Re} p(z) > 0$ for $z \in \mathbb{D}$.

Lemma 2.1. *For some complex valued y with $|y| \leqslant 1$, and some complex valued ζ with $|\zeta| \leqslant 1$,*

$$\begin{aligned} 2p_2 &= p_1^2 + y(4-p_1^2),\\ 4p_3 &= p_1^3 + 2(4-p_1^2)p_1 y - p_1(4-p_1^2)y^2 + 2(4-p_1^2)(1-|y|^2)\zeta. \end{aligned}$$

Lemma 2.2. $|p_n| \leqslant 2$ *for* $n \geqslant 1$, *and*

$$\begin{aligned} \left|p_2 - \frac{\mu}{2}p_1^2\right| &\leqslant \max\{2,\ 2|\mu-1|\}\\ &= \begin{cases} 2, & 0 \leqslant \mu \leqslant 2,\\ 2|\mu-1|, & \text{elsewhere.} \end{cases} \end{aligned}$$

Lemma 2.3. *If $0 \leqslant B \leqslant 1$, and $B(2B-1) \leqslant D \leqslant B$, then*

$$\left|p_3 - 2Bp_1p_2 + Dp_1^3\right| \leqslant 2.$$

Lemma 2.4.

$$\begin{aligned} \left|p_3 - (1+\mu)p_1p_2 + \mu p_1^3\right| &\leqslant \max\{2,\ 2|2\mu-1|\}\\ &= \begin{cases} 2, & 0 \leqslant \mu \leqslant 1,\\ 2|2\mu-1|, & \text{elsewhere.} \end{cases} \end{aligned}$$

3. The coefficients of $f(z)$

Theorem 3.1. *Let $f \in \mathcal{M}^\gamma$ for $\gamma \geqslant 0$, and be given by* (1.1). *Then*

$$|a_2| \leqslant \frac{2}{1+\gamma},$$

$$|a_3| \leqslant \begin{cases} \frac{3(1+3\gamma)}{(1+\gamma)^2(1+2\gamma)}, & \gamma \leqslant \frac{1}{2}(7+\sqrt{57}), \\ \frac{1}{1+2\gamma}, & \gamma \geqslant \frac{1}{2}(7+\sqrt{57}), \end{cases}$$

$$|a_4| \leqslant \begin{cases} \frac{2(18+113\gamma+292\gamma^2+7\gamma^3+2\gamma^4)}{9(1+\gamma)^3(1+2\gamma)(1+3\gamma)}, & 0 \leqslant \gamma \leqslant \gamma_0, \\ \frac{2}{3(1+3\gamma)}, & \gamma \geqslant \gamma_0, \end{cases}$$

where $\gamma_0 = 6.794\ldots$ *is the unique positive root of* $15+98\gamma+265\gamma^2-14\gamma^3-4\gamma^4 = 0$.
All the inequalities are sharp.

Proof. From (1.2), write

$$\Big(1 + \frac{zf''(z)}{f'(z)}\Big)^{\gamma} \Big(\frac{zf'(z)}{f(z)}\Big)^{1-\gamma} = p(z),$$

where $p \in \mathcal{P}$. Then equating coefficients gives

$$\begin{aligned} a_2 &= \frac{p_1}{1+\gamma}, \\ a_3 &= \frac{(2+7\gamma-\gamma^2)p_1^2}{4(1+\gamma)^2(1+2\gamma)} + \frac{p_2}{2(1+2\gamma)}, \\ a_4 &= \frac{(6+23\gamma+154\gamma^2-47\gamma^3+8\gamma^4)p_1^3}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)}, \\ &\quad + \frac{(3+19\gamma-4\gamma^2)p_1p_2}{6(1+\gamma)(1+2\gamma)(1+3\gamma)} + \frac{p_3}{3(1+3\gamma)}. \end{aligned} \tag{3.1}$$

The inequality for $|a_2|$ is trivial.

The first inequality for $|a_3|$, is obvious on noting that the coefficient of p_1^2 is positive when $\gamma \leqslant \frac{1}{2}(7+\sqrt{57})$, and applying the inequalities $|p_1| \leqslant 2$ and $|p_2| \leqslant 2$. The second inequality follows by a simple application of Lemma 2.2.

For $|a_4|$, write

$$a_4 = \frac{1}{3(1+3\gamma)}\Big(p_3 - \frac{(4\gamma^2-19\gamma-3)p_1p_2}{2(1+\gamma)(1+2\gamma)} + \frac{(6+23\gamma+154\gamma^2-47\gamma^3+8\gamma^4)p_1^3)}{12(1+\gamma)^3(1+2\gamma)}\Big).$$

Then since the coefficients of p_1p_2 and p_1^3 are positive when $\gamma \leqslant \frac{1}{8}(19+\sqrt{409})$, the first inequality for $|a_4|$ is valid on this interval on using the inequalities $|p_n| \leqslant 2$ for $n = 1, 2$ and 3.
We now use Lemma 2.3 (in the case $B = D$), with $B = \frac{4\gamma^2-19\gamma-3}{4(1+\gamma)(1+2\gamma)}$.

First note that $0 \leqslant B \leqslant 1$, when $\gamma \geqslant \frac{1}{8}(19+\sqrt{409})$, and so writing

$$a_4 = \frac{1}{3(1+3\gamma)}\Big(p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3\Big),$$

with $D = \frac{6+23\gamma+154\gamma^2-47\gamma^3+8\gamma^4}{12(1+\gamma)^3(1+2\gamma)}$, we see that $D - B \geqslant 0$ when $\frac{1}{8}(19+\sqrt{409}) \leqslant \gamma \leqslant \gamma_0$.

Applying the inequality $|p_3 - 2Bp_1p_2 + Bp_1^3| \leqslant 2$ from Lemma 2.3, and again using the inequalities $|p_n| \leqslant 2$ for $n = 1, 2$ and 3, gives the first inequality for $|a_4|$ on the interval $\frac{1}{8}(19 + \sqrt{409}) \leqslant \gamma \leqslant \gamma_0$.

Thus it remains to establish the inequality for $|a_4|$ on the interval $\gamma \geqslant \gamma_0$. We again use Lemma 2.3.

It is easy to see that both $0 \leqslant B \leqslant 1$ and $B(2B-1) \leqslant D \leqslant B$ hold when $\gamma \geqslant \gamma_0$, and so applying Lemma 2.3 gives the second inequality for $|a_4|$ at once.

To see that the above inequalities are sharp, we note that equality is attained in the inequality for $|a_2|$, and the first inequalities for $|a_3|$ and $|a_4|$ when $f(z)$ in (3.1) is chosen so that

$$\Big(1 + \frac{zf''(z)}{f'(z)}\Big)^{\gamma}\Big(\frac{zf'(z)}{f(z)}\Big)^{1-\gamma} = \frac{1+z}{1-z}.$$

The second inequality for $|a_3|$ is sharp when

$$\Big(1 + \frac{zf''(z)}{f'(z)}\Big)^{\gamma}\Big(\frac{zf'(z)}{f(z)}\Big)^{1-\gamma} = \frac{1+z^2}{1-z^2},$$

and the second inequality for $|a_4|$ is sharp when

$$\Big(1 + \frac{zf''(z)}{f'(z)}\Big)^{\gamma}\Big(\frac{zf'(z)}{f(z)}\Big)^{1-\gamma} = \frac{1+z^3}{1-z^3}.$$

□

4. The coefficients of $\log(f(z)/z)$

The logarithmic coefficients δ_n of a function $f \in \mathcal{S}$ are defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \delta_n z^n, \tag{4.1}$$

and play a central role in the theory of univalent functions. On differentiating $\log \frac{f(z)}{z}$, it is a trivial consequence of the inequality $|p_n| \leqslant 2$, that for $n \geqslant 1$, $|\delta_n| \leqslant 1/n$ when $f \in \mathcal{S}^*$, and $|\delta_n| \leqslant 1/2n$ when $f \in \mathcal{C}$.

However when $f \in \mathcal{M}^{\gamma}$, the same procedure does not give a convenient expression in terms of $1 + zf''(z)/f'(z)$, or $zf'(z)/f(z)$, unless $\gamma = 0$ or 1. We show next that it is however possible to obtain sharp estimates for the modulus of the initial coefficients of $\log \frac{f(z)}{z}$ when $f \in \mathcal{M}^{\gamma}$.

We prove the following.

Theorem 4.1. *Let $f \in \mathcal{M}^{\gamma}$ for $\gamma \geqslant 0$, and the coefficients of $\log \frac{f(z)}{z}$ be given by (4.1).*
Then

$$|\delta_1| \leqslant \frac{1}{1+\gamma},$$

$$|\delta_2| \leqslant \begin{cases} \frac{1+5\gamma}{2(1+\gamma)^2(1+2\gamma)}, & 0 \leqslant \gamma \leqslant 3, \\ \frac{1}{2(1+2\gamma)}, & \gamma \geqslant 3, \end{cases}$$

$$|\delta_3| \leqslant \begin{cases} \frac{3+11\gamma+121\gamma^2+7\gamma^3+2\gamma^4}{9(1+\gamma)^3(1+2\gamma)(1+3\gamma)}, & 0 \leqslant \gamma \leqslant \gamma_1, \\ \\ \frac{1}{3(1+3\gamma)}, & \gamma \geqslant \gamma_1, \end{cases}$$

where $\gamma_1 = 3.3751\ldots$ *is the unique positive root of the equation* $2-47\gamma+7\gamma^2+2\gamma^3 = 0$.
All the inequalities are sharp.

Proof. First note that differentiating (4.1), and equating coefficients gives

$$\begin{aligned}\delta_1 &= \frac{1}{2}a_2,\\ \delta_2 &= \frac{1}{2}(a_3 - \frac{1}{2}a_2^2),\\ \delta_3 &= \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3).\end{aligned}$$

Using Theorem 2.1, the inequality for $|\delta_1|$ is trivial.

For $|\delta_2|$ substituting for a_2 and a_3 we obtain

$$\delta_2 = \frac{1}{4(1+2\gamma)}\Big(p_2 - \frac{(-3+\gamma)\gamma p_1^2}{2(1+\gamma)^2}\Big),$$

and applying Lemma 2.2 with $\mu = \frac{(-3+\gamma)\gamma}{(1+\gamma)^2}$, easily gives the inequalities for $|\delta_2|$.

For δ_3, we again substitute from (3.1) to obtain

$$\delta_3 = \frac{\gamma(-17+23\gamma-10\gamma^2+4\gamma^3)p_1^3}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)} + \frac{\gamma(5-2\gamma)p_1p_2}{6(1+\gamma)(1+2\gamma)(1+3\gamma)} + \frac{p_3}{6(1+3\gamma)}.$$

First note that since the coefficients of p_1^3, p_1p_2 and p_3 are all positive on $1 \leqslant \gamma \leqslant 5/2$, using the inequality $|p_n| \leqslant 2$ for $n = 1, 2, 3$, the first inequality for $|\delta_3|$ in Theorem 4.1 follows when $1 \leqslant \gamma \leqslant 5/2$.

Next write the above expression for δ_3 as

$$\delta_3 = \frac{1}{6(1+3\gamma)}(p_3 - 2Bp_1p_2 + Dp_1^3),$$

where

$$B = \frac{\gamma(2\gamma-5)}{(1+\gamma)(1+2\gamma)} \quad \text{and} \quad D = \frac{\gamma(-17+23\gamma-10\gamma^2+4\gamma^3)}{6(1+\gamma)^3(1+2\gamma)}.$$

We now use Lemma 2.3, so that $0 \leqslant B \leqslant 1$, when $\gamma \geqslant 5/2$, and $B(2B-1) \leqslant D \leqslant B$, when $\gamma \geqslant \gamma_1$, and so Lemma 2.3 gives the second bound for $|\delta_3|$ in Theorem 4.1 when $\gamma \geqslant \gamma_1$.

Next write

$$\delta_3 = \frac{1}{6(1+3\gamma)}(p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3),$$

and note that $D - B \geqslant 0$ when $0.428\ldots \leqslant \gamma \leqslant \gamma_1$.

We now use Lemma 2.3 with $B = D$, and recalling that since $0 \leqslant B \leqslant 1$, we also require that $\gamma \geqslant 5/2$, to obtain the first inequality for $|\delta_3|$ on the interval $5/2 \leqslant \gamma \leqslant \gamma_1$.

Thus we are left to prove the first inequality for $|\delta_3|$ on the interval $0 \leqslant \gamma \leqslant 1$.

We now use Lemma 2.1 to express the coefficients p_2 and p_3 in terms of p_1 to obtain, after simplification, normalizing the coefficient p_1 so that $p_1 = p$ where $0 \leqslant p \leqslant 2$, and finally using the triangle inequality,

$$|\delta_3| \leqslant \frac{(3+11\gamma+121\gamma^2+7\gamma^3+2\gamma^4)p^3}{72(1+\gamma)^3(1+2\gamma)(1+3\gamma)} + \frac{(1+8\gamma)p(4-p^2)|y|}{12(1+\gamma)(1+2\gamma)(1+3\gamma)} + \frac{p(4-p^2)|y|^2}{24(1+3\gamma)} + \frac{(4-p^2)(1-|y|^2)}{12(1+3\gamma)} := \phi(p,|y|).$$

We now use elementary calculus to find the maximum of the above expression.

It is easily verified that differentiating $\phi(p,|y|)$ with respect to p and then $|y|$ and equating to zero shows that the only admissible turning points when $0 \leqslant \gamma \leqslant 1$ are when $p = |y| = 0$, and when $p = 2$ and $|y| = \frac{(1+\gamma+103\gamma^2-7\gamma^3-2\gamma^4)}{4(1+\gamma)^2(1+8\gamma)}$, which correspond to a maximum point and a saddle point respectively.

Thus when $p = |y| = 0$ we are led to the second required inequality for $|\delta_3|$, and when $p = 2$ and $|y| = \frac{(1+\gamma+103\gamma^2-7\gamma^3-2\gamma^4)}{4(1+\gamma)^2(1+8\gamma)}$ to the first inequality.

Finally we consider the end points of $[0,2] \times [0,1]$.

First note that for any value of γ, $\phi(0,|y|) = \frac{1-|y|^2}{3(1+3\gamma)} \leqslant \frac{1}{3(1+3\gamma)}$, and

$$\phi(2,|y|) = \frac{(3+11\gamma+121\gamma^2+7\gamma^3+2\gamma^4)}{9(1+\gamma)^3(1+2\gamma)(1+3\gamma)}.$$

Next

$$\phi(p,0) = \frac{(3+11\gamma+121\gamma^2+7\gamma^3+2\gamma^4)p^3}{72(1+\gamma)^3(1+2\gamma)(1+3\gamma)} + \frac{(4-p^2)}{12(1+3\gamma)},$$

whose derivative increases with p when $0 \leqslant \gamma \leqslant 1$, again giving the first inequality for $|\delta_3|$.

Finally

$$\phi(p,1) = \frac{(3+11\gamma+121\gamma^2+7\gamma^3+2\gamma^4)p^3}{72(1+\gamma)^3(1+2\gamma)(1+3\gamma)} + \frac{p(4-p^2)}{24(1+3\gamma)} + \frac{(1+8\gamma)p(4-p^2)}{12(1+\gamma)(1+2\gamma)(1+3\gamma)}.$$

The only critical point of this expression when $0 \leqslant \gamma \leqslant 1$ is when $p = 0$, and so checking the values at the end points gives the first inequality for $|\delta_3|$ once more.

The first inequality is sharp when $p_1 = p_2 = p_3 = 2$, and the second is sharp when $p_1 = 0$ and $p_3 = 2$. $\square$

5. The coefficients of the inverse function

Since $\mathcal{M}^\gamma \subset \mathcal{S}$, inverse functions f^{-1} exist defined in some disk $|\omega| < r_0(f)$. Let

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + A_4\omega^4 + \cdots .$$

Then since $f(f^{-1}(\omega)) = \omega$, equating coefficients gives

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= 2a_2^2 - a_3, \\ A_4 &= -5a_2^3 + 5a_2a_3 - a_4. \end{aligned} \tag{5.1}$$

Incomplete estimates were given for these coefficients in [2]. We give the complete solution.

Theorem 5.1. *Let $f \in \mathcal{M}^\gamma$ for $\gamma \geqslant 0$, and f^{-1} be the inverse function of f. Then*

$$|A_2| \leqslant \frac{2}{1+\gamma},$$

$$|A_3| \leqslant \begin{cases} \frac{5+7\gamma}{(1+\gamma)^2(1+2\gamma)}, & 0 \leqslant \gamma \leqslant \frac{1}{2}(5+\sqrt{41}), \\ \frac{1}{1+2\gamma}, & \gamma \geqslant \frac{1}{2}(5+\sqrt{41}), \end{cases}$$

$$|A_4| \leqslant \begin{cases} \frac{2(63+77\gamma+3\gamma^2+\gamma^3)}{9(1+\gamma)^3(1+3\gamma)}, & 0 \leqslant \gamma \leqslant 5, \\ \frac{2}{3(1+3\gamma)}, & \gamma \geqslant 5. \end{cases}$$

All the inequalities are sharp.

Proof. The inequality for $|A_2|$ is trivial.

Using (3.1) and (5.1) we obtain

$$\begin{aligned} A_3 &= \frac{p_2}{2(1+2\gamma)} - \frac{(6+9\gamma+\gamma^2)p_1^2}{4(1+\gamma)^2(1+2\gamma)} \\ &= \frac{1}{2(1+2\gamma)}\Big(p_2 - \frac{(6+9\gamma+\gamma^2)p_1^2}{2(1+\gamma)^2}\Big). \end{aligned}$$

A simple application of Lemma 2.2 with $\mu = \frac{6+9\gamma+\gamma^2}{(1+\gamma)^2}$, gives the inequalities for $|A_3|$.

Again from (3.1) and (5.1) we can write the expression for A_4 as

$$A_4 = \frac{1}{3(1+3\gamma)}\Big(p_3 - 2Bp_1p_2 + Dp_1^3\Big),$$

where

$$B = \frac{6+\gamma}{2(1+\gamma)}, \quad \text{and} \quad D = \frac{48+73\gamma+21\gamma^2+2\gamma^3}{6(1+\gamma)^3}.$$

First note that $0 \leqslant B \leqslant 1$, when $\gamma \geqslant 4$, and $B(2B-1) \leqslant D \leqslant B$, when $\gamma \geqslant 5$, and so applying Lemma 2.3 gives the second inequality for $|A_4|$.

Next write

$$A_4 = \frac{1}{3(1+3\gamma)}\Big(p_3 - 2Bp_1p_2 + Bp_1^3 + (D-B)p_1^3\Big).$$

Then since $D - B \geqslant 0$, when $0 \leqslant \gamma \leqslant 5$, and since $|p_3 - 2Bp_1p_2 + Bp_1^3| \leqslant 2$, (Lemma 2.3 with $D = B$), we obtain the first inequality for $|A_4|$ on the interval $4 \leqslant \gamma \leqslant 5$.

For the remaining interval $0 \leqslant \gamma \leqslant 4$, we use Lemma 2.4.

Write

$$A_4 = \frac{1}{3(1+3\gamma)}\Big(p_3 - (1+\mu)Bp_1p_2 + \mu p_1^3 + \frac{(18+13\gamma-9\gamma^2+2\gamma^3)}{6(1+\gamma)^3}p_1^3\Big),$$

with $\mu = 5/(1+\gamma)$.

Since μ lies outside $[0,1]$, when $0 \leqslant \gamma \leqslant 4$, and noting that $18 + 13\gamma - 9\gamma^2 + 2\gamma^3 \geqslant 0$, when $\gamma \geqslant 0$, applying Lemma 2.4 gives the first inequality for $|A_4|$ on this interval, which completes the proof of the theorem.

We note as before that equality is attained in the inequality for $|A_2|$, and the first inequalities for $|A_3|$ and $|A_4|$ when $p_1 = 2$, the second inequality for $|A_3|$ is sharp when $p_1 = p_2 = 2$, and the second inequality for $|A_4|$ is sharp when $p_1 = p_2 = p_3 = 2$. □

6. The second Hankel determinant

The problem of finding sharp bounds for the second Hankel determinant $H_2(2) = |a_2a_4 - a_3^2|$ for subclasses of univalent functions has received much attention in recent years. Most authors have employed the technique developed in [3], which was used to find the sharp bounds for functions in $\mathcal{S}^*$ and $\mathcal{C}$.

We now use the same method to give the sharp bounds for $H_2(2)$ when $f \in \mathcal{M}^\gamma$ when $0 \leqslant \gamma \leqslant 1$, noting that $\gamma = 0$ and $\gamma = 1$ correspond to $\mathcal{S}^*$ and $\mathcal{C}$ respectively [3].

Theorem 6.1. *Let $f \in \mathcal{M}^\gamma$ for $0 \leqslant \gamma \leqslant 1$, and be given by* (1.1)*. Then*

$$H_2(2) \leqslant \begin{cases} \frac{(1-\gamma)(9+142\gamma+257\gamma^2+80\gamma^3+16\gamma^4)}{9(1+\gamma)^4(1+2\gamma)^2(1+3\gamma)} & \gamma \neq 1, \\ \frac{1}{8} & \gamma = 1. \end{cases}$$

The inequalities are sharp.

Proof. First note that since $f \in \mathcal{M}^0 = \mathcal{S}^*$ and $f \in \mathcal{M}^1 = \mathcal{C}$, the first inequality when $\gamma = 0$, and second inequality are proved in [3].

From (3.1) we have

$$\begin{aligned} H_2(2) = {} & \frac{(-12-220\gamma-361\gamma^2-45\gamma^3+25\gamma^4+37\gamma^5)}{144(1+\gamma)^4(1+2\gamma)^2(1+3\gamma)} p_1^4 \\ & + \frac{\gamma(11+8\gamma-7\gamma^2)}{12(1+\gamma)^2(1+2\gamma)^2(1+3\gamma)} p_1^2 p_2 - \frac{p_2^2}{4(1+2\gamma)^2} \\ & + \frac{p_1 p_3}{3(1+\gamma)(1+3\gamma)}. \end{aligned}$$

We now use Lemma 2.1 to express p_2 and p_3 in term of p_1, simplify the resulting expression, and normalizing the coefficient $p_1 = p$ so that $0 \leqslant p \leqslant 2$, to obtain, using the triangle inequality

$$\begin{aligned} H_2(2) \leqslant {} & \frac{(1-\gamma)(9+142\gamma+257\gamma^2+80\gamma^3+16\gamma^4)}{144(1+\gamma)^4(1+2\gamma)^2(1+3\gamma)} p^4 + \\ & + \frac{(1+16\gamma+19\gamma^2)p^2(4-p^2)|y|}{24(1+\gamma)^2(1+2\gamma)^2(1+3\gamma)} + \frac{p^2(4-p^2)|y|^2}{12(1+\gamma)(1+3\gamma)} \\ & + \frac{(4-p^2)^2|y|^2}{16(1+2\gamma)^2} + \frac{p(4-p^2)(1-|y|^2)}{6(1+\gamma)(1+3\gamma)} := \Phi(p,|y|). \end{aligned}$$

Thus we need to maximize $\Phi(p, |y|)$ over the rectangle $[0, 2] \times [0, 1]$.

Differentiating $\Phi(p, |y|)$ with respect to p and then $|y|$ and equating to zero, shows that the only admissible critical point is when

$$p = 2, \quad y = \frac{(3+91\gamma+\gamma^2-327\gamma^3-160\gamma^4-40\gamma^5)}{3(1+\gamma)^2(1+16\gamma+19\gamma^2)},$$

which gives the required inequality for $H_2(2)$, provided $\gamma \neq 1$.
It remains therefore to check the values of $\Phi(p, |y|)$ at the end points of $[0, 2] \times [0, 1]$, and simple calculus shows that at each of these point, the maximum value taken by $\Phi(p, |y|)$ gives the correct bound for $H_2(2)$.

Finally note that the inequalities are sharp when $p_1 = p_2 = p_3 = 2$. □

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