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THE k-ALMOST RICCI SOLITONS AND CONTACT GEOMETRY

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Abstract. The aim of this article is to study the k-almost Ricci soliton and k-almost gradient Ricci soliton on contact metric manifold. First, we prove that if a compact K-contact metric is a k-almost gradient Ricci soliton, then it is isometric to a unit sphere S^{2n+1}. Next, we extend this result on a compact k-almost Ricci soliton when the flow vector field X is contact. Finally, we study some special types of k-almost Ricci solitons where the potential vector field X is point wise collinear with the Reeb vector field ξ of the contact metric structure.

1. Introduction

A Riemannian manifold (M^n, g) is said to be a Ricci soliton if there exists a vector field X on M^n and a constant λ satisfying the equation $S + \frac{1}{2}\pounds_X g = \lambda g$, where $\pounds_X g$ denotes the Lie-derivative of g along the vector field X on M^n and S is the Ricci tensor of g. In general, X and λ are known as the potential vector field and the soliton constant, respectively. Ricci solitons are the fixed points of Hamilton's Ricci flow: $\frac{\partial}{\partial t}g(t) = -2S(g(t))$ (where $g(t)$ a one-parameter family of metrics on M^n) viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings (cf. [12]). Recently, the notion of Ricci soliton was generalized by Pigoli-Rigoli-Rimoldi-Setti [16] to almost Ricci soliton by allowing the soliton constant λ to be a smooth function.

Recently, Wang-Gomes-Xia [18] extended the notion of almost Ricci soliton to k-almost Ricci soliton which is defined as:

Definition 1.1. A complete Riemannian manifold (M^n, g) is said to be a k-almost Ricci soliton, denoted by (M^n, g, X, k, λ), if there exists a smooth vector field X on M^n, a soliton function $\lambda \in C^\infty(M^n)$ and a positive real valued function k on M^n such that

$$S + \frac{k}{2}\pounds_X g = \lambda g. \tag{1.1}$$

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This notion has been justified as follows. Suppose (M^n, g_0) be a complete Riemannian manifold of dimension n and let $g(t)$ be a solution of the Ricci flow equation defined on $[0, \epsilon)$, $\epsilon > 0$, such that ψ_t is a one-parameter family of diffeomorphisms of M^n, with $\psi_0 = id_M$ and $g(t)(x) = \rho(x,t)\psi_t^* g_0(x)$ for every $x \in M^n$, where $\rho(x,t)$ is a positive smooth function on $M^n \times [0, \epsilon)$. Then one can deduce

$$\frac{\partial}{\partial t} g(t)(x) = \frac{\partial}{\partial t}\rho(x,t)\psi_t^* g_0(x) + \rho(x,t)\psi_t^* \pounds_{\frac{\partial}{\partial t}\psi(x,t)} g_0(x).$$

When $t = 0$, the foregoing equation reduces to

$$S_{g_0} + \frac{k}{2}\pounds_X g_0 = \lambda g_0,$$

where $X = \frac{\partial}{\partial t}\psi(x, 0)$, $\lambda(x) = -\frac{1}{2}\frac{\partial}{\partial t}\rho(x, 0)$ and $k(x) = \rho(x, 0)$.

A k-almost Ricci soliton is said to be *shrinking, steady* or *expanding* accordingly as λ is positive, zero or negative, respectively. It is trivial (Einstein) if the flow vector field X is homothetic, i.e., $\pounds_X g = cg$, for some constant c. Otherwise, it is non-trivial. A k-almost Ricci soliton is said to be a k-almost gradient Ricci soliton if the potential vector field X can be expressed as a gradient of a smooth function u on M^n, i.e., $X = Du$, where D is the gradient operator of g on M^n. In this case, we denotes (M^n, g, Du, k, λ) as a k-almost gradient Ricci soliton with potential function u. Further, the fundamental equation (1.1) takes the form

$$S + k\nabla^2 u = \lambda g, \tag{1.2}$$

where $\nabla^2 u$ denotes the Hessian of u.

In particular, a Ricci soliton is the 1-almost Ricci soliton with constant λ, and an almost Ricci soliton is just the 1-almost Ricci soliton. Barros and Ribeiro Jr. proved (cf. [2]) that a compact almost Ricci soliton with constant scalar curvature is isometric to a Euclidean sphere. An analogous result has also been proved by Wang-Gomes-Xia [18] for the case of k-almost Ricci soliton.

Theorem [WGX]**.** *Let (M^n, g, X, k, λ), $n \geq 3$ be a non-trivial k-almost Ricci soliton with constant scalar curvature r. If M^n is compact, then it is isometric to a standard sphere $S^n(c)$ of radius $c = \sqrt{\frac{2n(2n+1)}{r}}$.*

Recall that a smooth manifold M^n together with a Riemannian metric g is said to be a generalized quasi-Einstein manifold if there exist smooth functions f, μ and λ such that (cf. [7])

$$S + \nabla^2 f - \mu df \otimes df = \lambda g.$$

For $\mu = \frac{1}{m}$, the generalized quasi-Einstein manifold is known as generalized m-quasi-Einstein manifold (cf. [1,3]), and when λ is constant the generalized quasi-Einstein manifold is simply known as m-quasi-Einstein manifold. Case-Shu-Wei [6] proved that any complete quasi-Einstein-metric with constant scalar curvature is trivial (Einstein). Subsequently, this has been extended by Barros-Gomes [1]. In fact, they proved that any compact generalized m-quasi-Einstein metric with constant scalar curvature is isometric to a standard Euclidean sphere S^n. Particularly, Barros-Ribeiro [3] construct a family of nontrivial generalized m-quasi-Einstein metric on the unit sphere $S^n(1)$ that are rigid in the class of constant scalar curvature. It is interesting to note that by a suitable choice of the function f it is possible to reduce any generalized m-quasi-Einstein metric to a k-almost Ricci soliton. For instance, if we take $u = e^{\frac{f}{m}}$ and $k = -\frac{m}{u}$, then (1.2) reduces to

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g.$$

Thus, in one hand k-almost Ricci soliton generalizes generalized m-quasi-Einstein metric, on the other it covers gradient Ricci soliton and gradient almost Ricci soliton. For details we refer to [18]. Recently, Yun-Co-Hwang [20] studied Bach-flat k-almost gradient Ricci solitons.

During the last few years Ricci soliton and almost Ricci soliton have been studied by several authors (cf. [8], [9], [10], [11], and [17]) within the frame-work of contact geometry. In [17], Sharma initiated the study of gradient Ricci soliton within the frame-work of K-contact manifold and prove that *"any complete K-contact metric admitting a gradient Ricci soliton is Einstein and Sasakian"*. Later on, this has been generalized by the second author [9] who proved that *"if a complete K-contact metric (in particular, Sasakian) represents a gradient almost Ricci soliton, then is it isometric to the unit sphere S^{2n+1}"*. Inspired by these results, here we consider contact metric manifolds whose metric is a k-almost Ricci soliton. Following [3], one can construct a family of non-trivial examples of generalized m-quasi-Einstein metrics on the odd dimensional unit sphere S^{2n+1}. Another motivation arises from the fact that any odd dimensional unit sphere satisfies the generalized m-quasi-Einstein condition and hence it satisfies the gradient k-almost Ricci soliton equation (1.2). Since any odd dimensional unit sphere S^{2n+1} admits standard K-contact (Sasakian) structure we are interested in studying K-contact metric as a gradient k-almost Ricci soliton. We address this issue in Section 3 and prove that if a compact K-contact manifold admits a k-almost gradient Ricci soliton then it is isometric to a unit sphere S^{2n+1}. Next, we study k-almost Ricci soliton in the frame-work of compact K-contact manifold when the potential vector field is contact. Finally, a couple of results on contact metric manifolds admitting k-almost Ricci soliton are presented under the assumption that the potential vector field X is point wise collinear with the Reeb vector field ξ of the contact metric structure.

2. Preliminaries

First, we recall some basic definitions and formulas on a contact metric manifold. By a contact manifold we mean a Riemannian manifold M^{2n+1} of dimension $(2n+1)$ which carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. The form η is usually known as the contact form on

M^{2n+1}. It is well known that a contact manifold admits an almost contact metric structure on (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$, ξ a global vector field known as the characteristic vector field (or the Reeb vector field) and g is Riemannian metric, such that

$$\varphi^2 Y = -Y + \eta(Y)\xi, \tag{2.1}$$

$$\eta(Y) = g(Y, \xi), \tag{2.2}$$

$$g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z), \tag{2.3}$$

for all vector fields Y, Z on M. It follows from the above equations that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$ (see [4, p. 43]). A Riemannian manifold M^{2n+1} together with the almost contact metric structure (φ, ξ, η, g) is said to be a contact metric if it satisfies ([4, p. 47])

$$d\eta(Y, Z) = g(Y, \varphi Z) \tag{2.4}$$

for all vector fields Y, Z on M. In this case, we say that g is an associated metric of the contact metric structure. On a contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, we consider two self-adjoint operators $h = \frac{1}{2}\pounds_\xi \varphi$ and $l = R(\cdot, \xi)\xi$, where $\pounds_\xi$ is the Lie-derivative along ξ and R is the Riemann curvature tensor of g. The two operators h and l satisfy (e.g., see [4, p. 84, p. 85])

$$Tr\ h = 0, \quad Tr\ (h\varphi) = 0, \quad h\xi = 0, \quad l\xi = 0, \quad h\varphi = -\varphi h.$$

We now recall the following:

Lemma 2.1 ([4, p. 84; p. 112; p. 111])**.** *On a contact metric manifold* $M^{2n+1}(\varphi, \xi, \eta, g)$ *we have*

$$\nabla_Y \xi = -\varphi Y - \varphi h Y, \tag{2.5}$$

$$Ric_g(\xi, \xi) = g(Q\xi, \xi) = Tr\ l = 2n - Tr\ (h^2), \tag{2.6}$$

$$(\nabla_Z \varphi)Y + (\nabla_{\varphi Z}\varphi)\varphi Y = 2g(Y, Z)\xi - \eta(Y)(Z + hZ + \eta(Z)\xi) \tag{2.7}$$

for all vector fields Y, Z *on* M; *where* ∇ *is the operator of covariant differentiation of* g *and* Q *the Ricci operator associated with the* $(0,2)$ *Ricci tensor given by* $S(Y, Z) = g(QY, Z)$ *for all vector fields* Y, Z *on* M.

A contact metric manifold is said to be K-contact if the vector field ξ is Killing, equivalently if $h = 0$ ([4, p. 87]). Hence on a K-contact manifold Eq. (2.5) becomes

$$\nabla_Y \xi = -\varphi Y \tag{2.8}$$

for any vector field Y on M. Moreover, on a K-contact manifold the following formulas are also valid.

Lemma 2.2 (see Blair [4, p. 113; p. 116])**.** *On a* K*-contact manifold* $M^{2n+1}(\varphi, \xi, \eta, g)$ *we have*

$$Q\xi = 2n\xi, \tag{2.9}$$

$$R(\xi, Y)Z = (\nabla_Y \varphi)Z \tag{2.10}$$

for all vector fields Y, Z on M.

An almost contact metric structure on M is said to be normal if the almost complex structure J on $M \times \mathbb{R}$ defined by (e.g., see Blair [4, p. 80])

$$J(X, fd/dt) = (\varphi X - f\xi, \eta(X)d/dt),$$

where f is a real function on $M \times R$, is integrable. A normal contact metric manifold is said to be Sasakian. On a Sasakian manifold (e.g., [4, p. 86])

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$$

for all vector fields X, Y on M. Further, a contact metric manifold is Sasakian if and only if the curvature tensor R satisfies (e.g., [4, p. 114])

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.11}$$

for all vector fields X, Y on M. A Sasakian manifold is K-contact but the converse is true only in dimension 3 (e.g., [4, p. 87]).

A contact metric manifold is said to be η-Einstein if the Ricci tensor S satisfies $S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z)$ for any vector fields Y, Z on M and are arbitrary functions a, b on M. The functions a and b are constant for a K-contact manifold of dimension > 3 (cf. [19]).

Let T^1M be the unit tangent bundle of a compact orientable Riemannian manifold (M,g) equipped with the Sasaki metric g_s. Any unit vector field U determines a smooth map between (M,g) and (T^1M, g_s). The energy $E(U)$ of the unit vector field U is defined by

$$E(U) = \frac{1}{2}\int \parallel dU \parallel^2 dM = \frac{n}{2}vol(M,g) + \frac{1}{2}\int_M \parallel \nabla U \parallel^2 dM,$$

where dU denotes the differential of the map U and dM denotes the volume element of M. U is said to be a harmonic vector field if it is a critical point of the energy functional E defined on the space χ^1 of all unit vector fields on (M,g). A contact metric manifold is said to be an H-contact manifold if the Reeb vector field ξ is harmonic. In [15], Perrone proved that *"A contact metric manifold is an H-contact manifold, that is ξ is a harmonic vector field, if and only if ξ is an eigenvector of the Ricci operator."* On a contact metric manifold, ξ is an eigenvector of the Ricci operator implies that $Q\xi = (Tr\ l)\xi$. This is true for many contact metric manifolds. Such as, η-Einstein contact metric manifolds, K-contact (in particular Sasakian) manifolds, (k,μ)-contact manifolds and the tangent sphere bundle of a Riemannian manifold of constant curvature. In particular, this condition holds on the unit sphere S^{2n+1} with standard contact metric structure.

Definition 2.1. A vector field X on a contact manifold is said to be a contact vector field if it preserve the contact form η, i.e.,

$$\pounds_X \eta = f\eta \tag{2.12}$$

for some smooth function f on M. When $f = 0$ on M, the vector field X is called a strict contact vector field.

Example 2.2. It is well know [4] that any odd dimensional unit sphere S^{2n+1} admits a standard K-contact (Sasakian) structure (φ, ξ, η, g) and hence the Reeb vector field satisfies (2.8), for any vector field Y on S^{2n+1}. We now recall the theorem of Obata [14] that *a complete connected Riemannian manifold* (M, g) *of dimension* > 2 *is isometric to a sphere of radius* $\frac{1}{c}$ *if and only if it admits a non-trivial solution* k *of the equation* $\nabla\nabla k = -c^2 kg$. For unit sphere this transforms to $\nabla\nabla k = -kg$, where k is the eigenfunction of the Laplacian on S^{2n+1}. Let X be a vector field on S^{2n+1} such that $X = -Dk + \mu\xi$, where μ is a constant. Differentiating this along an arbitrary vector field Y on S^{2n+1} and using (2.8) we obtain $\nabla_Y X = -\nabla_Y Dk - \mu\varphi Y$. Then by Obata's theorem and (2.8) we see that

$$\frac{k}{2}(\pounds_X g)(Y, Z) + S(Y, Z) = (k^2 + 2n)g(Y, Z) \tag{2.13}$$

for all vector fields Y, Z on S^{2n+1}. This shows that $(S^{2n+1}, g, X, \lambda)$ is a almost Ricci soliton with $\lambda = k^2 + 2n$. Moreover, if we take $X = Du$ for some smooth non constant function u on S^{2n+1}, then from (2.13) it follows that S^{2n+1} also admits k-almost gradient Ricci soliton.

3. *K*-contact metric as *k*-almost gradient Ricci soliton and *k*-almost Ricci soliton

We assume that a K-contact metric g is a k-almost gradient Ricci soliton with the potential function u. Then the k-almost gradient Ricci soliton Eq. (1.2) can be exhibited as

$$k\nabla_Y Du + QY = \lambda Y \tag{3.1}$$

for any vector field Y on M; where D is the gradient operator of g on M. Taking the covariant derivative of (3.1) along an arbitrary vector field Z on M yields

$$k\nabla_Z\nabla_Y Du = \frac{1}{k}(Zk)(QY - \lambda Y) - (\nabla_Z Q)Y - Q(\nabla_Z Y) + (Z\lambda)Y + \lambda\nabla_Z Y$$

for any vector field Y on M. Using this and (3.1) in the well known expression of the curvature tensor $R(Y, Z) = [\nabla_Y, \nabla_Z] - \nabla_{[Y,Z]}$, we can easily find out the curvature tensor which is given by

$$\begin{aligned} kR(Y, Z)Du = {} & \frac{1}{k}(Yk)(QZ - \lambda Z) - \frac{1}{k}(Zk)(QY - \lambda Y) \\ & + (\nabla_Z Q)Y - (\nabla_Y Q)Z + (Y\lambda)Z - (Z\lambda)Y \end{aligned}$$

for all vector fields Y, Z on M.

Before entering into our main results we prove the following.

Lemma 3.1. *On a* K*-contact manifold* $M^{2n+1}(\varphi, \xi, \eta, g)$*, we have*

$$\nabla_\xi Q = Q\varphi - \varphi Q. \tag{3.2}$$

Proof. Since ξ is Killing on a K-contact manifold, we have $(\pounds_\xi Q)Y = 0$ for any vector field Y on M. Taking into account (2.8) it follows that

$$\begin{aligned} 0 &= \pounds_\xi(QY) - Q(\pounds_\xi Y) \\ &= \nabla_\xi QY - \nabla_{QY}\xi - Q(\nabla_\xi Y) + Q(\nabla_Y \xi) \\ &= (\nabla_\xi Q)Y + \varphi QY - Q\varphi Y \end{aligned}$$

for any vector field Y on M. This completes the proof. □

Theorem 3.1. *Let $(M^{2n+1}, g, Du, k, \lambda)$ be a k-almost gradient Ricci soliton with the potential function u. If (M, g) is a compact K-contact manifold, then it is isometric to a unit sphere S^{2n+1}.*

Proof. Firstly, taking covariant differentiation of (2.9) along an arbitrary vector field Y on M and using (2.8), we get

$$(\nabla_Y Q)\xi = Q\varphi Y - 2n\varphi Y. \tag{3.3}$$

Now, replacing ξ instead of Y in (3.2) and making use of the K-contact condition (2.9), (3.3) and (3.2), we get

$$\begin{aligned} kR(\xi, Z)Du = (\frac{\lambda - 2n}{k})(Zk)\xi + \frac{1}{k}(\xi k)(QZ - \lambda Z) - 2n\varphi Z \\ + \varphi QZ + (\xi\lambda)Z - (Z\lambda)\xi \end{aligned}$$

for any vector field Z on M. Scalar product of the last equation with an arbitrary vector field Y on M and using (2.10), we obtain

$$\begin{aligned} &kg((\nabla_Z\varphi)Y, Du) + (\frac{\lambda - 2n}{k})(Zk)\eta(Y) + (\xi\lambda - \frac{\lambda}{k}(\xi k))g(Y, Z) \\ &+ \frac{1}{k}(\xi k)g(QY, Z) + 2ng(\varphi Y, Z) - g(Q\varphi Y, Z) - (Z\lambda)\eta(Y) = 0 \end{aligned} \tag{3.4}$$

for any vector field Z on M. Next, substituting Y by φY and Z by φZ in (3.4) and using (2.1), $\eta o \varphi = 0$ and $\varphi\xi = 0$ provides

$$\begin{aligned} &kg((\nabla_{\varphi Z}\varphi)\varphi Y, Du) + (\xi\lambda - \frac{\lambda}{k}(\xi k))\{g(Y, Z) - \eta(Y)\eta(Z)\} \\ &+ \frac{1}{k}(\xi k)g(Q\varphi Y, \varphi Z) + 2ng(\varphi Y, Z) - g(\varphi QY, Z) = 0 \end{aligned}$$

for all vector fields Y, Z on M. Adding the preceding Eq. with (3.4) and using (2.7) (where $h = 0$, as M is K-contact) yields

$$\begin{aligned} &2\{k(\xi u) + (\xi\lambda) - \frac{\lambda}{k}(\xi k)\}g(Y, Z) + \{\frac{\lambda}{k}(\xi k) - (\xi\lambda) - k(\xi u)\}\eta(Y)\eta(Z) \\ &+ \{(\frac{\lambda - 2n}{k})(Zk) - k(Zu) - (Z\lambda)\}\eta(Y) + \frac{1}{k}(\xi k)g(QY, Z) \\ &+ 4ng(\varphi Y, Z) - g(Q\varphi Y + \varphi QY, Z) + \frac{1}{k}(\xi k)g(\varphi QY, \varphi Z) = 0 \end{aligned}$$

for all vector fields Y, Z on M. Anti-symmetrizing the foregoing equation provides

$$\begin{aligned}&\{(\frac{\lambda-2n}{k})(Zk)-k(Zu)-(Z\lambda)\}\eta(Y)-2g(Q\varphi Y+\varphi QY,Z)\\&-\{(\frac{\lambda-2n}{k})(Yk)-k(Yu)-(Y\lambda)\}\eta(Z)+8ng(\varphi Y,Z)=0\end{aligned}$$

for all vector fields Y, Z on M. Moreover, substituting Y by φY and Z by φZ in the last equation and using the K-contact condition (2.9), (2.1), $\eta o\varphi = 0$ and $\varphi\xi = 0$ gives

$$g(Q\varphi Y+\varphi QY,Z)=4ng(\varphi Y,Z)$$

for all vector fields Y, Z on M. It follows from last Eq. that

$$Q\varphi Y+\varphi QY=4n\varphi Y \tag{3.5}$$

for any vector field Y on M. Let $\{e_i, \varphi e_i, \xi\}$, $i = 1, 2, 3, \ldots, n$, be an orthonormal φ-basis of M such that $Qe_i = \sigma_i e_i$. Thus, we have $\varphi Qe_i = \sigma_i\varphi e_i$. Substituting e_i for Y in (3.5) and using the foregoing equation, we obtain $Q\varphi e_i = (4n - \sigma_i)\varphi e_i$. Using the φ-basis and (2.9), the scalar curvature r is given by

$$\begin{aligned}r&=g(Q\xi,\xi)+\sum_{i=1}^{n}[g(Qe_i,e_i)+g(Q\varphi e_i,\varphi e_i)]\\&=g(Q\xi,\xi)+\sum_{i=1}^{n}[\sigma_i g(e_i,e_i)+(4n-\sigma_i)g(\varphi e_i,\varphi e_i)]\\&=2n(2n+1).\end{aligned}$$

Therefore, the scalar curvature r is constant. As M is compact, Theorem [WGX] shows that M is isometric to $S^{2n+1}(c)$, where $c = \sqrt{\frac{2n(2n+1)}{r}}$ is the radius of the sphere. Since $r = 2n(2n + 1)$, we have $c = 1$. Hence, M is isometric to a unit sphere S^{2n+1}. This completes the proof. □

Remark 3.1. From the last theorem we see that any compact K-contact manifold M admitting a gradient k-almost Ricci soliton is isometric to a unit sphere and hence of constant curvature 1. Consequently, M is Sasakian. Since k-almost Ricci soliton covers Einstein manifold, we may compare this as an extension of the odd dimensional Goldberg conjecture which states that any compact Einstein K-contact manifold is Sasakian. For details, we refer to Boyer-Galicki [5].

In particular, the above result is also true for complete Sasakian manifolds.

Corollary 3.1. *Let $(M^{2n+1}, g, Du, k, \lambda)$ be a k-almost gradient Ricci soliton with the potential function u. If (M, g) is a complete Sasakian manifold, then it is compact and isometric to a unit sphere S^{2n+1}.*

Proof. On a Sasakian manifold the Ricci operator Q and φ commutes, i.e., $Q\varphi = \varphi Q$ (see [4, p. 116]). Using this in (3.5) implies $Q\varphi Y = 2n\varphi Y$ for any vector field Y on M. Substituting Y by φY in the last equation and using (2.9) gives $QY = 2nY$ for any vector field Y on M. This shows that M is Einstein with Einstein constant $2n$. As (M, g) is complete, M is compact by Myers' Theorem [13]. The rest of the proof follows from the last theorem. □

Next, we extend Theorem 3.1 and consider K-contact metric as a k-almost Ricci soliton when its potential vector field is a contact vector field and prove:

Theorem 3.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a compact K-contact manifold with X as a contact vector field. If g is a k-almost Ricci soliton with X as the potential vector field, then M is isometric to a unit sphere S^{2n+1}.*

Proof. Firstly, taking Lie-derivative of (2.4) along X and using (1.1) we have

$$k(\pounds_X d\eta)(Y, Z) = 2g(-QY + \lambda Y, \varphi Z) + kg(Y, (\pounds_X \varphi)Z) \tag{3.6}$$

for all vector fields Y, Z on M. As X is a contact vector field, we deduce from (2.12) that

$$\pounds_X d\eta = d\pounds_X \eta = (df) \wedge \eta + f(d\eta). \tag{3.7}$$

Now, making use of (3.7) in (3.6), we obtain

$$2k(\pounds_X \varphi)Z = 4Q\varphi Z + 2(fk - 2\lambda)\varphi Z + k(\eta(Z)Df - (Zf)\xi) \tag{3.8}$$

for any vector field Z on M. Next, replacing ξ instead of Z in the last equation and using $\varphi\xi = 0$ we have

$$2(\pounds_X \varphi)\xi = Df - (\xi f)\xi, \tag{3.9}$$

where we use k is positive. Further, tracing (1.1) gives

$$k\,divX = (2n + 1)\lambda - r. \tag{3.10}$$

Let ω be the volume form of M, i.e., $\omega = \eta \wedge (d\eta)^n \neq 0$. Taking Lie-derivative of this along the vector field X and using the formula $\pounds_X \omega = (divX)\omega$ and equation (3.7) yields $divX = (n + 1)f$. By virtue of this, (3.10) provides

$$r = (2n + 1)\lambda - (n + 1)kf. \tag{3.11}$$

Also, Lie-differentiation of $g(\xi, \xi) = 1$ along an arbitrary vector field X on M and by the use of the equations (1.1), (2.9) yields

$$kg(\pounds_X \xi, \xi) = 2n - \lambda. \tag{3.12}$$

Now, taking Lie-derivative of (2.2) on X and using (1.1), (2.9) and (2.12) we obtain $k\pounds_X \xi = (kf - 2\lambda + 4n)\xi$. Making use of this in (3.12) yields $kf = \lambda - 2n$. and therefore we have $k\pounds_X \xi = (2n - \lambda)\xi$. Next, taking Lie-derivative of $\varphi\xi = 0$ along X and using the foregoing equation we get $(\pounds_X \varphi)\xi = 0$. In view of this, the Eq. (3.9) becomes $Df = (\xi f)\xi$, i.e., $df = (\xi f)\eta$. Exterior derivative of the preceding equation gives $d^2 f = d(\xi f) \wedge \eta + (\xi f)d\eta$. Using $d^2 = 0$ in the last equation and then taking the wedge product with η we get $(\xi f)\eta \wedge d\eta = 0$.

By the definition of contact structure we know that $\eta \wedge d\eta$ is non-vanishing everywhere on M. Hence the previous equation provides $\xi f = 0$. This implies that $df = 0$, and therefore f is constant on M. Integrating both sides of $divX = (n+1)f$ over M and applying the divergence theorem we get $f = 0$. Since $kf = \lambda - 2n$, it follows that $\lambda = 2n$. Consequently, equation (3.11) gives $r = 2n(2n+1)$. This shows that the scalar curvature is constant. As M is compact, we may invoke Theorem [WGX] to conclude that M is isometric to $S^{2n+1}(c)$, where $c = \sqrt{\frac{2n(2n+1)}{r}}$ is the radius of the sphere. Since $r = 2n(2n+1)$, we have $c = 1$, and hence M is isometric to a unit sphere S^{2n+1}. This completes the proof. □

Waiving the compactness assumption and imposing a commutativity condition we have

Theorem 3.3. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$, $n > 1$, be a K-contact manifold with $Q\varphi = \varphi Q$. If g is a k-almost Ricci soliton such that X is a contact vector field, then it is trivial and the soliton vector field is Killing.*

Proof. As f is constant and $fk = \lambda - 2n$, the equation (3.8) becomes

$$k(\pounds_X \varphi)Z = 2Q\varphi Z - (\lambda + 2n)\varphi Z \tag{3.13}$$

for all vector field Z on M. Also, from (2.12) we have $k\pounds_X \eta = (\lambda - 2n)\eta$. Now, taking Lie-derivative of $\varphi^2 Z = -Z + \eta(Z)\xi$ along X and then multiplying k on both sides and using the forgoing Eq., we obtain $k\varphi(\pounds_X \varphi)Z + k(\pounds_X \varphi)\varphi Z = 0$ for any vector field Z on M. In view of (3.13), the last Eq. becomes

$$\varphi Q \varphi Z + Q\varphi^2 Z = (\lambda + 2n)\varphi^2 Z$$

for any vector field Z on M. Since $Q\varphi = \varphi Q$, the last equation reduces to

$$QZ = \left(\tfrac{\lambda+2n}{2}\right)Z + \left(\tfrac{2n-\lambda}{2}\right)\eta(Z)\xi \tag{3.14}$$

for any vector field Z on M. This shows that (M, g) is η-Einstein. Now, differentiating (3.14) along an arbitrary vector field Y on M and using (2.8), we get

$$(\nabla_Y Q)Z = \left(\tfrac{Y\lambda}{2}\right)Z - \left(\tfrac{Y\lambda}{2}\right)\eta(Z)\xi - \left(\tfrac{2n-\lambda}{2}\right)\{g(Z, \varphi Y)\xi + \eta(Z)\varphi Y\}$$

for any vector field Z on M. Tracing the foregoing equation over Y and Z, respectively, we have $Zr = Z\lambda - (\xi\lambda)\eta(Z)$ and $Zr = nZ(\lambda)$ for any vector field Z on M. Since ξ is Killing, $\xi r = 0$. Hence, the last equation provides $\xi\lambda = 0$. Consequently, we have $Zr = Z\lambda$ and $Zr = nZ(\lambda)$ for any vector field Z on M. As $n > 1$, these two equations imply that λ and r are constant. By virtue of (3.14), Eq. (3.13) reduces to $(\pounds_X \varphi)Z = 0$ for any vector field Z on M. At this point, we recall Lemma 1 (cf. [10]) "if a vector field X leaves the structure tensor φ of the contact metric manifold M invariant, then there exists a constant c such that $\pounds_X g = c(g + \eta \otimes \eta)$ " to conclude that

$$(\pounds_X g)(Y, Z) = c\{g(Y, Z) + \eta(Y)\eta(Z)\}$$

for all vector fields Y, Z on M. On the other hand, making use of (3.14) in the Eq. (1.1), we find

$$\frac{k}{2}(\pounds_X g)(Y,Z) = \lambda g(Y,Z) - S(Y,Z)$$

for all vector fields Y, Z on M. Comparing the last two equations, we deduce

$$\frac{ck}{2}\{g(Y,Z) + \eta(Y)\eta(Z)\} = \lambda g(Y,Z) - S(Y,Z). \tag{3.15}$$

Next, putting $Y = Z = \xi$ in (3.15) and using (2.9), we get $ck = 2(\lambda - 2n)$. Further, tracing (3.15) yields $ck(n+1) = (2n+1)\lambda - r$. These two equations together provides $r = 4n(n+1) - \lambda$. Moreover, using $kf = \lambda - 2n$ in (3.11) we have $r = (2n+1)\lambda - (n+1)(\lambda - 2n)$. Comparing the last two equations we see that $\lambda = 2n$. Utilizing this in (3.14) provides $QY = 2nY$ for any vector fields Y on M, i.e., the soliton is trivial. This completes the proof. □

For a Sasakian manifold the commutativity condition $Q\varphi = \varphi Q$ holds trivially (e.g., see Blair [4]). Thus, we have the following:

Corollary 3.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$, $n > 1$, be a Sasakian manifold with X is a contact vector field. If g is a k-almost Ricci soliton, then it is trivial and the soliton vector field is Killing.*

4. k-almost Ricci soliton where $X = \rho\xi$

In this section, we shall discuss about some special type of k-almost Ricci soliton where the potential vector field X is point wise collinear with the Reeb vector field ξ of the contact metric manifold.

Theorem 4.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a compact H-contact manifold. If g represents a non-trivial k-almost Ricci soliton with non-zero potential vector field X collinear with the Reeb vector field ξ, then M is Einstein and Sasakian.*

Proof. Since the potential vector field X on M is collinear with the Reeb vector field ξ, we have $X = \rho\xi$, where ρ is a non-zero smooth function on M (as X is non zero). Taking covariant derivative along an arbitrary vector field Y on M and using (2.5) and (2.6) we get

$$\nabla_Y X = (Y\rho)\xi - \rho(\varphi Y + \varphi hY). \tag{4.1}$$

By virtue of this the soliton equation (1.1) becomes

$$k(Y\rho)\eta(Z) + k(Z\rho)\eta(Y) - 2k\rho g(\varphi hY,Z) + 2S(Y,Z) = 2\lambda g(Y,Z) \tag{4.2}$$

for all vector fields Y, Z on M. Replacing ξ instead of Z in (4.2) gives

$$kD\rho + k(\xi\rho)\xi + 2(Q\xi - \lambda\xi) = 0. \tag{4.3}$$

At this point, putting $Y = Z = \xi$ in (4.2) and making use of (2.6) yields

$$k(\xi\rho) + Trl = \lambda. \tag{4.4}$$

Since M is H-contact, the Reeb vector field ξ is an eigenvector of the Ricci operator at each point of M, i.e., $Q\xi = (Trl)\xi$. Substituting this in (4.2) and then using (4.4), we have $kD\rho = k(\xi\rho)\xi$. Since the k-almost Ricci soliton is non trivial and k is a positive function, we have $D\rho = (\xi\rho)\xi$. Next, taking covariant derivative along an arbitrary vector field Y on M and using (2.5) yields $\nabla_Y D\rho = Y(\xi\rho)\xi - (\xi\rho)(\varphi Y + \varphi hY)$. In view of $g(\nabla_Y D\rho, Z) = g(\nabla_Z D\rho, Y)$, the foregoing equation yields

$$Y(\xi\rho)\eta(Z) - Z(\xi\rho)\eta(Y) + 2(\xi\rho)d\eta(Y,Z) = 0 \tag{4.5}$$

for all vector fields Y, Z on M. Choosing X, Y orthogonal to ξ and noting that $d\eta \neq 0$, the last equation provides $\xi\rho = 0$. Hence ρ is constant. Thus, the equation (4.2) reduces to

$$QZ + (k\rho)h\varphi Z = \lambda Z \tag{4.6}$$

for any vector field Z on M. Taking the trace of (4.6) we obtain $r = (2n+1)\lambda$. Further, covariant derivative of (4.6) along an arbitrary vector field Y on M gives

$$(\nabla_Y Q)Z + (k\rho)(\nabla_Y h\varphi)Z + \rho(Yk)h\varphi Z = (Y\lambda)Z.$$

Contracting this over Y yields

$$\frac{1}{2}(Zr) + \rho((h\varphi Z)k) + (k\rho)div(h\varphi)Z = (Z\lambda) \tag{4.7}$$

for any vector field Z on M. On a contact metric manifold it is known that $div(h\varphi)Z = g(Q\xi, Z) - 2n\eta(Z)$ for all vector field Z on M (see [4]). Using $Q\xi = (Trl)\xi$ in the previous equation we have $div(h\varphi)Z = (Trl - 2n)\eta(Z) = |h|^2\eta(Z)$. Hence, equation (4.7) reduces to

$$\tfrac{1}{2}(Zr) + \rho((h\varphi Z)k) + (k\rho)(Trl - 2n)\eta(Z) = (Z\lambda) \tag{4.8}$$

for any vector field Z on M. Setting $Z = \xi$ and making use of $r = (2n+1)\lambda$ and (2.6) equation (4.8) reduces to

$$\frac{2n-1}{2}(\xi r) - (k\rho)|h|^2 = 0. \tag{4.9}$$

Taking into account (4.1) and $X = \rho\xi$ we see that $div(rX) = \rho(\xi r) + r(\xi\rho) = \rho(\xi r)$, where we have also used $tr(h\varphi) = 0$. Using this equation in (4.9) gives $\frac{2n-1}{2}div(rX) = (k\rho^2)|h|^2$. Integrating this over M and using divergence theorem we obtain

$$\int k\rho^2 \, |h|^2 dM = 0.$$

Since the soliton is non-trivial with non-zero potential vector field X and k being positive, the foregoing equation implies $h = 0$ and hence M is K-contact. Therefore, equation (4.6) shows that $QZ = \lambda Z$. Using (2.9) it follows that $\lambda = 2n$. Thus, M is Einstein with Einstein constant $2n$. So, we can apply the result of Boyer-Galicki [5] which states that "any compact K-contact Einstein manifold is Sasakian" to conclude the proof. □

For a K-contact manifold it is known that $Q\xi = 2n\xi$. Thus, from the above theorem we have the following:

Corollary 4.1. *If a complete K-contact metric represents a non-trivial k-almost Ricci soliton with non-zero potential vector field X collinear with the Reeb vector field ξ, then it is Einstein and Sasakian.*

Next, replacing the "compact H-contact" of the previous theorem by the commutativity condition $Q\varphi = \varphi Q$ we prove:

Theorem 4.2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold satisfying $Q\varphi = \varphi Q$. If g represents a non-trivial k-almost Ricci soliton with nonzero potential vector field X collinear with the Reeb vector field ξ, then M is Einstein and K-contact. In addition, if M is complete, then it is compact Sasakian.*

Proof. The commutativity condition $Q\varphi = \varphi Q$ together with (2.6) and $\varphi\xi = 0$ shows that $Q\xi = (Trl)\xi$. Further, since the potential vector field X is collinear with the Reeb vector field ξ, from equations (4.1) to (4.9) are also valid here. Now we replace Z by φZ in (4.6) and using $h\xi = 0$ we get

$$Q\varphi Z - (k\rho)hZ = \lambda\varphi Z. \tag{4.10}$$

On the other hand, operating (4.6) by φ and using $h\varphi = -\varphi h$ we obtain

$$\varphi QZ + (k\rho)hZ = \lambda\varphi Z. \tag{4.11}$$

Adding (4.10) and (4.11) along with $Q\varphi = \varphi Q$ gives $Q\varphi Z = \lambda\varphi Z$. Therefore, replacing Z by φZ in the last equation and using $Q\xi = (Trl)\xi$, we deduce $QZ = \lambda Z + (Trl - \lambda)\eta(Z)\xi$. Since $\lambda = Trl$ (follows from (4.4), as $\xi\rho = 0$), the foregoing equation implies $QZ = \lambda Z$ and hence M is Einstein. Consequently, the scalar curvature r and λ are constant. Thus, from (4.9), it follows that $(k\rho)|h|^2 = 0$. Since k is positive and the soliton vector field X is non-zero, we can conclude that $h = 0$, and hence M is K-contact. From these, we see that M is K-contact and Einstein with Einstein constant $2n$. Now, if M is complete, then applying Myers' Theorem M becomes compact. Finally, using Boyer-Galicki's Theorem [5] we conclude the proof. ☐

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