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CARTIER OPERATORS ON COMPACT DISCRETE VALUATION RINGS AND APPLICATIONS

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Abstract. From an analytical perspective, we introduce a sequence of Cartier operators that act on the field of formal Laurent series in one variable with coefficients in a field of positive characteristic p. In this work, we discover the binomial inversion formula between Hasse derivatives and Cartier operators, implying that Cartier operators can play a prominent role in various objects of study in function field arithmetic, as a suitable substitute for higher derivatives. For an applicable object, the Wronskian criteria associated with Cartier operators are introduced. These results stem from a careful study of two types of Cartier operators on the power series ring $\mathbf{F}_q[[T]]$ in one variable T over a finite field $\mathbf{F}_q$ of q elements. Accordingly, we show that two sequences of Cartier operators are an orthonormal basis of the space of continuous $\mathbf{F}_q$-linear functions on $\mathbf{F}_q[[T]]$. According to the digit principle, every continuous function on $\mathbf{F}_q[[T]]$ is uniquely written in terms of a q-adic extension of Cartier operators, with a closed-form of expansion coefficients for each of the two cases. Moreover, the p-adic analogues of Cartier operators are discussed as orthonormal bases for the space of continuous functions on $\mathbf{Z}_p$.

1. Introduction

A Cartier operator is of great importance in characteristic-p algebraic geometry, which is a fundamental tool for working with Kähler differential forms in this geometry (see [6, 7]). It also plays a significant role in determining the criterion of algebraicity of power series over the field of rational functions in a field of characteristic $p > 0$ as is shown in studies conducted by Christol [8] and Sharif and Woodcock [27]. In this paper, from a purely analytical perspective, we consider two types of Cartier operators (or maps) which act on a non-Archimedean local field of any characteristic, while much emphasis is placed on the power series ring $\mathbf{F}_q[[T]]$ in one variable T over a finite field $\mathbf{F}_q$.

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The purpose of this work is to apply two types of Cartier maps to the characterization of continuous functions defined on the integer ring of a non-Archimedean local field of any characteristic. Accordingly, we first proceed to deduce a fundamental relation of what is known as a binomial inversion formula between Cartier operators and higher derivatives (or Hasse derivatives) on $\mathbf{F}_q[[T]]$. This binomial inversion formula holds over a more general field of formal Laurent series in one variable over a (perfect) field of characteristic $p > 0$, which shows that Cartier operators can play a role in various objects of study as a substitute for higher derivatives. For example, in Section 5, we present selected Wronskian criteria associated with two Cartier operators on more general fields, and these are parallel to the Wronskian criteria for the Hasse derivatives in [25] and [11]. As compared with the known properties of Hasse derivatives whose q-adic extension is referred to as digit derivatives [15, 17, 18], we show in Sections 2 and 3 that two sequences of Cartier operators are an orthonormal basis for the closed subspace $LC(\mathbf{F}_q[[T]], \mathbf{F}_q((T)))$ of $\mathbf{F}_q$-linear continuous functions on $\mathbf{F}_q[[T]]$. Added to this fact, Conrad's digit principle [10] enables us to prove that q-adic extensions of two Cartier operators are an orthonormal basis for the entire space, $C(\mathbf{F}_q[[T]], \mathbf{F}_q((T)))$, of continuous functions on $\mathbf{F}_q[[T]]$ together with a closed-form expression for expansion coefficients for two respective bases. At the same time, by analogy with the classical case, we provide two orthonormal bases for the space $C(\mathbf{Z}_p, \mathbf{Q}_p)$ of continuous functions on the ring $\mathbf{Z}_p$ of p-adic integers, consisting of p-adic extensions of two types of Cartier maps on $\mathbf{Z}_p$ with no closed-form formula for expansion coefficients.

2. Orthonormal bases for $LC(R, K)$

This section consists of two subsections. The first subsection is a quick review of orthonormal bases of a certain Banach space over the integer ring of a local field of any characteristic, with much emphasis on the positive characteristic. For such a Banach space, we mainly consider $LC(R, K)$ (see notational exposition after Lemma 2.2). Two types of Cartier operators are the main object of study in the second subsection, forming an orthonormal basis of the space $LC(R, K)$ and observing that all known orthonormal bases of $LC(R, K)$ are essentially equivalent.

2.1. Known bases for the subspace on $\mathbf{F}_q[[T]]$

Let $\mathcal{V}$ be a non-Archimedean local field of any characteristic, with an integer ring $\mathcal{O}$ and a maximal ideal M. In addition, let π be a uniformizer in $\mathcal{V}$ such that $M = (\pi)$, and let $\mathbf{F} := \mathcal{O}/M$ be the residue field of order q, and let $|\cdot|_\pi$ be the (normalized) absolute value on $\mathcal{V}$ associated with the additive valuation v_π on $\mathcal{V}$ such that $|x|_\pi = q^{-v_\pi(x)}$ for $x \in \mathcal{V}$.

The following are the cases for $(\mathcal{V}, \mathcal{O}, \pi)$ which are of greatest interest in this study:

(1) $(\mathbf{F}_q((T)), \mathbf{F}_q[[T]], T)$, where $\mathbf{F}_q((T))$ is the field of formal Laurent series in one variable T over a finite field $\mathbf{F}_q$ of q elements, where $q = p^e$ is a power of a prime number p and $\mathbf{F}_q[[T]]$ is the ring of the formal power series in T over $\mathbf{F}_q$.

(2) $(\mathbf{Q}_p, \mathbf{Z}_p, p)$, where $\mathbf{Q}_p$ is the field of p-adic numbers for a prime number p and $\mathbf{Z}_p$ is the ring of p-adic integers.

Definition 2.1. Let K be a non-Archimedean local field, and let E be a K-Banach space equipped with the sup-norm $\|f\| := \sup_{x\in R}\{|f(x)|\}$. Let both E and K be the same value group. We say that a sequence $\{f_n\}_{n\geq 0}$ in E is an orthonormal basis for E if and only if the following two conditions are satisfied:

(1) Every $f \in E$ can be expanded uniquely as $f = \sum_{n\geq 0} a_n f_n$, with $a_n \in K \to 0$ as $n \to \infty$.

(2) The sup-norm of f is given by $\|f\| = \max\{|a_n|\}$.

For subsequent use we state a simple, yet useful criterion of an orthonormal basis for a K-Banach space E which follows immediately from Serre's criterion [26, Lemme I].

Lemma 2.2. *Let K be a non-Archimedean local field with a nontrivial absolute value and let E be a K-Banach space with an orthonormal basis $\{e_n\}_{n\geq 0}$. If $f_n \in E$ with $\sup_{n\geq 0}\|e_n - f_n\| < 1$, then $\{f_n\}_{n\geq 0}$ is an orthonormal basis of E.*

Proof. See [9, Lemma 3.2] for an alternative proof. □

In what follows, let $R = \mathbf{F}_q[[T]]$ and $K = \mathbf{F}_q((T))$ and let $C(R,K)$ denote the K-Banach space of all continuous functions $f : R \to K$ equipped with the norm $\|f\|$ as defined above. Unlike the classical case, $C(R,K)$ contains a subspace of continuous $\mathbf{F}_q$-linear functions from R to K, which is denoted by $LC(R,K)$.

We now provide a brief review of three sets of well-known orthonormal bases for E in the case where $E = LC(R,K)$. First, the Hasse derivative $\{\mathcal{D}_n\}_{n\geq 0}$ on R is a sequence of functions defined by

$$\mathcal{D}_n(\sum_{i\geq 0} x_i T^i) = \sum_{i\geq n} \binom{i}{n} x_i T^{i-n}.$$

As is shown in [29], $\{\mathcal{D}_n\}_{n\geq 0}$ is a continuous $\mathbf{F}_q$-linear operator on R and satisfies various properties for higher differentiation rules. To recover the expansion coefficients let us recall the Carlitz difference operators $\{\Delta^{(n)}\}_{n\geq 0}$ on $LC(R,K)$, which are defined recursively by

$$(\Delta^{(n)} f)(x) = \Delta^{(n-1)} f(Tx) - T^{q^{n-1}} \Delta^{(n-1)} f(x) (n \geq 1); \Delta^0 = id.$$

For simplicity, Δ and Δ^n denote $\Delta^{(1)}$ and the nth iterate of Δ, respectively.

Theorem 2.3. (1) *$\{\mathcal{D}_n\}_{n\geq 0}$ is an orthonormal basis for $LC(R,K)$.*

(2) *Write* $f = \sum_{n=0}^{\infty} b_n \mathcal{D}_n \in LC(R, K)$. *Then, the coefficients can be recovered by iterating the Carlitz difference operator* Δ :

$$b_n = (\Delta^n f)(1) = \sum_{i=0}^{n} (-1)^{n-i} f(T^i) \mathcal{D}_i(T^n).$$

Proof. See [15, 17] or [28]. □

Secondly, we define the Carlitz $\mathbf{F}_q$-linear polynomial $\{E_n\}_{n\geq 0}$, which is given by

$$E_n(x) = e_n(x)/F_n \ (n \geq 1) \text{ and } E_0(x) = x,$$

where for $n \geq 1$,

$$e_n(x) = \prod_{\substack{\alpha \in \mathbf{F}_q[T] \\ \deg(\alpha) < n}} (x - \alpha)$$

and

$$F_n = [n][n-1]^q \cdots [1]^{q^{n-1}}; \ L_n = [n][n-1] \cdots [1] (n > 0); \ F_0 = L_0 = 1,$$

where $[n] = T^{q^n} - T (n > 0)$.

Theorem 2.4. (1) $\{E_n(x)\}_{n\geq 0}$ *is an orthonormal basis for* $LC(R, K)$.

(2) *Write* $f = \sum_{n\geq 0} a_n E_n(x) \in LC(R, K)$. *Then the coefficients can be recovered by the formula*:

$$a_n = (\Delta^{(n)} f)(1) = \sum_{i=0}^{n} C_i f(T^i), \tag{1}$$

where

$$C_n = 1; \ C_i = (-1)^{n-i} \sum_{e \in S_i} T^e \ (0 \leq i < n), \tag{2}$$

where S_i *is the set of all sums of distinct elements of* $\{1, q, \ldots, q^{n-1}\}$ *taken* $n - i$ *at a time.*

Proof. See [30, 31] or [10] and [15, 17]. □

We point out here that the coefficients in (1) are also recovered by the formula

$$a_n = \sum_{i=0}^{n} \sum_{r=i}^{n} \binom{r}{i} (-T)^{r-i} A_{n,r} f(T^i), \tag{3}$$

where $A_{n,1} = (-1)^{n-1} L_{n-1}$ and for $r > 1$,

$$A_{n,r} = (-1)^{n+r} L_{n-1} \sum_{0 < j_1 < \cdots < j_{r-1} < n} \frac{1}{[j_1][j_2] \cdots [j_{r-1}]},$$

with the convention that $A_{0,0} = 1$ and $A_{n,0} = 1$ for $n > 0$. Indeed, the formula in (3) follows from the identity $\Delta^{(n)} = \sum_{r=0}^{n} A_{n,r} \Delta^r$ in [16, Proposition 3] and the formula (2) in Theorem 2.3. By comparing two formulas for a_n in (1) and (3) the combinatorial sum in (2), for all $0 \le i \le n$, is given by the formula

$$\sum_{e\in S_i} T^e = \sum_{r=i}^{n}(-1)^{n-r}\binom{r}{i}T^{r-i}A_{n,r}.$$

Finally, as an $\mathbf{F}_q$-linear operator on R, the shift map $\{\mathbf{S}^{(n)}\}_{n\ge 0}$ is defined by

$$\mathbf{S}^{(n)}(\sum_{i\ge 0} x_i T^i) \quad = \quad \sum_{i\ge n} x_i T^{i-n}.$$

Theorem 2.5. (1) $\{\mathbf{S}^{(n)}\}_{n\ge 0}$ *is an orthonormal basis for* $LC(R,K)$.

(2) *Write* $f = \sum_{n\ge 0} c_n \mathbf{S}^{(n)}(x) \in LC(R,K)$. *Then the coefficients* c_n *are given by the formula*:

$$\begin{aligned} c_0 &= f(1);\\ c_n &= f(T^n) - Tf(T^{n-1}) \ (n \ge 1). \end{aligned}$$

Proof. See [20]. □

2.2. Cartier operators on $\mathbf{F}_q[[T]]$

We begin by introducing the Cartier operator on $\mathbf{F}_q[[T]]$ that is the main object of this study.

Definition 2.6. For m and r integers such that $0 \le r < q^m$, the Cartier operator $\Delta_{r,m}$ on $\mathbf{F}_q[[T]]$ is defined by

$$\Delta_{r,m}(\sum_{n\ge 0} x_n T^n) = \sum_{n\ge 0} x_{nq^m+r} T^n.$$

Observe that $\Delta_{r,m}$ is a complete generalization of $\Delta_{r,1}$, which is defined in [1, Definition 12.2.1]. Note also that $\Delta_{r,m}$ is an $\mathbf{F}_q$-linear operator on $\mathbf{F}_q[[T]]$ and that it can be defined, for a monomial $T^n (n \ge 0)$, by

$$\Delta_{r,m}(T^n) = \begin{cases} T^l & \text{if } n = lq^m + r \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and then it can be extended to $\mathbf{F}_q[[T]]$ by the $\mathbf{F}_q$-linearity and continuity in Lemma 2.9.

The relevant basic properties of the operators $\Delta_{r,m}$ are stated as follows.

Lemma 2.7. *For* $x, y \in \mathbf{F}_q[[T]]$,

(1) $x = \sum_{r=0}^{q^m-1} T^r \Delta_{r,m}^{q^m}(x)$.
(2) $\Delta_{r,m}(x^{q^m} y) = x\Delta_{r,m}(y)$.
(3) *For all integers* $s \ge 1$ *such that* $r + s < q^m$,

$$\Delta_{r,m}(x) = \Delta_{r+s,m}(T^s x).$$

Proof. The proofs of parts (1) and (2) follow from adapting those of the case $\Delta_{r,1}$ in [1, Lemma 12.2.2]. Part (3) follows from the repeated application of the identity $\Delta_{r,m}(x) = \Delta_{r+1,m}(Tx)$, which is obtained from (4). $\square$

The following result shows that $\Delta_{r,m}$ satisfies the product formula.

Lemma 2.8. *For $x, y \in \mathbf{F}_q[[T]]$,*

$$\Delta_{r,m}(xy) = \sum_{i+j=r} \Delta_{i,m}(x)\Delta_{j,m}(y) + T \sum_{i+j=q^m+r} \Delta_{i,m}(x)\Delta_{j,m}(y).$$

In particular,

$$\Delta_{q^m-1,m}(xy) = \sum_{i+j=q^m-1} \Delta_{i,m}(x)\Delta_{j,m}(y).$$

Proof. This follows from the identity in Lemma 2.7(1) which also provides the uniqueness of such a representation of any element in $\mathbf{F}_q[[T]]$, in terms of $\Delta_{r,m}^{q^m}$. This is left to the reader to verify. $\square$

Lemma 2.9. *For $0 \leq r < q^m$, $\Delta_{r,m}$ is continuous on $\mathbf{F}_q[[T]]$.*

Proof. Because $\Delta_{r,m}$ is linear, it suffices to show that it is continuous at $x = 0$, by checking

$$v(\Delta_{r,m}(x)) \geq [\frac{v(x)}{q^m}], \tag{5}$$

where $[a]$ is the greatest integer number $\leq a$. Setting $n = v(x)$, write $x = T^n y$ with $(T, y) = 1$ and $n = lq^m + s$ with $0 \leq s < q^m$ and $l \geq 0$. Lemma 2.7(2) gives

$$\Delta_{r,m}(x) = \Delta_{r,m}(T^n y) = T^{[n/q^m]}\Delta_{r,m}(T^s y).$$

Because $v((T^s y)) \geq 0$, the preceding equality yields the desired inequality in (5). $\square$

It is now of great interest to find explicit expansions of $\Delta_{r,m}$ and its qth powers, in terms of Hasse derivatives.

Theorem 2.10. *For t, m nonnegative integers and $0 \leq r < q^m$,*

$$\Delta_{r,m}^{q^t} = \sum_{n=0}^{\infty} C_{r,n}^{(t)} \mathcal{D}_n,$$

where $C_{r,n}^{(t)} = (-1)^{n-r}\binom{s}{r} T^{s-r}(T^{q^m} - T^{q^t})^l$ if $n = lq^m + s$ with $0 \leq s < q^m$ and $l \geq 0$.

Proof. By the formula for coefficients in Theorem 2.3, we have

$$C_{r,n}^{(t)} = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} T^{n-i} \Delta_{r,m}^{q^t}(T^i).$$

Writing $n = lq^m + s$, $i = l'q^m + s'$ with $0 \le s, s' < q^m$ and $l, l' \ge 0$, we obtain

$$\begin{aligned} C_{r,n}^{(t)} &= \sum_{l'=0}^{l} \sum_{s'=0}^{s} (-1)^{n-l'q^m-s'} T^{s-s'} T^{(l-l')q^m} \binom{lq^m+s}{l'q^m+s'} \Delta_{r,m}^{q^t}(T^{l'q^m+s'}) \\ &= \sum_{l'=0}^{l} (-1)^{n-l'q^m-r} T^{s-r} T^{(l-l')q^m} \binom{lq^m+s}{l'q^m+r} T^{q^t l'}. \end{aligned}$$

Because $\binom{lq^m+s}{l'q^m+r} = \binom{l}{l'}\binom{s}{r}$ in $\mathbf{F}_q$ from the Lucas congruence [22], we have

$$\begin{aligned} C_{r,n}^{(t)} &= (-1)^{n-r} \binom{s}{r} T^{s-r} T^{lq^m} \sum_{l'=0}^{l} (-1)^{l'} \binom{l}{l'} (T^{q^t-q^m})^{l'} \\ &= (-1)^{n-r} \binom{s}{r} T^{s-r} (T^{q^m})^l (1 - T^{q^t-q^m})^l \\ &= (-1)^{n-r} \binom{s}{r} T^{s-r} (T^{q^m} - T^{q^t})^l. \end{aligned}$$

The proof is complete. □

From Theorem 2.10, we select the most important case for which $t = m$, which gives that the right qth powers of $\Delta_{r,m}$ have a finite expansion in terms of Hasse derivatives.

Theorem 2.11. *For* $0 \le r < q^m$,

$$\Delta_{r,m}^{q^m} = \sum_{n=r}^{q^m-1} \binom{n}{r} (-T)^{n-r} \mathcal{D}_n.$$

In particular,

$$\Delta_{q^m-1,m}^{q^m} = \mathcal{D}_{q^m-1}.$$

Proof. From Theorem 2.10, $C_{r,n}^{(m)}$ vanishes only if $l \neq 0$ in the expression of $n = lq^m + s$. Hence, $C_{r,n}^{(m)} = (-1)^{n-r}\binom{n}{r}T^{n-r}$ for $r \le n < q^m$. The second assertion follows immediately from the first assertion. □

It is also of interest to find the inversion formula to the identity in Theorem 2.11. This indicates that the resulting formula provides an alternative way of calculating $\mathcal{D}_n(a)$ in terms of Cartier operators.

Theorem 2.12. *For* $0 \le n < q^m$,

$$\mathcal{D}_n = \sum_{r=n}^{q^m-1} \binom{r}{n} T^{r-n} \Delta_{r,m}^{q^m}.$$

Proof. Let $C(-T)$ be the q^m by q^m transition matrix from $\{\mathcal{D}_n\}_{0 \le n \le q^m-1}$ into $\{\Delta_{r,m}^{q^m}\}_{0 \le r \le q^m-1}$. From Theorem 2.11, $C(-T)$ is a matrix whose (n, r) entry is $C_{n,r}^{(m)} = \binom{n}{r}(-T)^{n-r}$. In addition, it is an upper triangular matrix whose diagonal entries are all 1; thus, it is invertible. We claim that the inverse of $C(-T)$ is $C(T)$ whose (n,r) entry is $B^{(m)}_{n,r} = \binom{n}{r}T^{n-r}$, equivalently $C(T)C(-T)$ is the identity matrix. For this, we compute the (n,r) entry, denoted $M_{n,r}$, of $C(T)C(-T)$ as follows:

$$\begin{aligned} M_{n,r} &= \sum_{k=0}^{q^m-1} B^{(m)}_{n,k} C^{(m)}_{k,r} \\ &= T^{n-r} \sum_{k=0}^{q^m-1} \binom{n}{k}\binom{k}{r}(-1)^{n-k} \\ &= \binom{n}{r}(-T)^{n-r}\sum_{k=r}^{n}(-1)^{k-r}\binom{n-r}{k-r} \\ &= \binom{n}{r}(-T)^{n-r}(1-1)^{n-r} = \delta_{n,r}, \end{aligned}$$

where $\delta_{n,r}$ is the Kronecker delta symbol, which is used extensively from this point onwards. Then the result follows. □

In Theorem 4.4 we will show that Theorems 2.11 and 2.12 hold over a field of Laurent series with coefficients in a more general field of characteristic $p > 0$. We are now ready to introduce two sets of Cartier operators on $R = \mathbf{F}_q[[T]]$.

Definition 2.13. Let n be an integer with $q^{k-1} \le n < q^k$, or $(n,k) = (0,0)$.

(1) We define a sequence of Cartier operators $\{\phi_n(x)\}_{n\ge 0}$ on R given by

$$\phi_n(x) = \Delta^{q^k}_{n,k}(x) = \sum_{i\ge 0} x_{iq^k+n}T^{iq^k}.$$

(2) We define a sequence of Cartier operators $\{\psi_n(x)\}_{n\ge 0}$ on R given by

$$\psi_n(x) = \Delta_{n,k}(x) = \sum_{i\ge 0} x_{iq^k+n}T^{i}.$$

From the definitions of both ϕ_n and ψ_n ($n \ge 1$), it follows that they implicitly involve the positive integer k, which is uniquely determined by $k = [\log_q n] + 1$,; that is, the number of q-adic digits of n. In what follows, the letter k is not appended to the notation of the two Cartier operators. Later on, it is shown that it is even easier to use ϕ_n than ψ_n. However, the latter is dealt with by way of the former and both ϕ_n and ψ_n play the same role in compositions of their q-adic extensions. It is now worth noting that for $q^{k-1} \le n < q^k$ and $x, y \in R$,

$$\phi_n = \psi_n^{q^k},$$

$$\phi_n(x^{q^k}y) = x^{q^k}\phi_n(y), \tag{6}$$

$$\phi_{q^k-1}(x) = \mathcal{D}_{q^k-1}(x), \tag{7}$$

$$\phi_n(x) = \phi_{q^k-1}(T^{q^k-1-n}x), \tag{8}$$

$$\psi_n(x^{q^k}y) = x\psi_n(y). \tag{9}$$

For $m = lq^k + m_0$ with $0 \le m_0 < q^k$ and $l \ge 0$, we have

$$\phi_n(T^m) = T^{m-m_0}\delta_{n,m_0} \text{ and } \psi_n(T^m) = T^{[m/q^k]}\delta_{n,m_0}. \tag{10}$$

For n, a positive integer written in q-adic form as

$$n = \sum_{i=0}^{k-1} n_i q^i \ (0 \le n_i < q \text{ and } n_{k-1} \ne 0),$$

we define $q(n)$ and n_-, respectively, as

$$q(n) = n_{k-1}q^{k-1} \text{ and } n_- = n - q(n). \tag{11}$$

As one of the main results, we state the following theorem.

Theorem 2.14. *Let $f : R \to K$ be an $\mathbf{F}_q$-linear continuous function. Set*

$$\begin{aligned} c_0 &= f(1), \\ c_n &= f(T^n) - T^{q(n)} f(T^{n_-}) \ (n \ge 1), \end{aligned} \tag{12}$$

where $q(n)$ and n_- are defined in (11)*. Then, $\{\phi_n\}_{n\ge 0}$ is an orthonormal basis for $LC(R,K)$. That is, $\sum_{n=0}^{\infty} c_n\phi_n(x)$ converges uniformly to $f(x)$.*

Proof. We provide two proofs. The first proof is based on standard arguments in non-Archimedean analysis, for example, [24, Theorem, p. 183].

Because $f(x)$ is continuous at $x = 0$, it is obvious from (12) that c_n is a null sequence in K because, as $n \to \infty$, $q(n) \to \infty$. As R is compact, the series $\sum_{n=0}^{\infty} c_n\phi_n(x)$ converges uniformly to the continuous function $f(x)$. Here we need to show that the two continuous functions are equal on R. By the $\mathbf{F}_q$-linearity and continuity, it now suffices to check that they agree on all monomials $T^m (m \ge 0)$. For this, write a positive integer $m = \sum_{j=0}^{k} m_{i_j} q^{i_j}$ in q-adic form such that $m_{i_j} \ne 0$ for all $0 \le j \le k$ and $0 \le i_0 < i_1 < \cdots < i_k$.

Using (10) we calculate

$$\begin{aligned} \sum_{n\ge 0} c_n\phi_n(T^m) = & \sum_{n=0}^{m} c_n\phi_n(T^m) \\ &= f(1)T^m + c_{m_{i_0}q^{i_0}} T^{m-m_{i_0}q^{i_0}} + c_{m_{i_0}q^{i_0}+m_{i_1}q^{i_1}} T^{m-m_{i_0}q^{i_0}-m_{i_1}q^{i_1}} \\ &\quad + \cdots + c_{m_{i_0}q^{i_0}+m_{i_1}q^{i_1}+\cdots+m_{i_{k-1}}q^{i_{k-1}}} T^{m_{i_k}q^{i_k}} + c_m. \end{aligned}$$

By the formula in (12), the right hand side of the preceding equality equals

$$\begin{aligned} & f(1)T^m + (f(T^{m_{i_0}q^{i_0}}) - T^{m_{i_0}q^{i_0}} f(1))T^{m-m_{i_0}q^{i_0}} \\ & + (f(T^{m_{i_0}q^{i_0}+m_{i_1}q^{i_1}}) - T^{q(m_{i_0}q^{i_0}+m_{i_1}q^{i_1})} f(T^{m_{i_0}q^{i_0}}))T^{m-m_{i_0}q^{i_0}-m_{i_1}q^{i_1}} \\ & + \cdots + (f(T^{\sum_{j=0}^{k-1} m_{i_j}q^{i_j}}) - T^{m_{i_{k-1}}q^{i_{k-1}}} f(T^{\sum_{i=0}^{k-2} m_{i_j}q^{i_j}}))T^{m_{i_k}q^{i_k}} \\ & + f(T^m) - T^{q(m)} f(T^{m_-}). \end{aligned}$$

It follows that the sum above becomes $f(T^m)$, because it is a telescoping sum, by (11).

In the usual way, from (12), we deduce that $\|f\| = \max\{|c_n|\}$. For an alternative proof, we use [26, Lemme I], as was done with Hasse derivatives in [10] and [15,20]. In this lemma, a necessary and sufficient condition for $\{\phi_n\}_{n\geq 0}$ to be an orthonormal basis for $LC(R,K)$ is that

(1) ϕ_n maps R into itself;

(2) the reduced functions modulo T, denoted $\overline{\phi_n}$, form a basis for $LC(R,\mathbf{F}_q)$ as an $\mathbf{F}_q$-vector space.

Since Part (1) is trivial, we only check Part (2) by showing that for any integer $n > 0$, the reduced functions $\overline{\phi_0}, \overline{\phi_1}, \ldots, \overline{\phi_{n-1}}$ are linearly independent in the $\mathbf{F}_q$-dual space $(\mathbf{F}_q[T]/T^n)^*$. In fact, using (10), these functions are the dual basis to $1, T, \ldots, T^{n-1}$. □

The qth power maps have an explicit expansion in terms of Cartier operators ϕ_n.

Corollary 2.15. *For any integer $m \geq 0$,*

$$x^{q^m} = x + \sum_{n\geq 1}(T^{nq^m} - T^{q(n)+q^m n_-})\phi_n(x).$$

Proof. The proof is immediate from Theorem 2.14. □

The expansions in Corollary 2.15 can be compared with Voloch's expansions of the qth power maps in terms of Hasse derivatives $\mathcal{D}_n$ (see [29]):

$$x^{q^m} = \sum_{n\geq 0}(T^{q^m} - T)^n \mathcal{D}_n(x). \tag{13}$$

Corollary 2.16. *For $q^{k-1} \leq r < q^k \leq q^m$ or $(r,k) = (0,0)$,*

$$\Delta^{q^m}_{r,m}(x) = \phi_r(x) - \sum_{\substack{k\leq i\leq m-1\\ 1\leq j\leq q-1}} T^{jq^i}\phi_{jq^i+r}(x).$$

Proof. By Theorem 2.14, writing $\Delta^{q^m}_{r,m}(x) = \sum_{n\geq 0} B^{(m)}_{r,n}\phi_n(x)$, we have $B^{(m)}_{r,0} = 0$ and for $n \geq 1$,

$$B^{(m)}_{r,n} = \Delta^{q^m}_{r,m}(T^n) - T^{q(n)}\Delta^{q^m}_{r,m}(T^{n_-}). \tag{14}$$

If $n \geq q^m$, then writing $n = lq^m + s$ with $l \geq 1$ and $0 \leq s < q^m$ we have

$$B^{(m)}_{r,n} = T^{n-s}\delta_{r,s} - T^{q(n)+n_- - s}\delta_{r,s} = 0.$$

If $n < r$, then $B^{(m)}_{r,n}$ also vanishes; thus, we may assume that $r \leq n < q^m$. From (14), we have

$$B^{(m)}_{r,n} = \delta_{r,n} - T^{q(n)}\delta_{r,n_-}.$$

From this relation we deduce that $B^{(m)}_{r,n} = 1$ if $n = r$, $-T^{q(n)}$ if $n > r$, and $n_- = r$, 0 if $n > r$ and $n_- = r$. This implies that $n \neq r$ is of the form $n = jq^i + r$

with $k \le i < m$ and $1 \le j \le q-1$. Therefore, we obtain the desired formula for $\Delta_{r,m}^{q^m}$. The proof of the case where $r = 0 = k$ follows in the similar way. $\square$

The following corollary can be deduced in a similar fashion as Corollary 2.16.

Corollary 2.17. *For $q^{k-1} \le r < q^k \le q^m$ or $(r,k) = (0,0)$,*

$$\Delta_{r,m}(x) = \phi_r(x) + \sum_{j>0}(T^j - T^{jq^m+j_-})\phi_{jq^m+r}(x) - \sum_{\substack{k\le i\le m-1\\ 1\le j\le q-1}} T^{jq^i}\phi_{jq^i+r}(x).$$

We state another main result related to Cartier operators ψ_n.

Theorem 2.18. *$\{\psi_n\}_{n\ge 0}$ is an orthonormal basis for $LC(R,K)$.*

Proof. We provide two proofs. For the first proof we invoke the following from Corollary 2.17 with $m = k$: For $q^{k-1} \le n < q^k$,

$$\psi_n(x) = \phi_n(x) + \sum_{j>0}(T^j - T^{jq^k+j_-})\phi_{jq^k+n}.$$

This identity implies that $\psi_n \equiv \phi_n \pmod T$; that is, $||\psi_n - \phi_n|| < 1$ for all $n \ge 0$. Then, the result follows from Lemma 2.2, together with Theorem 2.14.

The second proof follows immediately from the same argument applied to the second proof of Theorem 2.14. $\square$

Unlike ϕ_n, it is not easy to find a closed-form formula for coefficients in the representation of

$$f = \sum_{n\ge 0} B_n\psi_n(x) \in LC(R,K).$$

However, if $f \in LC(R,K)$ assumes values in R, it is very useful to observe the following simple relation on coefficients B_n from the proof of Theorem 2.14. For all $n \ge 1$,

$$B_n \equiv f(T^n) \pmod T. \tag{15}$$

This relation is sufficient to be used for proving that another sequence of operators is an orthonormal basis for $LC(R,K)$ in the next subsection. Refer to the formula for B_n in (19), which is indirectly derived from the coefficient formula for $f \in C(R,K)$.

Thus far, we introduced five orthonormal bases for $LC(R,K)$ among which two are new. Here we show the existence of a close relation between any two orthonormal bases among these five bases. Indeed, Theorem 2.19 below states that any two bases are equivalent to each other, which means that if one is an orthonormal basis for $LC(R,K)$, so is the other and vice versa.

Theorem 2.19. *All five orthonormal bases are equivalent.*

Proof. All equivalences follow from Lemma 2.2 and the results in Section 2.1, together with Theorems 2.14 and 2.18. So the detailed proofs are omitted here. $\square$

The product rule can be stated as follows.

Lemma 2.20. *For $q^{k-1} \le n < q^k$ and $x, y \in R$,*

$$\begin{aligned}\phi_n(xy) &= \phi_n(y)\phi_0(x) + \sum_{j=1}^{q^k-1} (\phi_n(T^j y) - T^{q(j)}\phi_n(T^{j-} y))\phi_j(x) \\ &= \phi_n(x)\phi_0(y) + \sum_{j=1}^{q^k-1} (\phi_n(T^j x) - T^{q(j)}\phi_n(T^{j-} x))\phi_j(y).\end{aligned}$$

Proof. We only prove the first identity because the interchanging roles of x and y provide the second identity. As an $\mathbf{F}_q$-linear continuous operator of x, write $\phi_n(xy) = \sum_{j=0}^{\infty} C_j(y)\phi_j(x)$. Then, by the formula in Theorem 2.14, $C_j(y) = \phi_n(T^j y) - T^{q(j)}\phi_n(T^{j-} y)$ for $j > 0$ and $C_0(y) = \phi_n(y)$. Now, it is easy to verify that $C_j(y)$ vanishes for all $j \ge q^k$. ☐

3. Orthonormal bases for $C(\mathcal{O}, \mathcal{V})$

Now, we redirect our attention to continuous functions from $\mathcal{O}$ to $\mathcal{V}$ where $\mathcal{O}$ is the integer ring of a local field $\mathcal{V}$ as in Section 2.1. We provide two sets of orthonormal bases of the entire space $C(\mathcal{O}, \mathcal{V})$ of all continuous functions from $\mathcal{O}$ to $\mathcal{V}$ for two distinguished cases where $\mathcal{O} = R = \mathbf{F}_q[[T]]$ or $\mathcal{O} = \mathbf{Z}_p$. These results essentially follow from Conrad's digit principle: [10, Theorem 2] for $\mathbf{F}_q[[T]]$ and [10, Theorem 11] for $\mathbf{Z}_p$.

3.1. Two q-adic digit Cartier bases of $C(R, K)$

The following definition is crucial for the construction of an orthonormal basis for $C(R, K)$ out of that of $LC(R, K)$.

Definition 3.1. Let $\{f_i(x)\}_{i\ge 0}$ be an orthonormal basis for $LC(R, K)$, and let

$$n = n_0 + n_1 q + \cdots + n_{w-1} q^{w-1}$$

be the q-adic expansion of any integer $n \ge 0$, with $0 \le n_i < q$. Set

$$\mathcal{F}_n(x) := \prod_{i=0}^{w-1} f_i^{n_i}(x) \ (n \ge 1), \ \mathcal{F}_0(x) = 1,$$

and

$$\mathcal{F}_n^*(x) := \prod_{i=0}^{w-1} \mathcal{F}_{n_i q^i}^*(x) \ (n \ge 1), \ \mathcal{F}_0^*(x) = 1,$$

where

$$\mathcal{F}_{n_i q^i}^*(x) = \begin{cases} f_i^{q-1}(x) - 1 & \text{if } n_i = q-1 \\ f_i^{n_i}(x) & \text{if } n_i < q-1. \end{cases}$$

Then, we say $\{\mathcal{F}_n(x)\}_{n\ge 0}$ in $C(R, K)$ is a q-adic extension of $\{f_i(x)\}_{i\ge 0}$ in $LC(R, K)$.

Well-known examples of such q-adic extensions include

$$(f_i, \mathcal{F}_n, \mathcal{F}_n^*) = (E_i, G_n, G_n^*), (\mathcal{D}_i, \mathsf{D}_n, \mathsf{D}_n^*), (\mathbf{S}^{(i)}, \mathbf{S}_n, \mathbf{S}_n^*), \tag{16}$$

where $E_i, \mathcal{D}_i$, and $\mathbf{S}^{(i)}$ are referred to as Carlitz linear polynomials, Hasse derivatives, and shift operators, respectively, as in Section 2.1, and G_n, D_n, and $\mathbf{S}_n$ are referred to as Carlitz polynomials, digit derivatives, and digit shifts, respectively. It is shown in [5] that Carlitz polynomials G_n are a prototypal q-adic extension of E_i. Besides these examples, by Theorems 2.14 and 2.18 we add to this list two more q-adic extensions

$$(f_i, \mathcal{F}_n, \mathcal{F}_n^*) = (\phi_i, \Phi_n, \Phi_n^*), (\psi_i, \Psi_n, \Psi_n^*), \tag{17}$$

where Φ_n and Ψ_n are referred to as digit Cartier I and II, respectively. For the remainder of this subsection, we only consider $(f_i, \mathcal{F}_n, \mathcal{F}_n^*)$ in (17) to emphasize both of these digit Cartier functions. We now examine some properties of these two functions, which are modeled on the properties such as the binomial and orthogonal formulas of Carlitz polynomials. Note that

$$f_i(T^j) = \begin{cases} 0 & \text{if } j < i; \\ 1 & \text{if } j = i; \\ \equiv 0 \pmod{T} & \text{if } j > i. \end{cases} \tag{18}$$

Proposition 3.2. *The binomial formulas for $\mathcal{F}_n$ and $\mathcal{F}_n^*$ are*

(1) $\mathcal{F}_n(\lambda x) = \lambda^n \mathcal{F}_n(x)$ *for* $\lambda \in \mathbf{F}_q$.
(2) $\mathcal{F}_n(x+y) = \sum_{i=0}^n \binom{n}{i} \mathcal{F}_i(x)\mathcal{F}_{n-i}(y)$.
(3) $\mathcal{F}_n^*(\lambda x) = \lambda^n \mathcal{F}_n(x)$ *for* $\lambda \in \mathbf{F}_q$.
(4) $\mathcal{F}_n^*(x+y) = \sum_{i=0}^n \binom{n}{i} \mathcal{F}_i(x)\mathcal{F}_{n-i}^*(y)$.

Proof. The proof follows by adopting the arguments in [5] or [12] for G_n and G_n^* to our case. □

Because $\binom{q^m-1}{i} = (-1)^i$ in $\mathbf{F}_q$, we obtain the following corollary of Propositions 3.2.

Corollary 3.3. (1) $\mathcal{F}_{q^m-1}(x+u) = \sum_{i+j=q^m-1} (-1)^i \mathcal{F}_i(x)\mathcal{F}_j(u)$.
(2) $\mathcal{F}_{q^m-1}(x-u) = \sum_{i+j=q^m-1} \mathcal{F}_i(x)\mathcal{F}_j(u)$.
(3) $\mathcal{F}_{q^m-1}^*(x+u) = \sum_{i+j=q^m-1} (-1)^i \mathcal{F}_i(x)\mathcal{F}_j^*(u)$.
(4) $\mathcal{F}_{q^m-1}^*(x-u) = \sum_{i+j=q^m-1} \mathcal{F}_i(x)\mathcal{F}_j^*(u)$.

The following is the orthogonality property of the two digit Cartier functions.

Proposition 3.4. (1) *For $l < q^n, k$ an arbitrary integer ≥ 0,*

$$\sum_{\substack{\alpha \in \mathbf{F}_q[T] \\ \deg(\alpha) < n}} \mathcal{F}_k(\alpha)\mathcal{F}_l^*(\alpha) = \begin{cases} 0 & \textit{if } k+l \neq q^n - 1 \\ (-1)^n & \textit{if } k+l = q^n - 1. \end{cases}$$

(2) *For $l < q^n, k < q^n$,*

$$\sum_{\substack{\alpha \text{ monic} \\ \deg(\alpha)=n}} \mathcal{F}_k(\alpha)\mathcal{F}_l^*(\alpha) = \begin{cases} 0 & \text{if } k+l \neq q^n-1 \\ (-1)^n & \text{if } k+l = q^n-1. \end{cases}$$

Proof. Two proofs are known for the case $(\mathcal{F}_n, \mathcal{F}_n^*) = (G_n, G_n^*)$. Indeed, Carlitz [5] provided the original proof, which is based on interpolations of his polynomials. Yang [32] established the same result in a direct yet elementary way. These two arguments are also applied to the two digit Cartier functions, as in the case of digit derivatives and digit shifts [15, 17]. In particular, Yang's argument operates with the digit principle once we have a basis $\{f_i\}_{i\geq 0}$ for $LC(R, K)$, having an additional property in (18). □

The digit principle leads to the following main result.

Theorem 3.5. *Let $(\mathcal{F}_n, \mathcal{F}_n^*) = (\Phi_n, \Phi_n^*)$ or (Ψ_n, Ψ_n^*).*

(1) *$\{\mathcal{F}_n(x)\}_{n\geq 0}$ is an orthonormal basis for $C(R, K)$.*
(2) *Write $f \in C(R, K)$ as $f(x) = \sum_{n\geq 0} c_n\mathcal{F}_n(x)$. Then, for any integer w such that $q^w > n$, c_n can be recovered by*

$$c_n = (-1)^w \sum_{\alpha\in A_w} \mathcal{F}^*_{q^w-1-n}(\alpha)f(\alpha),$$

where A_w denotes the set of all polynomials in T with coefficients in $\mathbf{F}_q$ of degree $< w$.

Proof. Part (1) follows from applying the digit principle to two bases, ϕ_n and ψ_n for $LC(R, K)$ in Theorems 2.14 and 2.18. For part (2), it follows from the orthogonality property in Proposition 3.4. For any integer w such that $q^w > n$, we have

$$(-1)^w \sum_{\alpha\in A_w} \mathcal{F}^*_{q^w-1-n}(\alpha)f(\alpha) = \sum_{j=0}^{\infty} c_j(-1)^w \sum_{\alpha\in A_w} \mathcal{F}^*_{q^w-1-n}(\alpha)\mathcal{F}_j(\alpha) = c_n.$$

□

Application of the digit principle to Theorem 2.19 produces the following result.

Theorem 3.6. *The q-adic extensions of all five bases of $LC(R, K)$ are equivalent as orthonormal bases for $C(R, K)$.*

Proof. The proof follows from the application of the digit principle to Theorem 2.19. For the individual proofs of the three bases in (16) we refer the reader to the following works: see [10, 12, 30, 32] for G_n and [9, 15, 17, 18, 28] for D_n and [20] for $\mathbf{S}_n$. □

For $f \in C(R, K)$ to lie in $LC(R, K)$, we provide the conditions in terms of its coefficients.

Corollary 3.7. *Write $f \in C(R, K)$ as $f(x) = \sum_{n=0}^{\infty} c_n \mathcal{F}_n(x)$. Then $f \in LC(R, K)$ if and only if $c_n = 0$ for $n \neq q^i$, where $i \geq 0$.*

Proof. This follows from Theorem 2.19. However, an alternative proof follows by adopting the arguments [30] or [17] to our case. □

From Corollary 3.7 and Theorem 3.5 we can indirectly retrieve the coefficients of $f(x) = \sum_{n=0}^{\infty} B_n \psi_n(x)$ by computing, for any w such that $q^n < q^w$,

$$B_n = (-1)^w \sum_{\alpha \in A_w} \Psi^*_{q^w - 1 - q^n}(\alpha) f(\alpha). \tag{19}$$

The formula for the coefficients c_n in Theorem 3.5 yields the following corollary.

Corollary 3.8. *Let $f(x) = \sum_{n \geq 0} c_n \mathcal{F}_n(x)$ be a continuous function from R to K. Then $f(x) \in C(R, R)$ if and only if $\{c_n\}_{n \geq 0} \subset R$.*

3.2. Two p-adic digit Cartier bases of $C(\mathbf{Z}_p, \mathbf{Q}_p)$

Here, we introduce analogues in $\mathbf{Z}_p$ of two Cartier operators on R and then show that their p-adic extensions form an orthonormal basis of $C(\mathbf{Z}_p, \mathbf{Q}_p)$. Now, the Cartier maps on $\mathbf{Z}_p$ can be defined in the same way as was with R in Definition 2.13, with the same notation to denote those maps.

Definition 3.9. Let n be an integer such that $p^{k-1} \leq n < p^k$, or $(n, k) = (0, 0)$.

(1) The Cartier map ϕ_n on $\mathbf{Z}_p$ is defined by

$$\phi_n\Big(\sum_{i \geq 0} x_i p^i\Big) = \sum_{i \geq 0} x_{ip^k + n} p^{ip^k}.$$

(2) The Cartier map ψ_n on $\mathbf{Z}_p$ is defined by

$$\psi_n\Big(\sum_{i \geq 0} x_i p^i\Big) = \sum_{i \geq 0} x_{ip^k + n} p^{i}.$$

Lemma 3.10. *For each $n \geq 0$, ϕ_n and ψ_n are continuous functions on $\mathbf{Z}_p$.*

Proof. It suffices to show that for each integer $n \geq 0$, all $x \in \mathbf{Z}_p$, and $m \geq n$,

$$\phi_n(x + p^m z)) \equiv \phi_n(x) \pmod{p^{m-n}};$$

$$\psi_n(x + p^m z)) \equiv \psi_n(x) \pmod{p^{[(m-n)/p^k]}}.$$

We leave the proof of these congruences to the reader because it follows from Definition 3.9. □

The following result gives the Mahler expansion of ϕ_n.

Lemma 3.11. *Let $\phi_n = \sum_{j=0}^{\infty} a_j^{(n)} \binom{x}{j}$ be the Mahler expansion of ϕ_n. Then the coefficients $a_j^{(n)}$ possess the following properties*:

(1) *$a_j^{(n)} = 0$ for $0 \leq j < p^n$;*

(2) *$a_j^{(n)} = 1$ for $j = p^n$;*

(3) *If $j > p^n$, then p divides $a_j^{(n)}$.*

Proof. We use Mahler's result to write $\phi_n = \sum_{j=0}^{\infty} a_j^{(n)} \binom{x}{j}$. Then, from the well-known formula for coefficients, we have

$$a_j^{(n)} = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \phi_n(i).$$

We observe from the definition of ϕ_n that $\phi_n(i) = 0$ for $0 \leq i < p^n$ and $\phi_n(p^n) = 1$. Then Parts (1) and (2) follow from these observations. Furthermore, they also give

$$a_j^{(n)} = \sum_{i=p^n}^{j} (-1)^{j-i} \binom{j}{i} \phi_n(i).$$

For part (3), we use Lucas's congruence to show that $a_j^{(n)} \equiv 0 \pmod{p}$ for $j > p^n$. Write $j \geq i \geq p^n$ in p-adic form as

$$j = j_0 + j_1 p + \cdots + j_n p^n + \cdots + j_s p^s;$$

$$i = i_0 + i_1 p + \cdots + i_n p^n + \cdots + i_s p^s.$$

Note that $\phi_n(i) \equiv i_n \pmod{p}$ if $i \geq p^n$ is of such p-adic form with $i_n \neq 0$. Application of Lucas's congruence to $a_j^{(n)}$ gives

$$a_j^{(n)} \equiv \sum_{\substack{0 \leq i_l \leq j_l, \forall l \neq n, \\ 1 \leq i_n \leq j_n}} (-1)^{j_0 - i_0} \cdots (-1)^{j_s - i_s} \binom{j_0}{i_0} \cdots \binom{j_s}{i_s} i_n \pmod{p}. \tag{20}$$

If $j_l \neq 0$ for some $l \neq n$, then it is easy to see that the sum in (20) vanishes from the identity

$$\sum_{0 \leq i_l \leq j_l} (-1)^{j_l - i_l} \binom{j_l}{i_l} = (1-1)^{j_l}.$$

If $j_l = 0$ for all l with $l \neq n$, then $j = j_n p^n > p^n$ with $j_n > 1$. Hence, the sum in (20) vanishes as

$$\sum_{1 \leq i_n \leq j_n} (-1)^{j_n - i_n} i_n \binom{j_n}{i_n} = j_n (-1)^{j_n - 1} (1-1)^{j_n - 1}.$$

We complete the proof. □

Parallel to Theorem 3.5, we have the following theorem for $\mathbf{Z}_p$.

Theorem 3.12. *For any integer $j \geq 0$, write $j = a_0 + a_1 p + \cdots + a_n p^n$ with $0 \leq a_i < p$. Set*

$$\Phi_j(x) = (\phi_0(x))^{a_0} (\phi_1(x))^{a_1} \cdots (\phi_n(x))^{a_n} \ (j > 0), \ \Phi_0(x) = 1.$$

Then, $\{\Phi_j\}_{j \geq 0}$ is an orthonormal basis for $C(\mathbf{Z}_p, \mathbf{Q}_p)$.

Proof. We provide two proofs which rely on Conrad's digit principle [10, Theorem 2]. First, we provide a direct proof by showing that for any integer $n > 0$, the map $\mathbf{Z}_p/p^n\mathbf{Z}_p \to (\mathbf{Z}_p/p\mathbf{Z}_p)^n$ defined by

$$x \mapsto (\phi_0(x), \phi_1(x), \ldots, \phi_{n-1}(x)) \pmod p$$

is a bijection. Then, the map is well defined because of the observation that if $x \equiv y \pmod{p^n}$, then $\phi_i(x) \equiv \phi_i(y) \pmod p$ for all $0 \le i < n$. Let us show that the map is bijective, which is equivalent to being surjective. Writing $x = x_0 + x_1 p + \cdots + x_{n-1}p^{n-1} \in \mathbf{Z}_p/p^n\mathbf{Z}_p$ with $0 \le x_i < p$, the image of the map is then simply

$$(\phi_0(x), \phi_1(x), \ldots, \phi_{n-1}(x)) \pmod p = (x_0, x_1, \ldots, x_{n-1}).$$

Then, the map is surjective, hence, bijective, completing the first proof. We provide a second proof using Lemma 3.11 by which we have

$$||\phi_n(x) - \binom{x}{p^n}|| \le 1/p < 1.$$

These inequalities also imply

$$||\Phi_j(x) - \{\begin{matrix} x \\ j \end{matrix}\}|| \le 1/p < 1,$$

where

$$\{\begin{matrix} x \\ j \end{matrix}\} = \binom{x}{1}^{a_0}\binom{x}{p}^{a_1}\cdots\binom{x}{p^n}^{a_n}$$

for the p-adic representation of j in Theorem 3.12.

Theorem 3.12 now follows by applying Lemma 2.2 to the inequality above, together with [10, Theorem 11] which reads that $\{\begin{matrix} x \\ j \end{matrix}\}_{j\ge 0}$ is an orthonormal basis for $C(\mathbf{Z}_p, \mathbf{Q}_p)$. □

The following result is also parallel to Theorem 3.5.

Theorem 3.13. *For any integer $j \ge 0$, write $j = a_0 + a_1 p + \cdots + a_n p^n$ with $0 \le a_i < p$. Set*

$$\Psi_j(x) = (\psi_0(x))^{a_0}(\psi_1(x))^{a_1}\cdots(\psi_n(x))^{a_n} \ (j > 0), \ \Psi_0(x) = 1.$$

Then, $\{\Psi_j\}_{j\ge 0}$ is an orthonormal basis for $C(\mathbf{Z}_p, \mathbf{Q}_p)$.

Proof. An independent proof follows by applying the same proof that was applied in Theorem 3.12 to ψ_n. An alternative proof follows from Theorem 3.12 and Lemma 2.2 together with the observation that for all $n \ge 0$,

$$\phi_n \equiv \psi_n \pmod p$$

equivalently for all $j \ge 0$,

$$\Phi_j \equiv \Psi_j \pmod p. \qquad □$$

Unlike the function field case, we were unable to determine a formula for the expansion coefficients in Theorems 3.12 and 3.13. Therefore, it may be interesting to find a closed-form formula for the coefficients in these two theorems.

4. Cartier operators on more general settings

In this section, we deal with the Cartier operators defined on more general settings than in the previous sections. Motivated by the relations in Theorems 2.11 and 2.12, we will show in Theorem 4.4 that Hasse and Cartier operators satisfy what is known as the binomial inversion formula in the sense that it resembles the well-known binomial inversion formula for two sequences of numbers. Moreover, we employ Cartier operators to present several Wronskian criteria for linear independence on the same settings.

4.1. Binomial inversion formula for Hasse and Cartier operators.

Let κ be a perfect field of positive characteristic p and let $\kappa[[t]]$ be the ring of formal power series in one variable t over κ. Let v be the additive valuation on $\kappa[[t]]$ such that the associated absolute value $|\cdot|$ on $\kappa[[t]]$ can be naturally extended to the quotient field of $\kappa[[t]]$, denoted $\kappa((t))$.

For a comparison with Cartier operators we first recall higher derivatives (also termed Hasse derivations). The k-linear higher derivative $\{\mathcal{D}_{n,t}\}_{n\geq 0}$ on $\kappa[[t]]$ ($\mathcal{D}_n = \mathcal{D}_{n,t}$ in abbreviated notation) is defined by

$$\mathcal{D}_n(t^m) = \binom{m}{n} t^{m-n}.$$

As continuous κ-linear operators, higher derivatives satisfy various properties such as the product formula and chain rule and the reader can consult [19] for additional background information in this regard. We define two Cartier operators on $\kappa[[t]]$. To this end, we fix q as a power of a prime number p, being characteristic of κ.

Definition 4.1. Let n be an integer such that $q^{k-1} \leq n < q^k$ for $k > 0$ an integer or $(n, k) = (0, 0)$. Then, the Cartier operators $\{\phi_n\}_{n\geq 0}$ and $\{\psi_n\}_{n\geq 0}$ on $\kappa[[t]]$ are respectively defined by

$$\phi_n(\sum_{i\geq 0} x_i t^i) = \sum_{i\geq 0} x_{iq^k+n} t^{iq^k};$$
$$\psi_n(\sum_{i\geq 0} x_i t^i) = \sum_{i\geq 0} x_{iq^k+n}^{1/q^k} t^i.$$

The assumption that κ is a perfect field of characteristic $p > 0$ is necessary because it guarantees that the coefficients of $\psi_n(x)$ lie in κ. Note that $\{\phi_n\}_{n\geq 0}$ is alternatively defined, for a monomial t^m, by

$$\phi_n(t^m) = t^{m-r}\delta_{n,r},$$

where r is the remainder of the division of m by q^k, that is, $m = lq^k + r$ with $0 \le r < q^k$ and $l \ge 0$.

As in the proof of Lemma 2.7(2), we deduce that for $q^{k-1} \le n < q^k$, and $x, y \in \kappa[[t]]$.

$$\phi_n(x^{q^k} y) = (x^{(k)})^{q^k} \phi_n(y);$$
$$\psi_n(x^{q^k} y) = x^{(k)} \psi_n(y),$$

where $x^{(k)} = \sum_{i \ge 0} x_i^{q^k} t^i$. The preceding identities imply that for $x \in \kappa[[t]]$,

$$\phi_n(t^{q^k} x) = t^{q^k} \phi_n(x); \tag{21}$$
$$\psi_n(t^{q^k} x) = t\psi_n(x).$$

Lemma 4.2. *For each $n \ge 0$, ϕ_n and ψ_n are continuous κ-linear operators on $\kappa[[t]]$.*

Proof. For ϕ_n to be continuous, it suffices to show from the κ-linearity that the following inequality holds:

$$v(\phi_n(x)) \ge v(x) - n, \tag{22}$$

implying the continuity of f at $x = 0$. Writing $m := v(x) = lq^k + s$ with $0 \le s < q^k$ and $l \ge 0$, and $x = t^m y$ with $(t, y) = 1$, using (21), we have

$$v(\phi_n(x)) \ge m - s + v(\phi_n(t^s y)).$$

A direct estimation of $v(\phi_n(t^s y))$ gives the desired result in (22). As for ψ_n, it follows from the inequality $v(\varphi_n(x)) = v(\varphi_n(t^m y)) \ge [m/q^k]$ as in Lemma 2.9. $\square$

Lemma 4.3. *For any integer $m > 0$ such that $m = lq^k + s$ with $0 \le s < q^k$ and $l \ge 0$ and for an integer n such that $1 \le n < q^k$,*

$$\sum_{r=n}^{q^k-1} \binom{r}{n}\binom{m+r-1}{r} \equiv \begin{cases} (-1)^n \pmod p & \textit{if } n + s = q^k, \\ 0 \pmod p & \textit{otherwise.} \end{cases}$$

Proof. By the well-known identities for binomial coefficients, we see

$$\sum_{r=n}^{q^k-1} \binom{r}{n}\binom{m+r-1}{r} = \binom{m+n-1}{m-1} \sum_{i=0}^{q^k-1-n} \binom{m+n+i-1}{m+n-1}.$$

Because the sum on the right above equals the coefficient of x^{m+n-1} in the polynomial of the form $(1+x)^{m+n-1} + \cdots + (1+x)^{m+q^k-1}$, we obtain

$$\begin{aligned} &\binom{m+n-1}{m-1} \sum_{i=0}^{q^k-1-n} \binom{m+n+i-1}{m+n-1} \\ &= \binom{m+n-1}{m-1}\binom{m+q^k-1}{m+n} \end{aligned}$$

$$= \binom{lq^k + s + n - 1}{lq^k + s - 1}\binom{lq^k + q^k + s - 1}{lq^k + s + n}.$$

If $n + s = q^k$ then by Lucas's congruence, we have

$$\binom{lq^k + s + n - 1}{lq^k + s - 1}\binom{lq^k + q^k + s - 1}{lq^k + s + n} \equiv \binom{q^k - 1}{s - 1} \equiv (-1)^{s-1} \equiv (-1)^n \pmod{p}.$$

In general, writing $n+s = \varepsilon q^k + j \neq q^k$ for some $\varepsilon \in \{0, 1\}$ and $0 \leq j < q^k - 1$, we note that if $\varepsilon = 0$, then $j \geq 1$, and that if $\varepsilon = 1$, then $j = 0$ is excluded. For these cases, Lucas's congruence gives

$$\begin{aligned} &\binom{lq^k + s + n - 1}{lq^k + s - 1}\binom{lq^k + q^k + s - 1}{lq^k + s + n} \\ \equiv &\binom{l + \varepsilon}{l}\binom{j - 1}{s - 1}\binom{l + 1}{l + \varepsilon}\binom{s - 1}{j} \equiv 0 \pmod{p}. \end{aligned}$$

Then we have the result, as desired. □

By extending Theorems 2.11 and 2.12 to a complete generality we have the binomial inversion formula for Hasse and Cartier operators.

Theorem 4.4. *Let $q^{k-1} \leq n < q^k$ or $(n, k) = (0, 0)$ and let $x \in \kappa((t))$. Then,*

(1) $\phi_n(x) = \sum_{r=n}^{q^k - 1} \binom{r}{n} (-t)^{r-n} \mathcal{D}_r(x)$.

(2) $\mathcal{D}_n(x) = \sum_{r=n}^{q^k - 1} \binom{r}{n} t^{r-n} \phi_r(x)$.

(3) $\phi_{q^k - 1}(x) = \mathcal{D}_{q^k - 1}(x)$.

Proof. For part (1), it suffices to show that the two functions on both sides are identical for $x = t^m (m \geq 0)$ by means of both continuity and linearity.

Case 1, in which $x \in \kappa[[t]]$. Writing $m = lq^k + s$ where $0 \leq s < q^k$ and $l \geq 0$, we have

$$\begin{aligned} \sum_{r=n}^{q^k-1} \binom{r}{n} (-t)^{r-n} \mathcal{D}_r(t^m) &= \sum_{r=n}^{q^k-1} \binom{r}{n} (-t)^{r-n} \binom{m}{r} t^{m-r} \\ &= t^{m-n} \sum_{r=n}^{q^k-1} (-1)^{r-n} \binom{r}{n}\binom{s}{r} \\ &= t^{m-n} \sum_{r=n}^{q^k-1} (-1)^{r-n} \binom{s-n}{r-n}\binom{s}{n} \\ &= t^{m-n} \delta_{n,s} = \phi_n(t^m). \end{aligned}$$

Case 2, in which $x \in \kappa((t)))$. Suppose that $x = \alpha/\beta$ for some $\alpha, \beta \in \kappa[[t]]$. Then, we may assume that $\alpha/\beta \notin \kappa[[t]]$; thus, there exists the smallest integer

$l > 0$ for which $t^l\alpha/\beta \in \kappa[[t]]$. Now, it suffices to verify that the identity (1) holds for t^{-m} where $0 < m \leq l$. We define $\phi_n(t^{-m})$ on $\kappa((t))$ so that for $m = lq^k + s$ with $0 \leq s < q^k$,

$$\phi_n(t^{-m}) = \begin{cases} t^{-(m+n)} & \text{if } n+s = q^k; \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Alternatively, ϕ_n can be extended to $\kappa((t))$ by setting

$$\phi_n(t^{-m}) = \phi_n(t^{m(q^k-1)})/t^{mq^k}.$$

Now, we use Lemma 4.3 to compute:

$$\begin{aligned}\sum_{r=n}^{q^k-1}\binom{r}{n}(-t)^{r-n}\mathcal{D}_r(t^{-m}) &= \sum_{r=n}^{q^k-1}(-1)^{r-n}\binom{r}{n}\binom{-m}{r}t^{-m-n} \\ &= \sum_{r=n}^{q^k-1}(-1)^{n}\binom{r}{n}\binom{m+r-1}{r}t^{-m-n} \\ &= (-1)^{n+s-1}t^{-m-n}\delta_{n+s,q^k} \\ &= \phi_n(t^{-m}),\end{aligned}$$

where the last equality follows from (23).

It is not difficult to establish that part (2) follows from part (1) by substituting (1) into (2) as in Theorem 2.12. Alternatively, the proof could be obtained by applying the same argument as in part (1). Moreover, part (3) follows from parts (1) or (2). □

The following identity is parallel to (9): For $q^{k-1} \leq n < q^k$,

$$\phi_n(x) = \phi_{q^k-1}(t^{q^k-1-n}x), \tag{24}$$

which follows from the definitions of ϕ_n on $\kappa[[t]]$ and its extension to $\kappa((t))$ in (23).

Remark 4.5. Formula (1) in Theorem 4.4 provides that ϕ_n can be a suitable alternative for calculating the higher derivatives of any functions in $\kappa[[t]]$; therefore, it can play a role in the study of rigid analytic functions occurring in a field of positive characteristic as a substitute for higher derivatives. For example, it was shown in [3] and [2] that higher derivatives are extensively used to investigate the differential properties of Drinfeld quasi-modular forms and to derive the arithmetic properties of the maximal extension of $\mathbf{F}(T)$ which is abelian and tamely ramified at the infinity prime. It would be interesting to establish the results parallel to these results by replacing higher derivatives with Cartier operators. As an illustration, the next section is devoted to some Wronskian criteria associated with Cartier operators on more general settings.

We use the binomial inversion formula in Theorem 4.4 to derive a product formula for ϕ_n in the case where $n < q$.

Theorem 4.6. *For $1 \le n < q$ and $x, y \in \kappa((t))$,*

$$\phi_n(xy) = \sum_{i+j=n} \phi_i(x)\phi_j(y) + t^q \sum_{i+j=q+n} \phi_i(x)\phi_j(y).$$

In particular,

$$\phi_{q-1}(xy) = \sum_{i+j=q-1} \phi_i(x)\phi_j(y).$$

Proof. For a positive integer $n < q$, we use the formula (1) in Theorem 4.4 to have

$$\phi_n(xy) = \sum_{r=n}^{q-1} \binom{r}{n} (-t)^{r-n} \mathcal{D}_r(xy).$$

The product formula of $\mathcal{D}_r$ enables us to rewrite it as

$$\phi_n(xy) = \sum_{r=n}^{q-1} \binom{r}{n} (-t)^{r-n} \sum_{\alpha+\beta=r} \mathcal{D}_\alpha(x)\mathcal{D}_\beta(y).$$

By formula (2) in Theorem 4.4 it equals

$$\begin{aligned}
\phi_n(xy) &= \sum_{r=n}^{q-1} \binom{r}{n} (-t)^{r-n} \sum_{\alpha+\beta=r} \left(\sum_{i=\alpha}^{q-1} \binom{i}{\alpha} t^{i-\alpha}\phi_i(x) \sum_{j=\beta}^{q-1} \binom{j}{\beta} t^{j-\beta}\phi_j(y) \right) \\
&= \sum_{r=n}^{q-1} \binom{r}{n} (-t)^{r-n} \sum_{\alpha+\beta=r} \left(\sum_{i=\alpha}^{q-1} \sum_{j=\beta}^{q-1} \binom{i}{\alpha} \binom{j}{\beta} t^{i+j-r}\phi_i(x)\phi_j(y) \right) \\
&= \sum_{r=n}^{q-1} (-1)^{r-n} \sum_{i=\alpha}^{q-1} \sum_{j=\beta}^{q-1} \left(\sum_{\alpha+\beta=r} \binom{i}{\alpha} \binom{j}{\beta} \right) t^{i+j-n}\phi_i(x)\phi_j(y).
\end{aligned}$$

From well-known formulas for binomial coefficients,

$$\begin{aligned}
\phi_n(xy) &= \sum_{i=\alpha}^{q-1} \sum_{j=\beta}^{q-1} \sum_{r=n}^{q-1} (-1)^{r-n} \binom{r}{n} \binom{i+j}{r} t^{i+j-n}\phi_i(x)\phi_j(y) \\
&= \sum_{i=\alpha}^{q-1} \sum_{j=\beta}^{q-1} \left(\binom{i+j}{n} \sum_{r=n}^{q-1} (-1)^{r-n} \binom{i+j}{r-n} \right) t^{i+j-n}\phi_i(x)\phi_j(y).
\end{aligned}$$

The identity $\binom{i+j}{n} \sum_{r=n}^{q-1} (-1)^{r-n} \binom{i+j-n}{r-n} = \delta_{i+j,n}$ or $\delta_{i+j,q+n}$ leads to the desired result.

Part (2) follows immediately from part (1). □

It is of interest to derive a general form of the product formula in Theorem 4.6 by removing the restriction on n.

Let us recall the qth power rule of Hasse derivatives: For $q^{k-1} \le n < q^k$,

$$\mathcal{D}_n(x^{q^m}) = \begin{cases} \mathcal{D}_j(x)^{q^m} & \text{if } n = jq^m; \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

Thanks to this qth power rule and the binomial inversion formula, we have the qth power rule for Cartier operators, which is a special case of the product formula in Theorem 4.6.

Proposition 4.7. *For $q^{k-1} \le n < q^k$,*

$$\phi_n(x^{q^m}) = \begin{cases} \phi_j(x)^{q^m} & \text{if } n = jq^m; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We may assume that $k > m$ otherwise $\phi_n(x^{q^m})$ vanishes from the definition of ϕ_n. From the binomial inversion formula (1) in Theorem 4.4, we have

$$\phi_n(x^{q^m}) = \sum_{r=n}^{q^k-1} \binom{r}{n} (-t)^{r-n} \mathcal{D}_r(x^{q^m}). \tag{26}$$

Case 1, in which q^m divides n, that is $n = jq^m$. Writing $r = r'q^m$, by (25) and Lucas's congruence, we derive

$$\begin{aligned} \phi_n(x^{q^m}) &= \sum_{r'=j}^{q^{k-m}-1} \binom{r'q^m}{jq^m} (-t)^{q^m(r'-j)} \mathcal{D}_{r'q^m}(x^{q^m}) \\ &= \sum_{r'=j}^{q^{k-m}-1} \binom{r'}{j} (-t)^{q^m(r'-j)} \mathcal{D}_{r'}(x)^{q^m} \\ &= \left(\sum_{r'=j}^{q^{k-m}-1} \binom{r'}{j} (-t)^{r'-j} \mathcal{D}_{r'}(x) \right)^{q^m} \\ &= \phi_j(x)^{q^m}, \end{aligned}$$

where the last equality follows from the formula (1) in Theorem 4.4.

Case 2, in which q^m does not divide n. This case follows on applying (25) and Lucas's congruence to (26). □

For $a(t) \in \kappa[[t]]$, the jth Hasse derivative $\mathcal{D}_{j,t}(a)$ of a with respect to t is defined alternatively as

$$a(\theta) = \sum_{j \ge 0} \mathcal{D}_{j,t}(a)(\theta - t)^j. \tag{27}$$

Cartier operators ϕ_i satisfy the following result parallel to the formula in (27).

Proposition 4.8. *For $a(t) \in \kappa[[t]]$,*

$$a(\theta) = \sum_{j \ge 0} \phi_{j,t}(a)(\theta^j - t^{q(j)}\theta^{j_-}).$$

Proof. From (27) we use the binomial inversion formula and Lucas's congruence to deduce the identity in the statement. Indeed,

$$\begin{aligned}
a(\theta) &= \sum_{j\geq 0} \mathcal{D}_{j,t}(a)(\theta - t)^j \\
&= \mathcal{D}_{0,t}(a) + \sum_{k\geq 1} \sum_{j=q^{k-1}}^{q^k-1} \sum_{i=j}^{q^k-1} \phi_i(a) \binom{i}{j} t^{i-j}(\theta - t)^j \\
&= \mathcal{D}_{0,t}(a) + \sum_{k\geq 1} \sum_{i=q^{k-1}}^{q^k-1} \phi_i(a) \sum_{j=q^{k-1}}^{q^k-1} \binom{i}{j} t^{i-j}(\theta - t)^j \\
&= \mathcal{D}_{0,t}(a) + \sum_{k\geq 1} \sum_{i=q^{k-1}}^{q^k-1} \phi_i(a) \left(\theta^i - \sum_{j=0}^{q^{k-1}-1} \binom{i}{j} t^{i-j}(\theta - t)^j\right) \\
&= \mathcal{D}_{0,t}(a) + \sum_{k\geq 1} \sum_{i=q^{k-1}}^{q^k-1} \phi_i(a) \left(\theta^i - \sum_{j=0}^{q^{k-1}-1} \binom{i_-}{j} t^{i_-+q(i)-j}(\theta - t)^j\right) \\
&= \mathcal{D}_{0,t}(a) + \sum_{k\geq 1} \sum_{i=q^{k-1}}^{q^k-1} \phi_i(a)(\theta^i - t^{q(i)}\theta^{i_-}).
\end{aligned}$$

Then the result follows. □

Papanikolas in [23, Proposition 2.3.25] used (27) to reprove Voloch's formula in (13). By the similar argument, Corollary 2.15 follows from Proposition 4.8 when κ and $a(\theta)$ are replaced with $\mathbf{F}_q$ and $a(t^{q^m})$, respectively. It is also shown in [23] that there are interesting formulas for polynomial power sums related to Hasse derivatives. In light of the binomial inversion formula and Proposition 4.8 it is also of interest to find some identities for polynomial power sums related to Cartier operators.

4.2. Wronskian criteria associated with Cartier operators

We retain all notation from the previous section except for q being a prime p. The purpose of this section is to present several Wronskian criteria for the linear independence of a finite number of element in $\kappa((t))$ over its subfields. To do this, let us define all the necessary notation.

As in [11], setting $K = \kappa((t))$, let

$$\begin{aligned}
K_m^{\mathcal{D}} &= \{x \in K \mid \mathcal{D}_i(x) = 0 \ \text{ for } \ 1 \leq i < p^m\}, \\
K_\infty^{\mathcal{D}} &= \{x \in K \mid \mathcal{D}_i(x) = 0 \ \text{ for all } \ i \geq 1\}.
\end{aligned}$$

Parallel to these, set

$$\begin{aligned}
K_m^{\phi} &= \{x \in K \mid \phi_i(x) = 0 \ \text{ for } \ 1 \leq i < p^m\}, \\
K_\infty^{\phi} &= \{x \in K \mid \phi_i(x) = 0 \ \text{ for all } \ i \geq 1\}.
\end{aligned}$$

From Theorem 4.4 we note that $K_m^\phi = K_m^{\mathcal{D}}$ and $K_\infty^\phi = K_\infty^{\mathcal{D}}$; thus, hereafter we denote by $K_m = K_m^{\mathcal{D}} = K_m^\phi$ and by $K_\infty = K_\infty^{\mathcal{D}} = K_\infty^\phi$.

Cartier operators has the following result analogous to that of Schmidt [13, Satz 10] for Hasse derivatives.

Theorem 4.9. *Let κ be a perfect field of positive characteristic p and let F be any field with $\kappa(t) \subset F \subset \kappa((t))$, a finite algebraic extension over $\kappa(t)$. Then, $\{\phi_n\}_{n\geq 0}$ and $\{\psi_n\}_{n\geq 0}$ extend uniquely to F such that $F_m = \kappa F^{p^m}$ and $F_\infty^\phi = \kappa$.*

Proof. First, we prove that $\{\psi_n\}_{n\geq 0}$ extends uniquely to F; hence, this also occurs for $\{\phi_n\}_{n\geq 0}$. Because F is separable, there exists $y \in F$ such that $F = \kappa(t)(y)$ with $f(T, y) = 0$ and $\frac{\partial f}{\partial y} \neq 0$. Then, it is well known in [25] that $\{\mathcal{D}_n\}_{n\geq 0}$ extends uniquely to F with $\mathcal{D}_n(F) \subset F$ for all $n \geq 0$. For $p^{k-1} \leq n < p^k$, using (24) and (3) in Theorem 4.4 we have

$$\phi_n(y) = \phi_{p^k-1}(T^{p^k-1-n}y) = \mathcal{D}_{p^k-1}(T^{p^k-1-n}y) \in F.$$

Equivalently,

$$\psi_n^{p^k}(y) = \mathcal{D}_{p^k-1}(T^{p^k-1-n}y).$$

As F is a separable extension, $\mathcal{D}_{p^k-1}(T^{p^k-1-n}y)$ is a p^k th power in F, such that $\psi_n(y)$ belongs to F. For additional properties, as $K_m^\phi = K_m^{\mathcal{D}}$ and $K_\infty^\phi = K_\infty^{\mathcal{D}}$, it is straightforward to verify that $F_m^\phi = \kappa F^{p^m}$ and $F_\infty^\phi = \kappa$. $\square$

For any vector $\varepsilon := (\varepsilon_0, \dots, \varepsilon_n)$ of integers ε_i with $0 \leq \varepsilon_0 < \cdots < \varepsilon_n$, we define two Wronskians of a family of Laurent series $x_0, \dots, x_n \in K$ as

$$W_\varepsilon^\phi(x_0, \dots, x_n) := \det(\phi_{\varepsilon_i}(x_j))_{0\leq i,j\leq n}$$

and

$$W_\varepsilon^\psi(x_0, \dots, x_n) := \det(\psi_{\varepsilon_i}(x_j))_{0\leq i,j\leq n}.$$

In parallel with the Wronskian criterion associated with $\mathcal{D}_n$, according to Schmidt [25], we provide Wronskian criteria associated with ϕ_n and ψ_n.

Theorem 4.10. *Let $x_0, \dots, x_n$ be elements in K. Then the following are equivalent*:

(1) *$x_0, \dots, x_n$ are linearly independent over κ.*

(2) *There exists a sequence of integers ε_i with $0 \leq \varepsilon_0 < \cdots < \varepsilon_n$ such that $W_\varepsilon^\phi(x_0, \dots, x_n) \neq 0$.*

(3) *There exists a sequence of integers ε_i with $0 \leq \varepsilon_0 < \cdots < \varepsilon_n$ such that $W_\varepsilon^\psi(x_0, \dots, x_n) \neq 0$.*

Proof. We provide two proofs of equivalence (1)$\Leftrightarrow$ (2); the first proof using the result of Schmidt [25], and a second independent proof. By using Theorem 4.4 we are able to determine that the former proof proceeds in the same way as that of Theorem 4.13. The proof of equivalence (1)$\Leftrightarrow$ (3) follows in the same way as the second proof of equivalence (1)$\Leftrightarrow$ (2). $\square$

Now, we provide the second proof of equivalence (1)$\Leftrightarrow$ (2), independent to the first, for which two lemmas concerning power series are needed. Recall that the order or the t-adic valuation of a nonzero power series is the smallest exponent with a nonzero coefficient in that series.

Lemma 4.11. *Let κ be a field of arbitrary characteristic and let $f_0, \ldots, f_n$ be a family of power series in $\kappa[[t]]$ which are linearly independent over κ. Then, there exists an invertible $(n+1)\times(n+1)$ matrix A with entries in κ such that the power series $g_0, \ldots, g_n$ defined by*

$$[g_0, \ldots, g_n] = [f_0, \ldots, f_n]A$$

are all nonzero and have mutually disjoint orders. Consequently, for any vector $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ of integers ε_i with $0 \le \varepsilon_0 < \cdots < \varepsilon_n$, the following equalities hold:

$$W_\varepsilon^\phi(g_0, \ldots, g_n) = W_\varepsilon^\phi(f_0, \ldots, f_n)\det(A)$$

and

$$W_\varepsilon^\psi(g_0, \ldots, g_n) = W_\varepsilon^\psi(f_0, \ldots, f_n)\det(A).$$

Proof. See [4, Lemma 2]. □

Lemma 4.12. *Let κ be a field of arbitrary characteristic. If the nonzero power series $g_0, \ldots, g_n$ in $\kappa[[T]]$ have orders $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ such that the orders ε_i of g_i's are arranged in increasing order, then $W_\varepsilon^\phi(g_0, \ldots, g_n) \neq 0$ and $W_\varepsilon^\psi(g_0, \ldots, g_n) \neq 0$.*

Proof. It follows by mimicking the proof of [4, Lemma 3]. □

Second proof of (1) $\Leftrightarrow$ (2) in Theorem 4.10.

Proof. We merely prove the "only if" part because the "if" part is obvious.

Case 1, in which x_i's are in $R = \kappa[[T]]$. Given κ-linearly independent vectors $x_i (0 \le i \le n)$, by Lemmas 4.11 and 4.12, we can take ε_i as the order of g_i such that

$$0 \le \varepsilon_0 < \varepsilon_1 \cdots < \varepsilon_n,$$
$$g_i = \sum_{j=0}^{n} a_{ji}x_j \text{ with } \det(a_{ji}) \neq 0.$$

As $W_\varepsilon^\phi(g_0, \ldots, g_n) \neq 0$, we deduce $W_\varepsilon^\phi(x_0, \ldots, x_n) \neq 0$ from the above equation.

Case 2, in which the x_i's are in K. Then, there exists a positive integer r such that $t^r x_i$ $(0 \le i \le n)$ are in R. Note that $t^r x_i \in R$ are linearly independent over κ if and only if this is also true for $x_i \in K$. By Case 1, there exists a vector $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$ with $0 \le \varepsilon_0 < \varepsilon_1 \cdots < \varepsilon_n$, and an invertible matrix A such that

$$W_\varepsilon^\phi(g_0, \ldots, g_n) = W_\varepsilon^\phi(t^r x_0, \ldots, t^r x_n) \det(A).$$

If we choose a pth power $r = p^l$ such that $r > e_n$, then we have

$$W_\varepsilon^\phi(t^r x_0, \ldots, t^r x_n) = t^{r(n+1)} W_\varepsilon^\phi(x_0, \ldots, x_n).$$

From this equality we deduce $W_\varepsilon^\phi(x_1, \ldots, x_n) \neq 0$ as $W_\varepsilon^\phi(g_0, \ldots, g_n) \neq 0$. Thus, the proof of the "only if" part is complete. $\square$

Cartier operators ϕ_n provide the following Wronskian criterion, analogous to the Wronskian criterion for higher derivatives $\mathcal{D}_n$ due to Garcia and Voloch [11, Theorem 1].

Theorem 4.13. *Let $x_0, \ldots, x_n$ $(n < p^m)$ be elements in K. Then, $x_0, \ldots, x_n$ are linearly independent over K_m if and only if there exists a sequence of integers ε_i with $0 \leq \varepsilon_0 < \cdots < \varepsilon_n < p^m$ such that $W_\varepsilon^\phi(x_0, \ldots, x_n) \neq 0$.*

Proof. Assuming that $x_0, \ldots, x_n$ are linearly dependent over K_m, then there exist $\alpha_0, \ldots, \alpha_n \in K_m$ not all of which are zero such that $\sum_{j=0}^n \alpha_j x_j = 0$. Because $K_m = \kappa K^{p^m}$ we have $\sum_{j=0}^n \alpha_j \phi_{\varepsilon_i}(x_j) =$ for all $0 \leq e_0, \ldots, \varepsilon_n < p^m$. This proves the "if" part of the theorem.

Conversely, if $x_0, \ldots, x_n$ are linearly independent over K_m then, by [11, Theorem 1], there exists a sequence of integers ε_i with $0 \leq \varepsilon_0 < \cdots < \varepsilon_n < p^m$ such that

$$W_\varepsilon^{\mathcal{D}}(x_0, \ldots, x_n) := \det(\mathcal{D}_{\varepsilon_i}(x_j))_{0 \leq i,j \leq n} \neq 0,$$

where $W_\varepsilon^{\mathcal{D}}$ represents the Wronskian associated with the higher derivatives. From the identity (2) in Theorem 4.4, the $(n+1) \times (n+1)$ invertible matrix $D := (\mathcal{D}_{\varepsilon_i}(x_j))_{0 \leq i,j \leq n}$ has the following decomposition:

$$D = M\Phi, \tag{28}$$

where $M = (M_{\varepsilon_i, j})$ is a $(n+1) \times p^m$ matrix whose (i, j) entry is given by formula (2) in Theorem 4.4 and the $p^m \times (n+1)$ matrix $\Phi = (\phi_i(x_j))_{0 \leq i \leq p^m - 1, 0 \leq j \leq n}$. Computing the rank of the transpose of the matrix equality in (28) yields

$$\operatorname{rank}(D^T) \leq \operatorname{rank}(\Phi^T),$$

implying Φ has full rank; equivalently, there exist integers ε_i with $0 \leq \varepsilon_0 < \cdots < \varepsilon_n < p^m$ for which the corresponding row vectors of Φ are linearly independent over K. $\square$

Several remarks on Wronskians are in order.

1. It is of interest to provide the proof of Theorem 4.13, independent of the result of Garcia and Voloch [11, Theorem 1].

2. It is much simpler and more practical to compute $W_\varepsilon^\phi(x_0, \ldots, x_n)$ than $W_\varepsilon^{\mathcal{D}}(x_0, \ldots, x_n)$, because the calculation of $W_\varepsilon^\phi(x_0, \ldots, x_n)$ does not concern binomial coefficients. It is interesting to check that $W_\varepsilon^\phi(x_0, \ldots, x_n) \neq 0$ if and only if $W_\varepsilon^{\mathcal{D}}(x_0, \ldots, x_n) \neq 0$ for the same integer vector $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$.

3. The question arises as to whether there exists a Wronskian criterion associated with Cartier operators ψ_n.

Remark 4.14. We close this paper by mentioning practical applications of digit Cartier I and II in Theorem 3.5 to non-Archimedean dynamical systems. Recently, Carlitz polynomials, digit derivatives/shifts were intensively used from a dynamical perspective by providing measure-preservation/ergodicity criterion of 1-Lipschitz functions on $\mathbf{F}_q[[T]]$ in terms of the expansion coefficients of the chosen basis [14, 20, 21]. For future work, digit Cartier I and II bases obtained in this paper can be also investigated for dynamic applications, as with Carlitz polynomials and digit derivatives/shifts.

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