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TRANSLATION THEOREMS FOR THE ANALYTIC FOURIER–FEYNMAN TRANSFORM ASSOCIATED WITH GAUSSIAN PATHS ON WIENER SPACE

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Abstract. In this article, we establish translation theorems for the analytic Fourier–Feynman transform of functionals in non-stationary Gaussian processes on Wiener space. We then proceed to show that these general translation theorems can be applied to two well-known classes of functionals; namely, the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick, and the space $\mathcal{B}_{\mathcal{A}}^{(p)}$ consisting of functionals of the form $F(x) = f(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle)$, where $\langle\alpha, x\rangle$ denotes the Paley–Wiener–Zygmund stochastic integral $\int_0^T \alpha(t)dx(t)$.

1. Introduction

Given a positive real $T > 0$, let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions x on the compact interval $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $\mathfrak{m}$ denote Wiener measure which is a Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$. Then, as is well-known, $(C_0[0,T], \mathcal{M}, \mathfrak{m})$ is a complete measure space.

It is well-known that there is no quasi-invariant measure on infinite dimensional linear spaces. Thus, there is no quasi-invariant probability measure on the Wiener space $(C_0[0,T], \mathcal{M}, \mathfrak{m})$. Based on such circumstance, numerous constructions and applications of the translation theorem (Cameron–Martin theorem) for integrals on infinite-dimensional spaces have been studied in various research fields in Mathematics and Physics. The most of the results in the

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literature are concentrated on the Wiener space $C_0[0,T]$. Translation theorems for Wiener integrals were given by Cameron and Martin in [3] and by Cameron and Graves in [2]. Translation theorems for analytic Feynman integrals were given by Cameron and Storvick in [4, 7] and translation theorems for analytic Feynman integrals on abstract Wiener and Hilbert spaces were given by Chung and Kang in [13].

On the other hand, the concepts of the generalized Wiener integral and the generalized analytic Feynman integral on $C_0[0,T]$ were introduced by Chung, Park and Skoug in [14]. The generalized Wiener integral was defined by the Wiener integral

$$\int_{C_0[0,T]} F(\mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx), \tag{1.1}$$

where $\mathcal{Z}_h(x,\cdot)$ is the Gaussian paths given by the stochastic integral $\mathcal{Z}_h(x,t) = \int_0^t h(s)dx(s)$ with $h \in L_2[0,T]$. For a precise definition of the process $\mathcal{Z}_h$, see equation (2.1) below. The Gaussian process $\mathcal{Z}_h : C_0[0,T]\times[0,T] \to \mathbb{R}$ is generally not stationary in time. Thus the meaning of the generalized Wiener integral (1.1) is an integral of functionals in sample paths of non-stationary Gaussian processes. Since then, the generalized analytic Fourier–Feynman transform (GFFT) associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$ also was introduced by Huffman, Park and Skoug in [18], and was further developed in [8, 11, 24].

In this article we establish translation theorems for the GFFT defined on the Wiener space $C_0[0,T]$. We then proceed to show that these general translation theorems for the GFFT can be applied to two well-known classes of functionals; namely, the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [6], and the space $\mathcal{B}_{\mathcal{A}}^{(p)}$ consisting of functionals of the form

$$F(x) = f(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle),$$

where $\langle\alpha,x\rangle$ denotes the Paley–Wiener–Zygmund (PWZ) stochastic integral $\int_0^T \alpha(t)dx(t)$.

2. Preliminaries

In this section, we first present a brief background and some well-known results about the Wiener space $C_0[0,T]$.

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mathfrak{m}(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\,\cdot\,)$ is Wiener-measurable for every $\rho > 0$. If two functionals F and G are equal s-a.e., we write $F \approx G$.

The PWZ stochastic integral [21] plays a key role throughout this article. Let $\{\phi_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $L_2[0,T]$, each of whose elements is of bounded variation on $[0,T]$. Then for each $v \in L_2[0,T]$, the PWZ stochastic integral $\langle v, x\rangle$ is defined by the formula

$$\langle v, x\rangle = \lim_{n\to\infty} \int_0^T \sum_{j=1}^n (v,\phi_j)_2 \phi_j(t)dx(t)$$

for all $x \in C_0[0,T]$ for which the limit exists, where $(\cdot,\cdot)_2$ denotes the L_2-inner product. For each $v \in L_2[0,T]$, the limit defining the PWZ stochastic integral $\langle v, x\rangle$ is essentially independent of the choice of the complete orthonormal set $\{\phi_n\}_{n=1}^\infty$ and it exists for s-a.e. $x \in C_0[0,T]$. If v is of bounded variation on $[0,T]$, then $\langle v, x\rangle$ equals the Riemann–Stieltjes integral $\int_0^T v(t)dx(t)$ for s-a.e. $x \in C_0[0,T]$, and for all $v \in L_2[0,T]$, $\langle v, x\rangle$ is a Gaussian random variable on $C_0[0,T]$ with mean zero and variance $\|v\|_2^2$. For a more detailed study of the PWZ stochastic integral, see [20, 22].

Given a function h in $L_2[0,T]$ with $\|h\|_2 > 0$, let $\mathcal{Z}_h(x,t)$ be the PWZ stochastic integral

$$\mathcal{Z}_h(x,t) = \langle h\chi_{[0,t]}, x\rangle, \tag{2.1}$$

where $\chi_{[0,t]}$ denotes the indicator function of the set $[0,t]$. Next, let $\beta_h(t) = \int_0^t h^2(u)du$. The stochastic process $\mathcal{Z}_h$ on $C_0[0,T]\times[0,T]$, $(x,t)\mapsto \mathcal{Z}_h(x,t)$, is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0,T]} \mathcal{Z}_h(x,s)\mathcal{Z}_h(x,t)\mathfrak{m}(dx) = \beta_h(\min\{s,t\}).$$

In addition, by [25, Theorem 21.1], $\mathcal{Z}_h(\cdot,t)$ is stochastically continuous in t on $[0,T]$. If $h \in L_2[0,T]$ is of bounded variation on $[0,T]$, then for all $x \in C_0[0,T]$, $\mathcal{Z}_h(x,t)$ is continuous in t. Also, for any $h_1, h_2 \in L_2[0,T]$,

$$\int_{C_0[0,T]} \mathcal{Z}_{h_1}(x,s)\mathcal{Z}_{h_2}(x,t)\mathfrak{m}(dx) = \int_0^{\min\{s,t\}} h_1(u)h_2(u)du.$$

Of course if $h(t)\equiv 1$ on $[0,T]$, then the process $\mathcal{W}$ on $C_0[0,T]\times[0,T]$ given by $(w,t)\xrightarrow{\mathcal{W}}\mathcal{W}_t(x) = \mathcal{Z}_1(x,t) = x(t)$ is a Wiener process. We note that the coordinate process $\mathcal{Z}_1$ is stationary in time, whereas the stochastic process $\mathcal{Z}_h$ generally is not. For more detailed studies on the stochastic process $\mathcal{Z}_h$, see [11, 14, 23].

From [14, Lemma 1], it follows that for each $\varphi \in L_\infty[0,T]$ (resp. $L_2[0,T]$) and each $h \in L_2[0,T]$ (resp. $L_\infty[0,T]$),

$$\langle \varphi, \mathcal{Z}_h(x,\cdot)\rangle = \langle \varphi h, x\rangle \tag{2.2}$$

for s-a.e. $x \in C_0[0,T]$.

We finish this section by stating the Cameron–Martin translation theorem [2, 3].

Theorem 2.1 (Cameron–Martin Translation Theorem). *Let F be Wiener integrable over $C_0[0,T]$ and let $w_v \in C_0[0,T]$ be given by $w_v(t) = \int_0^t v(s)ds$ for some $v \in L_2[0,T]$. Then*

$$\int_{C_0[0,T]} F(x+w_v)\mathfrak{m}(dx) = \exp\left\{-\frac{1}{2}\|v\|_2^2\right\} \int_{C_0[0,T]} F(x)\exp\{\langle v, x\rangle\}\mathfrak{m}(dx). \tag{2.3}$$

3. Analytic Fourier–Feynman transform associated with Gaussian paths

For the definitions and related work involving the generalized analytic Feynman integral and the GFFT associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$ ($\mathcal{Z}_h$-GFFT), see [8, 9, 11, 18, 24].

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part and non-zero complex numbers with nonnegative real part, respectively. For each $\lambda \in \widetilde{\mathbb{C}}_+$, $\lambda^{1/2}$ denotes the principal square root of λ; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$.

Definition 3.1. Let h be a function in $L_2[0,T]$ and let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$\int_{C_0[0,T]} F(\lambda^{-1/2}\mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx) = J(h;\lambda)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(h;\lambda)$ analytic on $\mathbb{C}_+$ such that $J^*(h;\lambda) = J(h;\lambda)$ for all $\lambda > 0$, then $J^*(h;\lambda)$ is defined to be the generalized analytic Wiener integral (associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$) of F over $C_0[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(\mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx) = J^*(h;\lambda). \tag{3.1}$$

Let $q \neq 0$ be a real number and let F be a functional such that

$$\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(\mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized analytic Feynman integral (associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$) of F with parameter q and we write

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathcal{Z}_h(x,\cdot))d\mathfrak{m}(x) = \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(\mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx). \tag{3.2}$$

The concept of an L_1 analytic Fourier–Feynman transform for functionals on the Wiener space $C_0[0,T]$ was introduced by Brue in [1]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean spaces. Further work involving the L_2–L_2 theory and the L_p–$L_{p'}$ theory, $1/p + 1/p' = 1$, includes [5, 15–17, 19].

Next (see [11, 18]) we state the definition of the analytic $\mathcal{Z}_h$-GFFT.

Definition 3.2. Let h be a function in $L_2[0,T]$. For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let

$$T_{\lambda,h}(F)(y) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(y + \mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx).$$

For $p \in (1,2]$, we define the L_p analytic $\mathcal{Z}_h$-GFFT, $T_{q,h}^{(p)}(F)$ of F by the formula,

$$T_{q,h}^{(p)}(F)(y) = \underset{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}}{\text{l. i. m.}}\, T_{\lambda,h}(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} \int_{C_0[0,T]} \left|T_{\lambda,h}(F)(\rho y) - T_{q,h}^{(p)}(F)(\rho y)\right|^{p'} \mathfrak{m}(dy) = 0, \tag{3.3}$$

where $1/p + 1/p' = 1$. We define the L_1 analytic $\mathcal{Z}_h$-GFFT, $T_{q,h}^{(1)}(F)$ of F by the formula

$$T_{q,h}^{(1)}(F)(y) = \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} T_{\lambda,h}(F)(y) \tag{3.4}$$

for s-a.e. $y \in C_0[0,T]$ whenever this limit exists.

We note that for $p \in [1,2]$, $T_{q,h}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q,h}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,h}^{(p)}(G)$ exists and $T_{q,h}^{(p)}(G) \approx T_{q,h}^{(p)}(F)$. One can see that for each $h \in L_2[0,T]$, $T_{q,h}^{(p)}(F) \approx T_{q,-h}^{(p)}(F)$, since

$$\int_{C_0[0,T]} F(x)\mathfrak{m}(dx) = \int_{C_0[0,T]} F(-x)\mathfrak{m}(dx).$$

Moreover, from equations (3.1), (3.2) and (3.4), it follows that

$$\int_{C_0[0,T]}^{\text{anf}_q} F(\mathcal{Z}_h(x,\cdot))\mathfrak{m}(dx) = T_{q,k}^{(1)}(F)(0) \tag{3.5}$$

in the sense that if either side exists, then both sides exist and equality holds.

Remark 3.3. Note that if $h \equiv 1$ on $[0,T]$, then $\mathcal{Z}_h(x,t) = x(t)$ for all $x \in C_0[0,T]$. In this case the generalized analytic Feynman integral given by equation (3.2) with $h \equiv 1$ and the L_p analytic $\mathcal{Z}_1$-GFFT, $T_{q,1}^{(p)}(F)$, agree with the previous definitions of the analytic Feynman integral and the analytic Fourier–Feynman transform, $T_q^{(p)}(F)$, see [5, 7, 15, 19]. Thus we also denote the $\mathcal{Z}_1$-GFFT $T_{q,1}^{(p)}(F)$ of functionals F on $C_0[0,T]$ by $T_q^{(p)}(F)$ throughout this article.

Next we give the definition of our generalized convolution product (GCP) [9].

Definition 3.4. Let F and G be scale-invariant measurable functionals on $C_0[0,T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$ and $h_1, h_2 \in L_2[0,T]$, we define their GCP with respect to $\{\mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}\}$ (if it exists) by

$$(F * G)_\lambda^{(h_1,h_2)}(y) = \begin{cases} \int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F(y + \mathcal{Z}_{h_1}(x,\cdot))G(y + \mathcal{Z}_{h_2}(x,\cdot))\mathfrak{m}(dx), & \lambda \in \mathbb{C}_+ \\ \int_{C_0[0,T]}^{\mathrm{anf}_q} F(y + \mathcal{Z}_{h_1}(x,\cdot))G(y + \mathcal{Z}_{h_2}(x,\cdot))\mathfrak{m}(dx), & \\ & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases} \tag{3.6}$$

When $\lambda = -iq$, we denote $(F * G)_\lambda^{(h_1,h_2)}$ by $(F * G)_q^{(h_1,h_2)}$.

Remark 3.5. In [9], a more general definition for the GCP of functionals on $C_0[0,T]$ is presented and fundamental relationships between the $\mathcal{Z}_h$-GFFTs and the GCPs are investigated.

4. Translation theorems for generalized Fourier–Feynman transform

In [4,7], Cameron and Storvick showed translation theorems for the analytic Feynman integral of functionals on classical Wiener space and in [13], Chung and Kang established translation theorems for analytic Feynman integrals on abstract Wiener and Hilbert spaces. In Theorems 4.1 and 4.3 below we present translation theorems for the $\mathcal{Z}_h$-GFFTs of functionals F on the Wiener space $C_0[0,T]$. First we give a translation theorem for the L_1 analytic $\mathcal{Z}_h$-GFFT.

Theorem 4.1. *Let h_1 be a function in $L_2[0,T]$ (resp. $L_\infty[0,T]$) and let F be a functional on $C_0[0,T]$ such that $F(\mathcal{Z}_h(x,\cdot))$ is Wiener integrable over $C_0[0,T]$. Given $\varphi \in L_\infty[0,T]$ (resp. $L_2[0,T]$), let $w_{\varphi h_1} \in C_0[0,T]$ be defined by*

$$w_{\varphi h_1}(t) = \int_0^t \varphi(s)h_1(s)ds. \tag{4.1}$$

Furthermore, assume that given a non-zero real q, the L_1 analytic $\mathcal{Z}_h$-GFFT, $T_{q,h_1}^{(1)}(F)$ of F exists. Then for each $h_2 \in L_2[0,T]$ (resp. $L_\infty[0,T]$) and s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned} &T_{q,h_1}^{(1)}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1},\cdot)) \\ &= \exp\left\{\frac{iq}{2}\|\varphi h_2\|_2^2 + iq(\varphi, y)^\sim\right\}(F * R_{q,\varphi})_q^{h_1,h_2}(y), \end{aligned} \tag{4.2}$$

where the functional $R_{q,\varphi} : C_0[0,T] \to \mathbb{C}$ is given by

$$R_{q,\varphi}(x) = \exp\{-iq\langle\varphi, x\rangle\}. \tag{4.3}$$

Proof. By the assumption of the existence of $T_{q,h_1}^{(1)}(F)$, we may assume that the analytic $\mathcal{Z}_{h_1}$-Wiener transform $T_{\lambda,h_1}(F)$ exists for all $\lambda \in \mathbb{C}_+$. We first note that for each $\lambda > 0$,

$$\mathcal{Z}_{h_2}(w_{\varphi h_1}, t) + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,t) = \lambda^{-1/2}\mathcal{Z}_{h_1}(x + w_{\lambda^{1/2}\varphi h_2}, t). \tag{4.4}$$

Next, for $\lambda > 0$, let

$$G^{\lambda}_{y,h_1}(x) = F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)). \tag{4.5}$$

Using (4.4), (4.5), (2.3) with F and w_v replaced with G^{λ}_{y,h_1} and $w_{\lambda^{1/2}\varphi h_1}$ respectively, and (2.2) with h replaced with h_2, it follows that for $\lambda > 0$,

(4.6)

$$\begin{aligned}
&T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1},\cdot)) \\
&= \int_{C_0[0,T]} F\big(y + \mathcal{Z}_{h_2}(w_{\varphi h_1},\cdot) + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)\big)\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} F\big(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x + w_{\lambda^{1/2}\varphi h_2},\cdot)\big)\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} G^{\lambda}_{y,h_1}(x + w_{\lambda^{1/2}\varphi h_2})\mathfrak{m}(dx) \\
&= \exp\bigg\{-\frac{1}{2}\|\lambda^{1/2}\varphi h_2\|_2^2\bigg\}\int_{C_0[0,T]} G^{\lambda}_{y,h_1}(x)\exp\{\langle\lambda^{1/2}\varphi h_2, x\rangle\}\mathfrak{m}(dx) \\
&= \exp\bigg\{-\frac{\lambda}{2}\|\varphi h_2\|_2^2\bigg\} \\
&\qquad\times\int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\exp\{\lambda^{1/2}\langle\varphi, \mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx) \\
&= \exp\bigg\{-\frac{\lambda}{2}\|\varphi h_2\|_2^2 - \lambda\langle\varphi, y\rangle\bigg\} \\
&\qquad\times\int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\exp\{\lambda\langle\varphi, y\rangle + \lambda^{1/2}\langle\varphi, \mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx).
\end{aligned}$$

On the other hand, by Definition 3.4, it follows that for $\lambda > 0$,

(4.7)

$$\begin{aligned}
&(F * R_q)^{h_1,h_2}_{\lambda}(y) \\
&= \int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))R_{q,\varphi}(y + \lambda^{-1/2}\mathcal{Z}_{h_2}(x,\cdot))\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\exp\{-iq\langle\varphi, y + \lambda^{-1/2}\mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\exp\{-iq\langle\varphi, y\rangle - iq\lambda^{-1/2}\langle\varphi, \mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx).
\end{aligned}$$

Thus, using (3.4), (4.6), (4.7), and (3.6) with G replaced with $R_{q,\varphi}$, we have that

$$\begin{aligned}
&T^{(1)}_{q,h_1}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1},\cdot)) \\
&\quad= \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_2}(w_{\varphi h_1},\cdot))
\end{aligned} \tag{4.8}$$

$$= \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \exp\bigg\{-\frac{\lambda}{2}\|\varphi h_2\|_2^2 - \lambda\langle\varphi, y\rangle\bigg\}\int_{C_0[0,T]} F(y+\lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))$$
$$\times \exp\{\lambda\langle\varphi,y\rangle + \lambda^{1/2}\langle\varphi,\mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx)$$
$$= \exp\bigg\{\frac{iq}{2}\|\varphi h_2\|_2^2 + iq\langle\varphi,y\rangle\bigg\}\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{C_0[0,T]} F(y+\lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))$$
$$\times \exp\{\lambda\langle\varphi,y\rangle + \lambda^{1/2}\langle\varphi,\mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx)$$
$$= \exp\bigg\{\frac{iq}{2}\|\varphi h_2\|_2^2 + iq\langle\varphi,y\rangle\bigg\}\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{C_0[0,T]} F(y+\lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))$$
$$\times \exp\{-iq\langle\varphi,y\rangle - iq\lambda^{-1/2}\langle\varphi,\mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx)$$
$$= \exp\bigg\{\frac{iq}{2}\|\varphi h_2\|_2^2 + iq\langle\varphi,y\rangle\bigg\}\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}(F*R_{q,\varphi})_\lambda^{h_1,h_2}(y)$$
$$= \exp\bigg\{\frac{iq}{2}\|\varphi h_2\|_2^2 + iq\langle\varphi,y\rangle\bigg\}(F*R_{q,\varphi})_q^{h_1,h_2}(y)$$

as desired. □

Using equation (3.6) with $y = 0$ and with G replaced with $R_{q,\varphi}$, we have the following corollary.

Corollary 4.2. *Let h_1, F, φ and $w_{\varphi h_1}$ be as in Theorem 4.1. Assume that given a non-zero real q, the $\mathcal{Z}_{h_1}$-Feynman integral with parameter q of F,*

$$\int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathcal{Z}_{h_1}(x,\cdot))\mathfrak{m}(dx)$$

exists. Then for each $h_2 \in L_2[0,T]$ (resp. $L_\infty[0,T]$),

$$\begin{aligned}&\int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathcal{Z}_{h_1}(x,\cdot)+\mathcal{Z}_{h_2}(w_{\varphi h_1},\cdot))\mathfrak{m}(dx)\\ &= \exp\bigg\{\frac{iq}{2}\|\varphi h_2\|_2^2\bigg\}\int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathcal{Z}_{h_1}(x,\cdot))\exp\{-iq\langle\varphi,\mathcal{Z}_{h_2}(x,\cdot)\rangle\}\mathfrak{m}(dx).\end{aligned} \tag{4.9}$$

Next we present a translation theorem for the L_p analytic $\mathcal{Z}_h$-GFFT.

Theorem 4.3. *Let $p \in [1,2]$ be given and let h_1, F, φ and $w_{\varphi h_1}$ be as in Theorem 4.1. Assume that given a non-zero real q, the L_p analytic $\mathcal{Z}_h$-GFFT, $T_{q,h_1}^{(p)}(F)$ of F exists. Then for s-a.e. $y \in C_0[0,T]$,*

$$T_{q,h_1}^{(p)}(F)(y+\mathcal{Z}_{h_1}(w_{\varphi h_1},\cdot)) = \exp\bigg\{\frac{iq}{2}\|\varphi h_1\|_2^2 + iq\langle\varphi,y\rangle\bigg\}T_{q,h_1}^{(p)}(FR_{q,\varphi})(y), \tag{4.10}$$

where the functional $R_{q,\varphi}$ is given by (4.3) above.

Proof. Let $h_1 = h_2$ in (4.8). Then using the fifth expression of (4.8) with $h_1 = h_2$ and the definition of the L_1 analytic $\mathcal{Z}_h$-GFFT, it follows equation (4.10) with $p = 1$, immediately. But, to obtain equation (4.10) with $p \in (1, 2]$, we have to use the concept of the scale-invariant limit (see (3.3) above) for the proof. Thus we will proceed the proof in the case that $p \in (1, 2]$.

Proceeding as in the proof of Theorem 4.1, we observe that for $\lambda > 0$,

$$
\begin{aligned}
&T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_1}(w_{\varphi h_1}, \cdot)) \\
&= \exp\Big\{ -\frac{\lambda}{2}\|\varphi h_1\|_2^2 - \lambda\langle\varphi, y\rangle\Big\} \\
&\quad \times \int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \exp\{\lambda\langle\varphi, y\rangle + \lambda^{1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx).
\end{aligned}
$$

We also obtain that for $\lambda > 0$,

$$
\begin{aligned}
&T_{\lambda,h_1}(FR_{q,\varphi})(y) \\
&= \int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \exp\{-iq\langle\varphi, y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \exp\{-iq\langle\varphi, y\rangle - iq\lambda^{-1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx).
\end{aligned}
$$

On the other hand, using Hölder's inequality, we get that for $\lambda > 0$,

$$
\begin{aligned}
&\int_{C_0[0,T]} \big|F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \exp\{-iq\langle\varphi, y\rangle - iq\lambda^{-1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\} \\
&\qquad\qquad - F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \exp\{\lambda\langle\varphi, y\rangle + \lambda^{1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\big|\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} \big|F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\big| \big|\exp\{-iq\langle\varphi, y\rangle - iq\lambda^{-1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\big| \\
&\qquad \times \big|1 - \exp\{(iq+\lambda)\langle\varphi, y\rangle + (iq\lambda^{-1/2} + \lambda^{1/2})\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\big|\mathfrak{m}(dx) \\
&= \int_{C_0[0,T]} \big|F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\big| \\
&\qquad \times \big|1 - \exp\{(iq+\lambda)\langle\varphi, y\rangle + (iq\lambda^{-1/2} + \lambda^{1/2})\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\big|\mathfrak{m}(dx) \\
&\le \bigg(\int_{C_0[0,T]} \big|F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot))\big|^p \mathfrak{m}(dx)\bigg)^{1/p} \\
&\qquad \times \bigg(\int_{C_0[0,T]} \big|1 - \exp\{(iq+\lambda)\langle\varphi, y\rangle \\
&\qquad\qquad\qquad + (iq\lambda^{-1/2} + \lambda^{1/2})\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\big|^{p'} \mathfrak{m}(dx)\bigg)^{1/p'}.
\end{aligned}
$$

Note that each factor in the last expression has a limit as $\lambda \to -iq$ through $\mathbb{C}_+$, and that

$$\Big(\int_{C_0[0,T]} \big|1 - \exp\{(iq+\lambda)\langle\varphi, y\rangle + (iq\lambda^{-1/2} + \lambda^{1/2})\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\big|^{p'}\mathfrak{m}(dx)\Big)^{1/p'} \to 0$$

as $\lambda \to -iq$ through $\mathbb{C}_+$. Hence we have that

$$\begin{aligned}
&T^{(p)}_{q,h_1}(F)(y + \mathcal{Z}_{h_1}(w_\varphi,\cdot)) \\
&= \underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\, T_{\lambda,h_1}(F)(y + \mathcal{Z}_{h_1}(w_\varphi,\cdot)) \\
&= \underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\, \exp\Big\{-\frac{\lambda}{2}\|\varphi h_1\|_2^2 - \lambda\langle\varphi, y\rangle\Big\}\int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \\
&\qquad\qquad \times \exp\{\lambda\langle\varphi, y\rangle + \lambda^{1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx) \\
&= \exp\Big\{\frac{iq}{2}\|\varphi h_1\|_2^2 + iq\langle\varphi, y\rangle\Big\}\underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \\
&\qquad\qquad \times \exp\{\lambda\langle\varphi, y\rangle + \lambda^{1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx) \\
&= \exp\Big\{\frac{iq}{2}\|\varphi h_1\|_2^2 + iq\langle\varphi, y\rangle\Big\}\underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\int_{C_0[0,T]} F(y + \lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)) \\
&\qquad\qquad \times \exp\{-iq\langle\varphi, y\rangle - iq\lambda^{-1/2}\langle\varphi, \mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx) \\
&= \exp\Big\{\frac{iq}{2}\|\varphi h_1\|_2^2 + iq\langle\varphi, y\rangle\Big\}\underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\mathrm{l.i.m.}}\, T_{\lambda,h_1}(FR_{q,\varphi})(y) \\
&= \exp\Big\{\frac{iq}{2}\|\varphi h_1\|_2^2 + iq\langle\varphi, y\rangle\Big\}T^{(p)}_{q,h_1}(FR_{q,\varphi})(y)
\end{aligned}$$

as desired. □

Using (4.10) with $p = 1$ and $y = 0$, and (3.5), we have the following corollary. But equation (4.11) below also can be obtained from (4.9) with h_2 replaced with h_1.

Corollary 4.4. *Let h_1, F, φ and $w_{\varphi h_1}$ be as in Theorem 4.1. Assume that given a non-zero real q, the $\mathcal{Z}_{h_1}$-Feynman integral with parameter q of F,*

$$\int^{\mathrm{anf}_q}_{C_0[0,T]} F(\mathcal{Z}_{h_1}(x,\cdot))\mathfrak{m}(dx)$$

exists. Then

$$\int^{\mathrm{anf}_q}_{C_0[0,T]} F(\mathcal{Z}_{h_1}(x,\cdot) + \mathcal{Z}_{h_1}(w_{\varphi h_1},\cdot))\mathfrak{m}(dx) \tag{4.11}$$

$$= \exp\left\{\frac{iq}{2}\|\varphi h_1\|_2^2\right\}\int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathcal{Z}_{h_1}(x,\cdot))\exp\{-iq\langle\varphi,\mathcal{Z}_{h_1}(x,\cdot)\rangle\}\mathfrak{m}(dx).$$

The Gaussian process given by (2.1) with $h_1 \equiv 1$ is an ordinary Wiener process. Thus we have the following translation theorem for the analytic Feynman integral. This result subsumes a similar result obtained by Cameron and Storvick in [4].

Corollary 4.5. *Setting $h_1 \equiv 1$ in Corollary 4.4 yields the formula*

$$\begin{aligned}
&\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x+w_\varphi)\mathfrak{m}(dx) \\
&\equiv \int_{C_0[0,T]}^{\mathrm{anf}_q} F(\mathcal{Z}_1(x,\cdot)+\mathcal{Z}_1(w_\varphi,\cdot))\mathfrak{m}(dx) \qquad (4.12)\\
&= \exp\left\{\frac{iq}{2}\|\varphi\|_2^2\right\}\int_{C_0[0,T]}^{\mathrm{anf}_q} F(x)\exp\{-iq\langle\varphi,x\rangle\}\mathfrak{m}(dx)
\end{aligned}$$

for all real $q \in \mathbb{R}\setminus\{0\}$, where $w_\varphi(t) = \int_0^t \varphi(s)ds$.

5. Functionals on $C_0[0,T]$

In this section, we give various corollaries which show that Theorems 4.1 and 4.3, giving the translation theorems for the $\mathcal{Z}_h$-GFFT, are indeed very general theorem since the translation theorems hold for functionals F in large classes. Below we list results of two types.

5.1. Banach algebra $\mathcal{S}$

We will see that the translation formulas (4.2) and (4.10) hold for the $\mathcal{Z}_h$-GFFT of functionals in the Banach algebra $\mathcal{S}$ introduced in [6]. The Banach algebra $\mathcal{S}$ consists of functionals expressible in the form

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle v,x\rangle\}df(v) \qquad (5.1)$$

for s-a.e. $x \in C_0[0,T]$ where f is an element of $\mathcal{M}(L_2[0,T])$, the space of $\mathbb{C}$-valued, countably additive (and hence finite) Boreal measures on $L_2[0,T]$. Further work on $\mathcal{S}$, see [7, 8, 11, 18, 24].

Corollary 5.1. *Let $F \in \mathcal{S}$ be given by (5.1), and given $h_1 \in L_\infty[0,T]$ and $\varphi \in L_2[0,T]$, let $w_{\varphi h_1}$ be given by (4.1). Then for all non-zero real q and each $h_2 \in L_\infty[0,T]$, equations (4.2) and (4.9) above hold.*

Proof. This corollary follows from Theorem 4.1 above since, by [18, Theorem 3.1], the $\mathcal{Z}_h$-GFFT $T_{q,h}^{(1)}(F)$ exists for all $q \in \mathbb{R}\setminus\{0\}$ and each $h \in L_\infty[0,T]$. □

Corollary 5.2. *Let F, h_1, φ, and $w_{\varphi h_1}$ be as in Corollary 5.1. Then equations (4.10) and (4.11) above hold for all non-zero real q. In particular, setting $h_1 \equiv 1$ in equation (4.11) with F in $\mathcal{S}$, it follows equation (4.12).*

Remark 5.3. In [7, 8], the authors established equations (4.12) and (4.11) for the functionals F in the Banach algebra $\mathcal{S}$ using the concept of the Radon–Nikodym derivative and a direct calculation.

5.2. Cylinder functionals

Functionals that involve PWZ stochastic integrals are quite common. In this subsection we apply our results to the cylinder functionals F on $C_0[0,T]$ given by

$$F(x) = f(\langle g_1, x\rangle, \ldots, \langle g_n, x\rangle), \tag{5.2}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function and $\{g_1, \ldots, g_n\}$ is an independent set of functions in $L_2[0,T]$. It is well-known [12] that the functional F given by (5.2) is Wiener measurable if and only if f is Lebesgue measurable.

Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ is an orthogonal set of non-zero functions in $L_2[0,T]$, each of whose elements is of bounded variation on $[0,T]$. For $p \in [1,\infty)$, let $\mathcal{B}_{\mathcal{A}}^{(p)}$ be the space of all functionals on $C_0[0,T]$ of the form

$$F(x) = f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle) \tag{5.3}$$

for s-a.e. $x \in C_0[0,T]$ where f is in $L_p(\mathbb{R}^n)$. Let $\mathcal{B}_{\mathcal{A}}^{(\infty)}$ be the space of all functionals having the form (5.3) with f in $C_0(\mathbb{R}^n)$, the space of bounded continuous functions on $\mathbb{R}^n$ that vanish at infinity. It is quite easy to see that if F is in $\mathcal{B}_{\mathcal{A}}^{(p)}$, then F is scale-invariant measurable.

Given an orthogonal set $\mathcal{A} = \{\alpha_1, \ldots, \alpha_n\}$ of non-zero functions of bounded variation on $[0,T]$, let $\mathcal{O}_2(\mathcal{A})$ be the class of all functions h in $L_2[0,T]$ such that $\mathcal{A}h = \{\alpha_1 h, \ldots, \alpha_n h\}$ is orthonormal in $L_2[0,T]$.

In [10], the authors defined the class $\mathcal{B}_{\mathcal{A}}^{(p)}$, $1 \le p \le +\infty$, for the case that $\mathcal{A}$ is simply an orthogonal set of non-zero functions in $L_2[0,T]$. Consequently, in order to ensure the existence of the $\mathcal{Z}_h$-GFFT of functionals F in $\mathcal{B}_{\mathcal{A}}^{(p)}$, the functions h was taken to be in $\mathcal{O}_2(\mathcal{A}) \cap L_\infty[0,T]$. However, using the techniques similar to those used in [10], we can obtain the following theorem because each of the PWZ stochastic integrals $\langle \alpha_j, \mathcal{Z}_h(x,\cdot)\rangle \equiv \langle \alpha_j h, x\rangle$ always exists in our setting for $\mathcal{A}$ and h.

Theorem 5.4. *Let $p \in [1,2]$ and let $F \in \mathcal{B}_{\mathcal{A}}^{(p)}$ be given by equation (5.3). Then for all non-zero real number q and all $h \in \mathcal{O}_2(\mathcal{A})$, the L_p analytic $\mathcal{Z}_h$-GFFT, $T_{q,h}^{(p)}(F)(y)$ exists as an element of $\mathcal{B}_{\mathcal{A}}^{(p')}$ and is given by the formula*

$$T_{q,h}^{(p)}(F)(y) = (\Psi_{-iq} f)(\langle \alpha_1, y\rangle, \ldots, \langle \alpha_n, y\rangle)$$

for s-a.e. $y \in C_0[0,T]$, where

$$(\Psi_{-iq} f)(\xi_1, \ldots, \xi_n) = \left(\frac{-iq}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{\frac{iq}{2}\sum_{j=1}^n (u_j - \xi_j)^2\right\} d\vec{u}.$$

Corollary 5.5. *Let p and F be as in Theorem 5.4, and given $\varphi \in L_\infty[0,T]$ and $h_1 \in \mathcal{O}_2(\mathcal{A})$, let $w_{\varphi h_1}$ be given by* (4.1)*. Then for all $h_2 \in L_2[0,T]$, equations* (4.2)*,* (4.9) *and* (4.10) *above hold for all non-zero real q.*

Proof. This corollary follows from Theorems 4.1 and 4.3 since, by [10, Theorems 4.7 and 4.8], $T^{(p)}_{q,h}(F)$ of F in $\mathcal{B}^{(p)}_{\mathcal{A}}$ exist for all non-zero real q, all $h \in L_2[0,T]$ and all $p \in [1,2]$. □

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