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FACTORIZATION IN MODULES AND SPLITTING MULTIPLICATIVELY CLOSED SUBSETS

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Abstract. We introduce the concept of multiplicatively closed subsets of a commutative ring R which split an R-module M and study factorization properties of elements of M with respect to such a set. Also we demonstrate how one can utilize this concept to investigate factorization properties of R and deduce some Nagata type theorems relating factorization properties of R to those of its localizations, when R is an integral domain.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. We assume that all modules are nonzero. Also R denotes a ring and M is an R-module.

Theory of factorization in commutative rings which has a long history (see for example [18]), still gets a lot of attention from various researchers. To see some recent papers on this subject, the reader is referred to [1,2,5,10–15,17,19,20]. In [6,7], D. D. Anderson and S. Valdes-Leon generalized the theory of factorization in integral domains to commutative rings with zero divisors and to modules as well. These concepts are further studied in [1, 2, 8, 12, 20].

One of the longstanding questions in this subject is "what is the relationship between factorization properties of R and those of its localizations?", especially when R is a domain (see for example [4,8,18]). In particular, many have tried to give conditions under which, if R_S is a UFD (or has other types of factorization properties), then R is so, where S is a multiplicatively closed subset of R. For example [16, Corollary 8.32], says that if R is a Krull domain and S is generated by a set of primes and R_S is a UFD, then R is a UFD. This type of results, are called Nagata type theorems due to a theorem of Nagata in [18]. One can find some other similar results and a brief review of this subject in [4, Section 3].

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On the other hand, in [19] the concept of factorization with respect to a saturated multiplicatively closed subset (also called a divisor closed multiplicative submonoid) of R is introduced. If we apply Theorem 3.9 of that paper with $S' = M = R$ and assume that R is an integral domain, then we get a Nagata type result which states that if R_S is a bounded factorization domain and R as an R-module is an S-bounded factorization module, then R is a bounded factorization domain (for exact definitions, see Section 2). It is still unknown whether the similar result holds for other factorization properties such as unique or finite factorization (see [19, Question 3.11]).

The main aim of this research is to find partial answers to the above question and utilize them to find relations between factorization properties of R and R_S, especially when R is a domain. For this, we generalize the concept of an splitting multiplicatively closed subset in [4] which is of key importance in the results of that paper. Interestingly, we find out that this concept is equivalent to another one which is completely stated in terms of factorization properties with respect to a saturated multiplicatively closed subset.

In Section 2, we briefly review the concepts of factorization theory with respect to a saturated multiplicatively closed subset. Then in Section 3, we state the definition of a multiplicatively closed subset which splits M and study basic properties of such sets. In Section 4, we present our main results, which state how factorization properties of M and M_S are related when S splits M. Finally, in Section 5, we present an example in which $M = R$ is an integral domain to show how our result could be applied in order to study factorization properties of integral domains.

In the following, by $\mathrm{U}(R)$ and $\mathrm{J}(R)$ we mean the set of units and Jacobson radical of R, respectively. Furthermore, $\mathrm{Z}(N)$, where $N \subseteq M$, means the set of zero divisors of N, that is, $\{r \in R \mid \exists 0 \neq m \in N : rm = 0\}$. In addition, $\mathrm{Ann}(N)$ (resp. $\mathrm{Ann}_M(r)$) denotes the annihilator in R of $N \subseteq M$ (resp. in M of $r \in R$). Any other undefined notation is as in [9].

2. A brief review of factorization with respect to a saturated multiplicatively closed subset

In this section we recall the main concepts of factorization with respect to a saturated multiplicatively closed subset of R which is needed in this paper. For more details and several examples the reader is referred to [19]. In the sequel, S always denotes a saturated multiplicatively closed subset of R (we let S to contain 0, which means $S = R$).

We say that two elements, m and n of M, are *S-associates* and write $m \sim^S n$, if there exist $s, s' \in S$ such that $m = sn$ and $n = s'm$. They are called *S-strong associates*, if $m = un$ for some $u \in \mathrm{U}(R) \cap S$ and we denote it by $m \approx^S n$. Also we call them *S-very strong associates*, denoted by $m \cong^S n$, when $m \sim^S n$ and either $m = n = 0$ or from $m = sn$ for some $s \in S$ it follows that $s \in \mathrm{U}(R)$.

In the case that $S = R$, we drop the S prefixes. In this case, our notations coincide with that of [6, 7]. An $m \in M$ is called *S-primitive* (resp. *S-strongly primitive*, *S-very strongly primitive*), when $m = sn$ for some $s \in S$, $n \in M$ implies $n \sim^S m$ (resp. $n \approx^S m$, $n \cong^S m$). A nonunit element $a \in R$ is called *irreducible* (resp. *strongly irreducible*, *very strongly irreducible*) if $a = bc$ for some $b, c \in R$, implies $a \sim b$ or $a \sim c$ (resp. $a \approx b$ or $a \approx c$, $a \cong b$ or $a \cong c$). Note that here by being associates in R, we mean being associates in R as an R-module.

By an *S-factorization* of $m \in M$ with length k, we mean an equation $m = s_1 \cdots s_k n$ where s_i's are nonunits in S, $k \in \mathbb{N} \cup \{0\}$ and $n \in M$. If moreover, for some $\alpha \in$\{irreducible, strongly irreducible, very strongly irreducible\} and $\beta \in$\{primitive, strongly primitive, very strongly primitive\} s_i's are α and n is S-β, we call this an *(α, β)-S-factorization.* If every nonzero element of M has an (α, β)-S-factorization, we say that M is *(α, β)-S-atomic.*

By an *S-atomic factorization* we mean an (irreducible, primitive)-S-factorization and by an *S-atomic module* we mean a module which is (irreducible, primitive)-S-atomic. Also we say two S-atomic factorizations $m = s_1 \cdots s_k n = t_1 \cdots t_l n'$ are *isomorphic*, if $k = l$, $n \sim^S n'$ and for a permutation σ of $\{1, \ldots, k\}$, we have $s_i \sim t_{\sigma(i)}$ for all $1 \leq i \leq k$.

We say that M is S-présimplifiable when from $sm = m$ ($s \in S, m \in M$), we can deduce that $s \in \mathrm{U}(R)$ or $m = 0$. By [19, Theorem 2.7(ii)], this is equivalent to saying that the three relations $\sim^S$, $\approx^S$ and $\cong^S$ coincide or to asserting that $\cong^S$ is reflexive. In particular, all kinds of S-primitivity and also by [19, Theorem 2.7(iv)], all types of S-factorization are equivalent for a nonzero element of M, if M is S-présimplifiable.

We call a module M, an *S-unique factorization module* or *S-UFM* (resp. *S-finite factorization module* or *S-FFM*), when M is S-atomic and every nonzero element of M has exactly one (resp. finitely many) S-atomic factorization up to isomorphism. Also we say that M is an *S-bounded factorization module* or *S-BFM*, if for every $0 \neq m \in M$ there is an $N_m \in \mathbb{N}$ such that the length of every S-factorization of m is at most N_m and say that M is an *S-half factorial module* or *S-HFM* when M is S-atomic and for each element $0 \neq m \in M$ the length of all S-atomic factorizations of m are the same.

Note that in the cases that $S = R$ or $S = R = M$, these concepts coincide with the previously defined notations (see [6, 7, 20]). For example an integral domain R is a UFD if and only if it is an R-UFM over itself. Moreover, a BFR (bounded factorization Ring) means a ring which is a BFM over itself and a FFD (finite factorization domain) means a domain D which is a D-BFM. The notations UFR, FFR, BFD, HFD, ..., have similar meanings. Also, we sometimes say *M has unique factorization* (*or has finite factorization or is présimplifiable, ...*) *with respect to S* instead of saying M is an S-UFM (or S-FFM or S-présimplifiable, ...).

Furthermore, if $E \subseteq R$, we say that *R has unique factorization in E* when every nonzero nonunit element in E has unique factorization (with respect to

$S = R$). Similar notations are used for other factorization properties. In the following remark, we collect some observations which will be used in the paper without any further mention.

Remark 2.1. (i) Every S-UFM is both an S-FFM and an S-HFM by definition and every S-BFM is S-présimplifiable (see remarks on page 8 of [19]).

(ii) If R has unique factorization in E, then it is half factorial and has finite factorization in E and if R has bounded factorization in E, then it is présimplifiable in E. Also if E is a saturated multiplicatively closed subset or more generally, has the property that $xy \in E$ leads to $x \in E$ and $y \in E$, then being half factorial or having finite factorization in E results to having bounded factorization in E (see the second paragraph of [19, p. 8]).

(iii) It is straightforward to see if $M = R$ and $s \in S$, then all kinds of S-primitivity for s are equivalent to being a unit and for elements in S, any type of S-associativity is equivalent to the corresponding type of R-associativity, since S is assumed to be saturated.

(iv) An element $m \in M$ is S-very strongly primitive, if and only if from $m = sm'$ for some $s \in S$ and $m' \in M$, we can deduce $s \in \mathrm{U}(R)$.

It should be mentioned that one can define other kinds of isomorphisms using different types of associativity and also many forms of UFM, HFM, ... based on the choice of the type of irreducibility, primitivity and isomorphism (see [6, 7] for the case $S = R$). But in order not to make the paper too long, we just investigate the forms defined above, mentioning that similar techniques could be utilized to get similar results on the other forms.

3. M-splitting multiplicatively closed subsets

A main concept used in [4] to relate factorization in R and R_S is the notion of a splitting multiplicatively closed subset of R. A saturated multiplicatively closed subset S of a domain R is called a *splitting multiplicative set*, when for each $x \in R$, $x = as$ for some $a \in R$ and $s \in S$ such that $aR \cap tR = atR$ for all $t \in R$. An equivalent condition is that principal ideals of R_S contract to principal ideals in R [4, Lemma 1.2]. Here we will restate this condition using factorization properties of the R-module R with respect to S and generalize it to every R-module M. For this we need some more definitions.

Definition 3.1. By a compact S-atomic factorization of an element $m \in M$, we mean an equation of the form $m = sn$ for $s \in S$ and S-primitive element $n \in M$. We say that a subset $E \subseteq M$ is compactly S-atomic if every nonzero element of E has a compact S-atomic factorization. If E is compactly S-atomic and for every $0 \neq m \in E$ and compact S-atomic factorizations $m = sn = s'n'$ of m, we have $s \sim s'$ (resp. $s \sim s'$ and $n \sim^s n'$), then E is called semi-S-factorable (resp. S-factorable).

Clearly every S-atomic module is compactly S-atomic but the following example shows that the converse is not true. This example also shows that not all S-UFM's are factorable. Note that as usual when $S = R$, we drop the S prefixes.

Example 3.2. Let R be a ring with no irreducible elements (such as the domain D in [19, Example 2.14]) and $S = R$, then the R-module R is not atomic but is compactly atomic and even factorable, since $r = r1$ is the only compact atomic factorization of an $r \in R$ up to associates. Also if $M = R/\mathfrak{M}$ for a maximal ideal $\mathfrak{M}$ of R, then as in [19, Example 2.14], M is a UFM which is not even semi-factorable, since for any nonunit $r \in R \setminus \mathrm{Ann}(M)$ and any $m \in M \setminus \mathrm{Ann}_M(r)$, $m' = rm = 1m'$ are two compact atomic factorizations of m' and $r \nsim 1$. Note that if we choose R to be a valuation domain of Krull dimension 1 which is not a discrete valuation domain, then M is présimplifiable.

On the other hand, if M is an S-UFM and R is atomic in $S \setminus \mathrm{Ann}(M)$, then M is S-factorable. Because if $m = sn$ is a compact S-atomic factorization of a nonzero element $m \in M$, then by replacing s with its atomic factorization, we get the unique S-atomic factorization of m and hence s and n are unique up to S-associates. Some properties of semi-S-factorable modules is stated in the next proposition.

Proposition 3.3. *Suppose that E is a semi-S-factorable subset of M.*

(i) *If $m \in E$ is S-primitive, then $m \cong^S m$ and m is S-very strongly primitive.*
(ii) *If $S \cap \mathrm{Z}(M) = \emptyset$, then E is S-factorable.*
(iii) *If $E = M$, then R is présimplifiable in $S \setminus \mathrm{Z}(M)$, all kinds of irreducibility are equivalent for elements in $S \setminus \mathrm{Z}(M)$ and all kinds of associativity are equivalent for pairs of elements in $S \setminus \mathrm{Z}(M)$.*

Proof. (i) It is easy to see that $m \cong^S m$. Now if $m = sm'$ for some $s \in S, m' \in M$, then since m is S-primitive, there is an $s' \in S$ such that $m' = s'm$, hence $m = ss'm$ and by $m \cong^S m$, it follows that $s, s' \in \mathrm{U}(R)$ and m is S-very strongly primitive.

(ii) Let $0 \neq m = sn = s'n'$ be two compact S-atomic factorizations of m. By semi-S-factorability, $s \sim s'$ and hence $s = s''s'$ for some $s'' \in R$. Since S is saturated, $s'' \in S$. Now $s's''n = s'n'$ and as $s' \notin \mathrm{Z}(M)$, we get $s''n = n'$. Since n' is S-very strongly primitive by (i), we deduce that $s'' \in \mathrm{U}(R)$ and hence $n' \sim^S n$, as required.

(iii) Assume that $s = s's$ for some $s \in S \setminus \mathrm{Z}(M), s' \in R$ and let m be an S-primitive element of M. Since S is saturated, $s' \in S$ and because $sm = ss'm$ and $s \notin \mathrm{Z}(M)$, we get $m = s'm$ and by (i), it follows that $s' \in \mathrm{U}(R)$. Therefore, R is présimplifiable in $S \setminus \mathrm{Z}(M)$. Other parts of the claim follows from [6, Theorem 2.2(2)] or [19, Theorem 2.7(iv)]. □

Part (ii) of the above proposition shows that if $S \cap \mathrm{Z}(M) = \emptyset$, then semi-$S$-factorability and S-factorability are equivalent. Indeed, the author does not know an example of a semi-S-factorable module which is not S-factorable even when $S \cap Z(M)$ is nonempty.

The next theorem and the remark following it, state conditions under which S-factorization properties of M are determined by factorization properties of elements in $S \setminus \mathrm{Ann}(M)$.

Theorem 3.4. *Suppose that M is semi-S-factorable, $S \cap \mathrm{Z}(M) = \emptyset$ and let $\mathcal{P}$ be one of the following properties: being présimplifiable, having unique factorization, having finite factorization, being half factorial, having bounded factorization, being atomic. Then M has $\mathcal{P}$ with respect to S if and only if R has $\mathcal{P}$ in $S \setminus \mathrm{Ann}(M)$.*

Proof. ($\Rightarrow$) Let $s \in S \setminus \mathrm{Ann}(M)$ and $m \in M \setminus \mathrm{Ann}_M(s)$. By replacing m with an S-primitive element appearing in its compact S-atomic factorization, we can assume m is S-primitive. First assume $\mathcal{P}$ = atomicity. So M is S-atomic and sm has an S-atomic factorization $sm = s_1 \cdots s_l m'$ with each s_i irreducible and m', S-primitive. Since M is semi-S-factorable, we deduce that $s \sim s_1 \cdots s_l$. By (3.3)(iii), $s \cong s_1 \cdots s_l$, that is, $s = us_1 \cdots s_l$ for some $u \in \mathrm{U}(R)$ and s has an atomic factorization.

Now for the other factorization properties, note that multiplication by m turns any factorization of s into an S-factorization of sm with the same length. Also this operation preserves isomorphism, so factorization properties of M pass to $S \setminus \mathrm{Ann}(M)$. A similar argument takes care of $\mathcal{P}$ = being présimplifiable.

($\Leftarrow$) For $\mathcal{P}$ = atomicity, the result is clear. Suppose that R is présimplifiable in $S \setminus \mathrm{Ann}(M)$, $0 \neq m = sm$ for some $m \in M$ and $s \in S$ and let $m = s'm'$ be a compact S-atomic factorization of m. Then $s'm' = ss'm'$ are two compact S-atomic factorizations for m and hence $s' \sim ss'$. In particular, $s' = rss'$ for some $r \in R$. Since R is présimplifiable in S, we deduce that $r, s \in \mathrm{U}(R)$, as required.

So assume $\mathcal{P} \neq$ atomicity or being présimplifiable. In any of the cases, R is présimplifiable in $S \setminus \mathrm{Ann}(M)$ by (2.1)(ii) and hence M is S-présimplifiable by the previous paragraph. Now if $x = s_1 \cdots s_l m = s'_1 \cdots s'_{l'} m'$ are two S-atomic factorizations of $0 \neq x \in M$, then by semi-S-factorability of M, we get two S-atomic factorizations of $s = s_1 \cdots s_l \cong s'_1 \cdots s'_{l'}$ with lengths l, l'. Thus if $\mathcal{P}$= being half factorial, then $l = l'$ and hence M is an S-HFM. The case of bounded factorization is quite similar. For $\mathcal{P}$ = having unique factorization or finite factorization, note that according to (3.3)(ii), $m \sim^S m'$ in the above factorizations of x and so if these two factorizations are non-isomorphic, then the two factorizations of s are also non-isomorphic. So the number of non-isomorphic factorizations of x and s are the same. □

In several parts of the proof of the above result, we did not use the assumptions $S \cap \mathrm{Z}(M) = \emptyset$ or semi-S-factorability of M. So we get the following remark that states some weaker conditions under which, some S-factorization properties of M are determined by factorization properties of elements in $S \setminus \mathrm{Ann}(M)$.

Remark 3.5. Note that in the proof of (3.4)($\Rightarrow$), for $\mathcal{P}$= having bounded factorization or being présimplifiable, we did not use any of the two assumptions. Also for other properties, we could replace the condition "$S \cap \mathrm{Z}(M) = \emptyset$" with the weaker condition "R is présimplifiable in $S \setminus \mathrm{Ann}(M)$".

In the proof of (3.4)($\Leftarrow$), for $\mathcal{P}$= being présimplifiable, atomic or half factorial or having bounded factorization, we did not need the condition "$S \cap \mathrm{Z}(M) = \emptyset$" and for the other properties we could replace the two conditions with "M is S-factorable." □

Combining (3.3)(iii) with the case $\mathcal{P}$ = being présimplifiable of (3.4) we get:

Corollary 3.6. *If M is semi-S-factorable and $S \cap \mathrm{Z}(M) = \emptyset$, then M is S-présimplifiable.*

Next we define the main concept of this research, namely M-splitting sets.

Definition 3.7. Let $E \subseteq M$. We say that S splits E or S is E-splitting, when the following two conditions hold.

(i) E is semi-S-factorable.

(ii) For every S-primitive element $r \in R$ and S-primitive element $m \in M$ such that $0 \neq rm \in E$, the element rm is S-primitive.

To see an example of this concept, see Section 5. The following result shows that this definition generalizes the concept of splitting multiplicative sets as defined in [4].

Theorem 3.8. *Suppose that R is a domain. Then S is a splitting multiplicative set* (*in the sense of* [4]) *if and only if S splits R.*

Proof. ($\Rightarrow$) Note that since R is a domain, it is présimplifiable and all kinds of associativity are equivalent and also all kinds of primitivity are equivalent. Suppose that $r \in R$ is S-primitive. By assumption we can write $r = sr'$ with $s \in S, r' \in R$ such that $Rr' \cap Rt = Rtr'$ for all $t \in S$. By S-primitivity, $r = ur'$ for some $u \in \mathrm{U}(R)$. It follows that if $r \in R$ is S-primitive, then

$$(*) \qquad Rr \cap Rt = Rtr \text{ for all } t \in S.$$

Conversely, if $r \in R$ satisfies $(*)$ and $r = sr'$ for some $s \in S, r' \in R$, then $r \in Rr \cap Rs = Rrs$, that is, $r = r''rs$ for some $r'' \in R$ and hence $1 = r''s$, $s \in \mathrm{U}(R)$ and r is S-primitive. Thus satisfying $(*)$ is equivalent to being S-primitive. Consequently, according to [4, Corollary 1.4(a)], R is S-factorable. Now assume that r_1, r_2 are nonzero S-primitive elements of R. By the above remarks r_1 and r_2 satisfy $(*)$ and hence by [4, corollary 1.4(b)], r_1r_2 also satisfies $(*)$ and hence is S-primitive, as required.

($\Leftarrow$) It suffices to show that S-primitive elements of R, such as r, satisfy $(*)$. Let $0 \neq x \in Rr \cap Rt$, say $x = r_1r = r_2t$ for $r_1, r_2 \in R$. If $r_i = s_ir_i'$ is the compact S-atomic factorization of r_i $(i = 1, 2)$, then $x = s_1(r'_1 r) = (ts_2)r'_2$ are compact S-atomic factorizations of x, because by assumption $r'_1 r$ is S-primitive. So by semi-S-factorability, $s_1 \sim ts_2$ and $s_1 = t'ts_2$ for some $t' \in R$. Thus $x = t'ts_2r'_1 r \in Rtr$ and therefore, $Rr \cap Rt = Rtr$. □

At the end of this section we state a proposition which will be needed in the later sections.

Proposition 3.9. *Suppose that S is M-splitting, S' is a saturated multiplicatively closed subset of R and $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic. Then S splits $S' \setminus \mathrm{Ann}(M)$. If $S' = R$, then S splits the R-module $R/\mathrm{Ann}(M)$.*

Proof. Suppose $r = s_1r_1 = s_2r_2$ are two compact S-atomic factorizations of $r \in S' \setminus \mathrm{Ann}(M)$. There exists an S-primitive $m \in M \setminus \mathrm{Ann}_M(r)$. Then $0 \neq rm = s_1(r_1m) = s_2(r_2m)$ are two compact S-atomic factorizations of rm. So $s_1 \sim s_2$. Now assume $r, r' \in S' \setminus \mathrm{Ann}(M)$ are S-primitive and $rr' \in S' \setminus \mathrm{Ann}(M)$. If $rr' = sr''$ is a compact S-atomic factorization of rr' and $m \in M \setminus \mathrm{Ann}_M(rr')$ is S-primitive, then $rr'm$ and $r''m$ are both S-primitive by condition (ii) of definition of M-splitting saturated multiplicatively closed subsets and hence from $rr'm = sr''m$ we deduce that $s \sim^S 1$ is a unit. Therefore rr' is S-primitive and S splits $S' \setminus \mathrm{Ann}(M)$.

To prove the claim about $N = R/\mathrm{Ann}(M)$, it suffices to show that an $r \in R \setminus \mathrm{Ann}(M)$ is S-primitive if and only if its image $\bar{r}$ is S-primitive in N. Assume that $r \in R \setminus \mathrm{Ann}(M)$ is S-primitive and $\bar{r} = s\bar{r}'$. So $r = sr' + a$ with $a \in \mathrm{Ann}(M)$. Let $m \in M$ be such that $rm \neq 0$. Since M is compactly S-atomic, we can assume that m is S-primitive. Also suppose that $r' = s'r''$ is a compact S-atomic factorization of r'. Then $rm = sr'm = ss'(r''m)$ and it follows from S being M-splitting that $ss' \sim^S 1$, and $s \in \mathrm{U}(R)$. Thus $\bar{r}$ is S-primitive. The reverse implication is straightforward. □

4. Behavior of S-factorizations under localization

Throughout this section we assume that $S \subseteq S'$ are two saturated multiplicatively closed subsets of R and set T to be the saturated multiplicatively closed subset, $S^{-1}S' = \{\frac{s'}{s} \mid s' \in S', s \in S\}$ of R_S. We investigate how factorization properties of M with respect to S' is related to factorization properties of M_S with respect to T, under the assumption that S splits M. As we will see, in the case that $S' = R$, we get some Nagata type theorems and also our results serve as partial answers to [19, Question 3.11]. To this end, we first study how irreducibility behaves under localization.

Proposition 4.1. *Suppose that S splits $E = S' \setminus \mathrm{Ann}(M)$ and $S \cap \mathrm{Z}(M) = S \cap \mathrm{Z}(R) = \emptyset$. Let $\alpha \in${irreducible, strongly irreducible, very strongly irreducible} and $r = sa$ be the compact S-atomic factorization of $r \in E \setminus S$. Then $\bar{r} = r/1 \in R_S$ is α if and only if a is so in R.*

Proof. Suppose that a is very strongly irreducible and $\bar{r} = (r_1/s_1)(r_2/s_2)$. Note that as $r_1r_2 = s_1s_2r \in S'$, we must have $r_1, r_2 \in S'$. As $S \cap \mathrm{Z}(M) = \emptyset$ and $r \notin \mathrm{Ann}(M)$, we conclude that $s_1s_2r \notin \mathrm{Ann}(M)$ and hence $r_1, r_2 \in E$. If $r_i = s_i'r_i'$ is the compact S-atomic factorization of r_i $(i = 1, 2)$, then $s_1s_2sa = s_1's_2'(r_1'r_2')$ and thus $r_1'r_2' \in E$. So $r_1'r_2'$ is S-primitive by (3.7)(ii). According to (3.3)(ii) (applied with $M = R$), we have $a \sim^S r_1'r_2'$, in particular, $a = tr_1'r_2'$ for some $t \in S$. Because a is very strongly irreducible, one of r_i''s, say r_1' is a unit. Thus $r_1/s_1 = s_1'r_1'/s_1 \in \mathrm{U}(R_S)$ and as $\bar{r}$ is not a unit of R_S (since $r \notin S$), it is very strongly irreducible. Similar arguments show that if a is (strongly) irreducible, then $\bar{r}$ is so.

Conversely, suppose that $\bar{r}$ is very strongly irreducible and $a = bc$. Then $b, c \in E$. So they have compact S-atomic factorizations $b = s_1b'$ and $c = s_2c'$ with $b', c' \in E$. Since S splits E and $a = s_1s_2(b'c')$ is a compact S-atomic factorization of the S-primitive element a we deduce that $s_1, s_2 \in \mathrm{U}(R)$ and hence b, c are both S-primitive. On the other hand, $\bar{r} = (\bar{s}\bar{b})\bar{c}$ and it follows from very strongly irreducibility of $\bar{r}$ that for example $\bar{b} \in \mathrm{U}(R_S)$. This means that $b \in S$ and since b is S-primitive, we conclude that $b \in \mathrm{U}(R)$. Therefore, a is very strongly irreducible.

Now assume that $\bar{r}$ is strongly irreducible and $a = bc$. As in the above paragraph we see that b, c are S-primitive and $\bar{r} = (\bar{s}\bar{b})\bar{c}$. So by strongly irreducibility of $\bar{r}$ it follows that for example $\bar{r} \approx \bar{b}$, that is, $\bar{r} = (s_1/s_2)\bar{b}$ for some $s_1, s_2 \in S$. So $s_2sa = s_1b$ and by (3.3)(ii) $a \sim^S b$. But (3.3)(i) implies that $a \cong^S b$ and hence according to [19, Theorem 2.7(i)], $a \approx b$, as required. The case $\alpha =$ irreducible is similar. □

Next we consider how S'-primitivity behaves under localization. Recall that throughout this section $S \subseteq S'$ are saturated multiplicatively closed subsets and $T = S^{-1}S'$.

Proposition 4.2. *Suppose S splits M, $S \cap \mathrm{Z}(M) = \emptyset$ and $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic. Let $0 \neq m \in M$, $\beta \in${primitive, strongly primitive, very strongly primitive} and assume that $m = sn$ is the compact S-atomic factorization of m. Then $m/1$ is T-β in the R_S-module M_S if and only if n is S'-β in M.*

Proof. As $S \cap \mathrm{Z}(M) = \emptyset$, we have $M \subseteq M_S$, so we write m instead of $m/1$. Assume that m is T-β and $n = s'n'$ for $s' \in S', n' \in M$. If $n' = s_1n''$ and $s' = s_2s''$ are compact S-atomic factorizations of n' and s', then we get $n = s_1s_2(s''n'')$ and as S splits M, we deduce that $s_1s_2 \sim 1$. This means that $s_1, s_2 \in \mathrm{U}(R)$ and both n' and s' are S-primitive. Also $m = ss'n'$.

Consider the case that $\beta =$ primitive. Then we get $m \sim^T n'$, that is, $n' = (s_0'/s_0)m$ for $s_0' \in S'$, $s_0 \in S$. This leads to $s_0n' = s_0'sn = sv(s_1'n)$, where $s_0' = vs_1'$ is a compact S-atomic factorization of s_0' with $v \in S$. Consequently we get $n' \sim^S s_1'n$ and $n' = us_1'n$ for some $u \in S$. As $us_1' \in S'$, we conclude that $n' \sim^{S'} n$ and hence n is S'-primitive.

Now consider the case that $\beta =$ strongly primitive. Then we get $m \approx^T n'$, whence $n' = (s'_0/s_0)m$ with both $s_0, s'_0 \in S$. Thus $s_0 n = s'_0 sn$ and as S splits M, $n \sim^S n'$ which implies $n \cong^S n'$ by (3.3)(i) and hence by [19, Theorem 2.7(i)], $n \approx^{S'} n'$, as required. We leave the similar proof of the case $\beta =$ very strongly primitive to the reader.

Conversely, assume that n is S'-primitive and $m = tm'$ for some $t \in T$ and $m' \in M_S$. One can readily check that if $x = y/v$ for $y \in M, v \in S$ and $m \sim^T y$ (resp. $m \approx^T y$, $m \cong^T y$), then $m \sim^T x$ (resp. $m \approx^T x$, $m \cong^T x$). Therefore, we can assume that $m' \in M$ and also $t = u'/u$ for $u' \in S'$ and $u \in S$. If $u' = u_1 u''$ and $m' = u_2 m''$ are compact S-atomic factorizations of u' and m' with $u_1, u_2 \in S$, then it follows that $usn = um = u'm' = u_1 u_2 (u''m'')$ and hence $n \sim^S u''m''$. In particular, $n = s_0 u'' m''$ for some $s_0 \in S$. Note that u'' and hence $s_0 u''$ are in S', so as n is S'-primitive, we deduce that $n \sim^{S'} m''$, hence $m'' = s'n$ for some $s' \in S'$. Thus $sm' = su_2 m'' = su_2 s'n = u_2 s'm$ and $m' = (u_2 s'/s)m$. Since $u_2 s'/s \in T$, we see that $m \sim^T m'$ which shows that m is T-primitive. The proof for (very) strongly primitivity is similar. $\square$

To see how S'-atomicity of M and T-atomicity of M_S are related, we need a lemma.

Lemma 4.3. *Suppose that $s' \in S' \setminus (S \cup \mathrm{Ann}(M))$ and $0 \neq m \in M$ such that s' is α and m is S'-β where $\alpha \in${irreducible, strongly irreducible, very strongly irreducible}, $\beta \in${primitive, strongly primitive, very strongly primitive}. If M is semi-S-factorable, then m is S-primitive. If S splits M and $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic, then s' is S-primitive.*

Proof. If $m = sm'$ is a compact S-atomic factorization of m, then by S'-primitivity, $m \sim^{S'} m'$. So $m' = tm$ for some $t \in S'$. Thus $m' = s(tm')$ and since m' is S-very strongly primitive by (3.3), we deduce that s is a unit and hence m is S-primitive.

Now assume that S splits M, $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic and let $s' = s_1 a$ be a compact S-atomic factorization of s'. Since s' is irreducible, either $s' \sim s_1$ or $s' \sim a$. In the former case, it follows that $s_1 = rs'$ and as S is saturated, we get the contradiction $s' \in S$. So $s' \sim a$ and $a = rs'$ for some $r \in R$. Note that since $s' = s_1 s' r$ and S' is saturated, we must have $r \in S'$. If $r = s_2 r'$ is a compact S-atomic factorization of r, then $a = s_1 s_2 (r'a)$ and thus $s_1 s_2 \sim 1$, for both sides are compact S-atomic factorizations in $E = S' \setminus \mathrm{Ann}(M)$ and S splits E by (3.9). This means that $s_1 \in \mathrm{U}(R)$ and the result follows. $\square$

Theorem 4.4. *Suppose that S is M-splitting, $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic, $\alpha \in${irreducible, strongly irreducible, very strongly irreducible} and $\beta \in${primitive, strongly primitive, very strongly primitive}. Then the following hold.*

(i) *If M is (α, β)-S'-atomic, then M is (α, very strongly primitive)-S-atomic.*

(ii) *Assume that* $S \cap \mathrm{Z}(M) = S \cap \mathrm{Z}(R) = \emptyset$. *Then* M *is* (α, β)-S'-*atomic if and only if* M *is* (α, *primitive*)-S-*atomic and* M_S *is* (α, β)-T-*atomic.*

Proof. (i) If $m = s_1 \cdots s_k s'_1 \cdots s'_{k'} m'$ is an (α, β)-S'-atomic factorization of $0 \neq m \in M$ where $s_i \in S$ and $s'_i \in S' \setminus S$, then by (4.3), s'_i and m' are S-primitive and hence their product $0 \neq m'' = s'_1 \cdots s'_{k'} m'$ is also S-primitive and by (3.3)(i), indeed S-very strongly primitive. Therefore $m = s_1 \cdots s_k m''$ is an (α, very strongly primitive)-S-atomic factorization of m.

(ii) ($\Rightarrow$) It follows from (i) that M is (α, primitive)-S-atomic. Suppose $m = s_1 \cdots s_k s'_1 \cdots s'_{k'} m'$ is an (α, β)-S'-atomic factorization of $0 \neq m \in M$ where $s_i \in S$ and $s'_i \in S' \setminus S$, then by (4.3), (4.2) and (4.1), $s'_i/1$ is α and $m'/1$ is T-β. Hence we get the (α, β)-T-atomic factorization $m/1 = (s'_1/1) \cdots (s'_{k'}/1)(um'/1)$, where $u = (s_1 \cdots s_k)/1 \in \mathrm{U}(R_S)$. So M_S is (α, β)-T-atomic.

($\Leftarrow$) Let $0 \neq m \in M$. Then by assumption $m = s_1 \cdots s_k n$ where $s_i \in S$ are α and n is S-primitive. Assume $n/1 = (s'_1/u_1) \cdots (s'_{k'}/u_{k'})(m'/u_{k'+1})$ is an (α, β)-T-atomic factorization where $s'_i \in S' \setminus S$, $u_i \in S$ and let $s'_i = v_i s''_i$ and $m' = vm''$ be compact S-atomic factorizations of s'_i and m', respectively. Then as s'_i/u_i and hence $s'_i/1$ are α it follows from (4.1) that s''_i is α. Similarly by (4.2), m'' is S'-β. Since all s''_i and m'' are S-primitive and S splits M, $n' = s''_1 \cdots s''_{k'} m''$ is S-primitive. Now from the above factorization of $n/1$ it follows that $un = u'n'$ for some $u, u' \in S$ and since n and n' are both S-primitive and by (3.3)(ii), $n \sim^S n'$ and hence $n \cong^S n'$ according to (3.3)(i). This implies that $n = an'$ with $a \in \mathrm{U}(R)$. We conclude that $m = s_1 \cdots s_k s''_1 \cdots s''_{k'}(am'')$ is an (α, β)-S'-atomic factorization of m. $\square$

To establish a version of the above theorem for other factorization properties we need:

Lemma 4.5. *Suppose that* M *is semi-*S*-factorable and* $S \cap \mathrm{Z}(M) = \emptyset$. *If* M *is an* S'-*UFM,* S'-*HFM or* S'-*FFM, then* M *is an* S-*BFM. Also if* M *is an* S-*BFM, then* $E = R \setminus \mathrm{Ann}(M)$ *is compactly* S-*atomic.*

Proof. Let $0 \neq m \in M$. In either of the cases, the possible lengths of an S'-atomic factorization of m are finite and hence there is an upper bound N_m on these lengths. Now let $m = s_1 \cdots s_l m'$ where $s_i \in S \setminus \mathrm{U}(R)$. By replacing m' with one of its compact S-atomic factorizations, the length of this factorization does not decrease. Therefore, we can assume that m' is S-primitive.

Let $s_1 m' = s'_{1,1} \cdots s'_{1,k_1} m'_1$ be an S'-atomic factorization of $s_1 m'$. If $k_1 = 0$, then as m'_1 is S-very strongly primitive by (4.3) and (3.3)(i), we must have $s_1 \in \mathrm{U}(R)$ against our assumption. So $k_1 \geq 1$. Similarly, we can find S'-atomic factorizations $s_i m'_{i-1} = s'_{i,1} \cdots s'_{i,k_i} m'_i$ for each $2 \leq i \leq l$ with $k_i \geq 1$ for all i. Consequently, we get an S'-atomic factorization $m = s'_{1,1} \cdots s'_{1,k_1} \cdots s'_{l,1} \cdots s'_{l,k_l} m'_l$. Hence $l \leq \sum_{i=1}^{l} k_i \leq N_m$ and M is an S-BFM.

Now assume that $r \in E$ and M is an S-BFM. Then there is an S-primitive element $m \in M \setminus \mathrm{Ann}_M(r)$. If $r = s_1 \cdots s_l r'$ with $s_i \in S \setminus \mathrm{U}(R)$, then $rm =$

$s_1 \cdots s_l(r'm)$ and hence $l \leq N_{rm}$. If l is the largest possible length of S-factorizations of r, then r' is S-very strongly primitive and the result follows. □

Note that although under the conditions of the first part of the above lemma, M is an S-BFM, but it need not be an S'-BFM as demonstrated in [19, Example 2.14] with $S' = D$ and $S = \mathrm{U}(D)$.

Theorem 4.6. *Suppose that M is semi-S-factorable, $S \cap \mathrm{Z}(M) = \emptyset$ and let $\mathcal{P}$ be one of the following properties: being présimplifiable, having unique factorization, having finite factorization, being half factorial, having bounded factorization. If M has $\mathcal{P}$ with respect to S' then it has $\mathcal{P}$ with respect to S.*

Proof. For the case that $\mathcal{P}$ = being présimplifiable or having bounded factorization, the result is [19, Theorem 3.8] (and does not need semi-S-factorability or M-regularity). For other cases, note that by the previous lemma, M is an S-BFM and hence S-atomic and therefore R is atomic in $E = S' \setminus \mathrm{Ann}(M)$ by (3.4). Let $s \in E$ and $m \in M \setminus \mathrm{Ann}_M(s)$. By replacing m with an S'-primitive element appearing in an S'-atomic factorization of m, we can assume that m is S'-primitive. Now if $s = s_1 \cdots s_k$ is an atomic factorization, then $sm = s_1 \cdots s_k m$ is an S'-atomic factorization of sm with the same length and two factorizations of sm arising in this way are isomorphic if and only if the two factorizations of s are isomorphic. Consequently, R has $\mathcal{P}$ in E and hence by (3.4), M has $\mathcal{P}$ with respect to S. □

Another condition under which, an S'-UFM is an S-UFM is presented in [19, Theorem 3.8 & Notation 3.5].

Lemma 4.7. *Assume that S splits M, $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic and $S \cap \mathrm{Z}(M) = S \cap \mathrm{Z}(R) = \emptyset$. Let $y_1, y_2 \in S' \setminus (S \cup \mathrm{Ann}(M))$ be irreducible and $0 \neq m_1, m_2 \in M$ be S'-primitive. Then $y_1/1 \sim y_2/1$ in R_S if and only if $y_1 \sim y_2$ in R and $m_1/1 \sim^T m_2/1$ if and only if $m_1 \sim^{S'} m_2$.*

Proof. ($\Leftarrow$) Trivial. ($\Rightarrow$) Suppose $y_1/1 = (r/s)(y_2/1)$. Then $sy_1 = ry_2 = s_0(r'y_2)$, where $r = s_0 r'$ is a compact S-atomic factorization of r. Hence by (4.3), (3.9) and (3.3)(ii), $y_1 \sim^S r'y_2$ and $y_1 \in Ry_2$. Similarly $y_2 \in Ry_1$ and $y_1 \sim y_2$. The proof of the other statement is similar. □

Theorem 4.8. *Suppose that S splits M, $S \cap \mathrm{Z}(M) = S \cap \mathrm{Z}(R) = \emptyset$ and let $\mathcal{P}$ be one of the following properties: having unique factorization, having finite factorization, being half factorial, having bounded factorization. Then M has $\mathcal{P}$ with respect to S' if and only if M has $\mathcal{P}$ with respect to S and M_S has $\mathcal{P}$ with respect to T. A similar statement holds for $\mathcal{P}$ = being présimplifiable, if we further assume that $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic.*

Proof. ($\Rightarrow$) According to (4.6), we just need to show that M_S has $\mathcal{P}$ with respect to T. Note that by (4.5), in all cases $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic. First assume that $\mathcal{P}$ = being présimplifiable and $m/s_0 = (s'/s_1)(m/s_0)$ for some

$s' \in S', s_1, s_0 \in S, 0 \neq m \in M$. If $m = v_0 m'$ and $s' = v_1 s''$ are compact S-atomic factorizations of m and s' with $v_i \in S$, then $s_1 m' = v_1(s'' m')$ and as S splits M and by (3.3), it follows that $m' \cong^S s'' m'$. Then $m' = (us'')m'$ for some $u \in \mathrm{U}(R)$ and since M is S'-présimplifiable and $s'' \in S'$, we deduce $s'' \in \mathrm{U}(R)$ and $s' \in S$, as required.

For $\mathcal{P} \neq$ being présimplifiable, since M is S'-atomic, it follows form (4.4) that M_S is T-atomic. Let $0 \neq m \in M$ be S-primitive. If $(m/1) = (s'_1/s_1)\cdots(s'_k/s_k)(m'/1)$ is any T-factorization of $m/1$ and $s'_i = v_i s''_i$ and $m' = vm''$ are compact S-atomic factorization of s'_i and m', then $s_1 \cdots s_k m = v_1 \cdots v_k(s''_1 \cdots s''_k m'')$ which by (3.3)(ii) implies that $m \cong^S s''_1 \cdots s''_k m''$. Also according to (4.1) and (4.2), s''_i is irreducible if and only if s'_i/s_i is so and m'' is S'-primitive if and only if $m'/1$ is so. Therefore, if for example the number of S'-atomic factorizations of m is finite, then so is the number of T-atomic factorizations of $m/1$. Similarly other S'-factorization properties of m pass to T-factorization properties of $m/1$. Noting that each $0 \neq x \in M_S$ is a unit multiple of some $m/1$ where $0 \neq m \in M$ is S-primitive, the result is concluded.

($\Leftarrow$) The cases $\mathcal{P} =$ being présimplifiable or having bounded factorization is [19, Theorem 3.9] (with much less assumptions). So assume that $\mathcal{P} =$ having unique factorization or finite factorization or being half factorial. Note that by (4.5) applied with $S' = S$, we see that M is S-BFM and $R \setminus \mathrm{Ann}(M)$ is compact S-atomic. Thus it follows (4.4)(ii), that M is S'-atomic. We prove the result for $\mathcal{P} =$ having finite factorization and the other cases follow similarly.

Let $m = s_1 \cdots s_k s'_1 \cdots s'_{k'} m'$ be an S'-atomic factorization of $0 \neq m \in M$ with $s_i \in S$ and $s'_i \in S' \setminus S$. Then by (4.3), each s'_i, m' and hence their product $m'' = s'_1 \cdots s'_{k'} m'$ are S-primitive. Consequently, $m = s_1 \cdots s_k m''$ is S-isomorphic (and thus S'-isomorphic) to one of the finite S-atomic factorizations of m. So if we show S-primitive elements of M have finitely many S'-atomic factorizations, we are done. Thus we assume m is S-primitive and $k = 0$. Then $m/1 = (s'_1/1)\cdots(s'_{k'}/1)(m'/1)$ is a T-atomic factorization of $m/1$ by (4.1) and (4.2). Therefore, each S'-atomic factorization of m leads to a T-atomic factorization of $m/1$ and according to (4.7) if two such T-atomic factorizations of $m/1$ are T-isomorphic, then the original S'-atomic factorizations of m are S'-isomorphic. Consequently, as $m/1$ has finitely many T-atomic factorizations up to T-isomorphisms, m also has finitely many S'-atomic factorizations up to S'-isomorphism, as claimed. $\square$

It should be mentioned that ($\Leftarrow$) of the above theorem, is a partial answer to [19, Question 3.11]. Summing up Theorems (4.4)(ii), (4.8) and (3.4), we get:

Corollary 4.9. *Let $S \subseteq S'$ be two saturated multiplicatively closed subsets of R and set $T = S^{-1}S'$. Suppose that S splits M and $S \cap \mathrm{Z}(M) = S \cap \mathrm{Z}(R) = \emptyset$. For $\mathcal{P} \in \{$having unique factorization, having finite factorization, being half factorial, having bounded factorization$\}$, the following are equivalent.*

(i) *M has $\mathcal{P}$ with respect to S'.*

(ii) *M has $\mathcal{P}$ with respect to S and M_S has $\mathcal{P}$ with respect to T.*

(iii) *R has $\mathcal{P}$ in $S \setminus \mathrm{Ann}(M)$ and M_S has $\mathcal{P}$ with respect to T.*

If we further assume that $S' \setminus \mathrm{Ann}(M)$ is compactly S-atomic, then the above conditions are also equivalent for $\mathcal{P}$ = being présimplifiable or atomic.

If we set $S' = R$ in the above corollary, we get some Nagata type theorems. If we further assume that $M = R$ and R is an integral domain, then this corollary implies [4, Theorems 3.1 & 3.3] (except for ACCP and idf-domain, which are not investigated in this research). Because if S is generated by primes, then every element of S has a unique factorization as a product of primes and hence irreducibles, that is, R has unique factorization (and whence has finite and bounded factorization and is présimplifiable, atomic and half factorial) in S. Indeed, even when $M = S' = R$ and R is a domain, this corollary is slightly stronger than [4, Theorem 3.1], since in our results S need not be generated by primes. An example in which S is not generated by primes is presented in the next section.

5. An application

We present an example which shows how our main result (4.9), could be applied in the case that $M = S' = R$ and R is an integral domain, the classical and most important situation in the factorization theory. Note that in this case, R is présimplifiable and hence all types of associativity (resp. irreducibility, primitivity) are equivalent to each other. Also if S splits R, then R is compactly S-atomic and hence (4.9) could be applied for all $\mathcal{P} \in \{$having unique factorization, having finite factorization, being half factorial, having bounded factorization, being atomic$\}$. First let's state the setting of our example as a notation.

Notation 5.1. In this section, we assume that $M = S' = R = A + XB[X]$ where $A \subseteq B$ are integral domains and $S = (\mathrm{U}(B) \cap A) \cup \{uX^n | u \in \mathrm{U}(B), n \in \mathbb{N}\}$. Also we set $S_0 = \mathrm{U}(B) \cap A$ and denote the quotient field of A by K.

It is easy to see that S is a saturated multiplicatively closed subset of R, indeed, it is the saturated multiplicatively closed subset generated by X. Also if $B \neq A$ and $b \in B \setminus A$, then $(bX)^2 \in RX$, while $bX \notin RX$. So X is not prime and S is not generated by primes if $A \neq B$.

Theorem 5.2. *The set S splits R if and only if all of the following conditions hold*:

(i) *S_0 splits A;*
(ii) *for every $b \in B$ there are $u \in \mathrm{U}(B)$ and $a \in A$ such that $b = ua$;*
(iii) $\mathrm{U}(B) \cap K \subseteq A_{S_0}$.

In particular, if either B is a filed or S_0 is any A-splitting saturated multiplicatively closed subset of A and $B = A_{S_0}$, then S splits R.

Proof. ($\Rightarrow$) Note that if $a \in A$, and $a = sf$ for some $s \in S, f \in R$, then $s \in S_0$ and $f \in A$. Thus a is S-primitive if and only if it is S_0-primitive and so (i) follows as S splits A. To see (ii), let $b \in B$. If bX is S-primitive, then b^2X^2 should be S-primitive by (ii) of Definition (3.7). But $b^2X^2 = (b^2X)X$ is an S-factorization of b^2X^2, a contradiction. So bX is not S-primitive. Thus if its compact S-atomic factorization is $bX = sf$, with $s \in S$ and S-primitive $f \in R$, then $\deg f = 0$. Whence $f \in A$ and $s = uX$ for some $u \in B$ and (ii) follows with $a = f$.

Now let $u \in \mathrm{U}(B)$ and $u = a/a'$ for some $0 \neq a, a' \in A$. Assume that $a = s_0a_0$ and $a' = s_0'a_0'$ are compact S-atomic (hence S_0-atomic) factorizations of a and a'. Then $((us_0'/s_0)X)a_0' = (X)a_0$ are two compact S-atomic factorizations of a_0X, therefore by uniqueness of such factorizations we have $X \cong (us_0'/s_0)X$, that is, $v = us_0'/s_0 \in \mathrm{U}(R) = \mathrm{U}(A)$. Consequently, $u = vs_0/s_0' \in A_{S_0}$.

($\Leftarrow$) Suppose that $f = \sum_{i=k}^n b_iX^i \in R$ with $k \leq n$ and $b_k \neq 0$. If $k = 0$, then $b_0 \in A$ has a compact S_0-atomic factorization $b_0 = sa$ where $s \in \mathrm{U}(B) \cap A$. Therefore $f = s(a + \sum_{i=2}^n (b_i/s)X^i)$ is a compact S-atomic factorization of f (note that $a + \sum_{i=2}^n (b_i/s)X^i$ is S-primitive, since a is so). If $k > 0$, then by (ii), there are $u \in \mathrm{U}(B)$ and $a_0 \in A$ such that $b = ua_0$. If $a_0 = s_0a$ is the compact S_0-atomic factorization of a_0, then set $s = us_0 \in \mathrm{U}(B)$. Thus we get the compact S-factorization $f = (sX^k)(a + \sum_{i=k+1}^n (b_i/s)X^{i-k})$. Thus R is compactly S-atomic and f is S-primitive if and only if $b_0 = f(0)$ is nonzero and S_0-primitive in A. In particular, if $f, g \in R$ are S-primitive, then $0 \neq (fg)(0)$ is S_0-primitive in A by (i) and fg is S-primitive.

It remains to show the uniqueness of compact S-factorizations of f. Assume that $f = u_1X^{k_1}f_1$ and $u_2X^{k_2}f_2$ are two S-atomic factorizations of f where f_1 and f_2 are S-primitive and $u_1, u_2 \in \mathrm{U}(B)$. Therefore, by the previous paragraph, $a_1 = f_1(0) \neq 0$, $a_2 = f_2(0) \neq 0$ are S_0-primitive and hence $k_1 = k_2 = k$. Also $u_1a_1 = u_2a_2$ and hence $u = u_1/u_2 = a_2/a_1 \in \mathrm{U}(B) \cap K \subseteq A_{S_0}$, by (iii). Consequently, $u = a/s_0$ for some $a \in A$ and $s_0 \in S_0$. If $a = s_0'a'$ is the compact S_0-atomic factorization of a, then $s_0a_2 = aa_1 = s_0'(a'a_1)$. Since both a_2 and $a'a_1$ are S_0-primitive, we must have $a_2 \sim^{S_0} a'a_1$ and in particular, $a_2 \in Aa_1$. Similarly $a_1 \in Aa_2$, hence $Aa_1 = Aa_2$ and as A is a domain, $a_1 = va_2$ for some $v \in \mathrm{U}(A)$. It follows that $u_2 = vu_1$ and hence $u_1X^k \sim^S u_2X^k$, as required. □

Thus in particular, we can apply (4.9) with $M = S' = R$, in the case $A = B$. Since in this case X is prime, $R = B[X]$ has unique factorization in S and hence we get

Corollary 5.3. *If B is an integral domain, then $B[X]$ is atomic (resp. a BFD, a FFD, a HFD, a UFD) if and only if $B[X, X^{-1}]$ is so.*

Theorem 5.4. *Using Notation 5.1 and assuming that the conditions* (i)–(iii) *of* (5.2) *hold, then we have*

(i) *R is atomic (resp. a BFD, a HFD) if and only if $U(B) \cap A = \mathrm{U}(A)$ and $B[X]$ is atomic (resp. a BFD, a HFD).*
(ii) *R is a FFD if and only if $|\frac{\mathrm{U}(B)}{\mathrm{U}(A)}| < \infty$ and $B[X]$ is a FFD.*

(iii) *R is a UFD if and only if* $|\frac{\mathrm{U}(B)}{\mathrm{U}(A)}| = 1$ *and* $B[X]$ *is a UFD.*

Proof. In all cases, by applying (4.9), we deduce that R has the desired property $\mathcal{P}$ if and only if R has $\mathcal{P}$ in S and $R_S = B[X, X^{-1}]$ has $\mathcal{P}$. Thus according to (5.3), we just need to show that in each case, R has $\mathcal{P}$ in S if and only if the stated condition on $\mathrm{U}(B)$ and $\mathrm{U}(A)$ is satisfied.

(i) Assume R is atomic in S. Then X has decomposition $X = a_1 \cdots a_n(uX)$ with irreducible $a_i \in A$ and $u \in \mathrm{U}(B)$ such that uX is irreducible in R. If $a \in \mathrm{U}(B) \cap A$, then $uX = ((u/a)X)a$ and as uX is irreducible, it follows that $a \in \mathrm{U}(R) = \mathrm{U}(A)$. Conversely, if $\mathrm{U}(B) \cap A = \mathrm{U}(A)$, then uX is irreducible for each $u \in \mathrm{U}(B)$ and hence $f = uX^n = (uX)X^{n-1}$ is an atomic factorization of $f \in S \setminus \mathrm{U}(R)$. Also in any atomic factorization of f exactly n terms of the form vX appear where $v \in \mathrm{U}(B)$. Since irreducible elements of S are exactly those of the form vX for $v \in \mathrm{U}(B)$, it follows that any atomic factorization of f has length n and R is half factorial and has bounded factorization in S.

(ii) If R has finite factorization in S, then it is atomic in S and hence by (i), $\mathrm{U}(B) \cap A = \mathrm{U}(A)$ and the irreducible elements in S are of the form vX for $v \in \mathrm{U}(B)$. Note that $vX \cong v'X$ if and only if $v \in \mathrm{U}(A)v'$ if and only if the image of v, v' in the quotient group $\mathrm{U}(B)/\mathrm{U}(A)$ are equal. So if $v_1, v_2, \ldots$ are infinite elements of $\mathrm{U}(B)$, such that $v_i\mathrm{U}(A) \neq v_j\mathrm{U}(A)$ for each $i \neq j$, then we can find an infinite set of non-isomorphic atomic factorizations of $X^2 = (v_1X)(v_1^{-1}X) = (v_2X)(v_2^{-1}X) = \cdots$.

Conversely, suppose that $n = |\mathrm{U}(B)/\mathrm{U}(A)| < \infty$. If $a \in \mathrm{U}(B) \cap A$, then $a^n \in \mathrm{U}(A)$ and hence $a \in \mathrm{U}(A)$. Therefore, by (i) R is atomic and half factorial in S and irreducibles of R in S are of the form vX with $v \in \mathrm{U}(B)$. Since every atomic factorization of $uX^k \in S$ has the same length k, to show that it has finitely many factorizations it suffices to show that it has only finitely many non-associate irreducible divisors. But there are n non-associate irreducible elements in S, because $vX \cong v'X$ if and only if $v\mathrm{U}(A) = v'\mathrm{U}(A)$, and we are done. The proof of (iii) is similar. □

This theorem generalizes the following previously known result (for example, it is an immediate consequence of the propositions considering the $D + M$ construction in [3]) which follows from (5.4) in the case that B is a field.

Corollary 5.5. *Assume that B is a field. Then R is atomic if and only if R is a BFD if and only if R is a HFD if and only if A is a field. Also R is a FFD (resp. UFD) if and only if A is a field and $|B^*/A^*| < \infty$ (resp. $B = A$).*

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