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CONSTANT CURVATURE FACTORABLE SURFACES IN 3-DIMENSIONAL ISOTROPIC SPACE

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Abstract. In the present paper, we study and classify factorable surfaces in a 3-dimensional isotropic space with constant isotropic Gaussian (K) and mean curvature (H). We provide a non-existence result relating to such surfaces satisfying $\frac{H}{K} = const$. Several examples are also illustrated.

1. Introduction

Let $\mathbb{E}^3$ be a 3-dimensional Euclidean space and (x, y, z) rectangular coordinates. A surface in $\mathbb{E}^3$ is said to be *factorable* (so-called *homothetical*) if it is a graph of the form $z(x, y) = f(x) g(y)$, where f and g are smooth functions (see [4, 14]). Such surfaces in $\mathbb{E}^3$ with constant Gaussian (K) and mean curvature (H) were obtained in [10, 14, 24].

As more general case, Zong et al. [25] defined that an *affine factorable surface* in $\mathbb{E}^3$ is a graph of the form

$$z(x, y) = f(x) g(y + ax), \ a \neq 0$$

and classified these ones with K, H constants.

A surface in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ is said to be *factorable* if it can be expressed by one of the explicit forms ([15])

$$\Phi_1 : z(x, y) = f(x) g(y), \ \Phi_2 : y(x, z) = f(x) g(z), \ \Phi_3 : x(y, z) = f(y) g(z).$$

Up to the causal characters of the directions, six different classes of these surfaces in $\mathbb{E}^3_1$ appear. The surfaces in $\mathbb{E}^3_1$ with K, H constants were described in [9, 15, 21].

In 3-dimensional context, the factorable surfaces are closely connected with translation surfaces, namely the surfaces generated by translating of two curves. For instance; in the homogeneous Riemannian space $\mathbb{H}^2 \times \mathbb{R}$ that is a Lie group, up to its group operation, a translation surface of type 2 is a graph of the form

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$y(x,z) = f(x)g(z)$ (see [22]). For more details, we refer to [7, 8], [11–13], [17, 23].

Besides the Minkowskian space, a 3-dimensional isotropic space $\mathbb{I}^3$ provides two different types of the factorable surfaces. This special ambient space which is one of the real Cayley-Klein spaces is the product of the xy-plane and the isotropic z-direction with a degenerate parabolic distance metric (cf. [5]).

Due to the absolute figure of $\mathbb{I}^3$, the factorable surface Φ_1 distinctly behaves from others. We call it *factorable surface of type 1* (see [1–3]). The surfaces Φ_2, Φ_3 in $\mathbb{I}^3$ are locally isometric and, up to a sign, have same second fundamental form. This means to have same isotropic Gaussian K and, up to a sign, mean curvature H. These surfaces are said to be of *type 2*.

In this manner we are mainly interested in the factorable surfaces of type 2 in $\mathbb{I}^3$. We describe such surfaces in $\mathbb{I}^3$ with $K, H, H/K$ constants by the following results:

Theorem 1.1. *A factorable surface of type 2* (Φ_3) *in* $\mathbb{I}^3$ *has constant isotropic mean curvature* H_0 *if and only if, up to suitable translations and constants, one of the following occurs*:

(i) *If* Φ_3 *is isotropic minimal, i.e.,* $H_0 = 0$;
  (i.1) Φ_3 *is a non-isotropic plane,*
  (i.2) $x(y,z) = y\tan(cz)$,
  (i.3) $x(y,z) = c\frac{z}{y}$.

(ii) *Otherwise* ($H_0 \neq 0$), $x(y,z) = \pm\sqrt{\frac{-z}{H_0}}$,

where c is some nonzero constant.

Theorem 1.2. *A factorable surface of type 2* (Φ_3) *in* $\mathbb{I}^3$ *has constant isotropic Gaussian curvature* K_0 *if and only if, up to suitable translations and constants, one of the following holds*:

(i) *If* Φ_3 *is isotropic flat, i.e.,* $K_0 = 0$;
  (i.1) $x(y,z) = c_1 g(z)$, $\frac{dg}{dz} \neq 0$,
  (i.2) $x(y,z) = c_1 e^{c_2 y + c_3 z}$,
  (i.3) $x(y,z) = c_1 y^{c_2} z^{c_3}$, $c_2 + c_3 = 1$.

(ii) *Otherwise* ($K_0 \neq 0$);
  (ii.1) K_0 *is negative and* $x(y,z) = \pm\frac{z}{\sqrt{-Ky}}$,
  (ii.2) $x(y,z) = \frac{c_1}{y} g(z)$ *for*

$$z = \pm \int \left(c_2 g^{-1} - \frac{K_0}{c_1^2} \right)^{1/2} dg,$$

where c_1, c_2, c_3 *are some nonzero constants.*

Theorem 1.3. *There does not exist a factorable surface of type 2 in* $\mathbb{I}^3$ *that satisfies* $\frac{H}{K} = const. \neq 0$.

We point out that the above results are also valid for the factorable surface Φ_2 in $\mathbb{I}^3$ by replacing x with y as well as taking $y = \pm\sqrt{\frac{z}{H_0}}$ in the last statement of Theorem 1.1.

2. Preliminaries

For detailed properties of isotropic spaces, see [6,16], [18–20].

Let $P\left(\mathbb{R}^3\right)$ be a real 3-dimensional projective space and $(x_0 : x_1 : x_2 : x_3)$ denote the projective coordinates in $P\left(\mathbb{R}^3\right)$. A 3-dimensional *isotropic space* $\mathbb{I}^3$ is a Cayley-Klein space obtained from $P\left(\mathbb{R}^3\right)$ such that its *absolute figure* consists of a plane (*absolute plane*) ω and complex-conjugate straight lines (*absolute lines*) l_1, l_2 in ω. In coordinate form, ω is given by $x_0 = 0$ and l_1, l_2 by $x_0 = x_1 \pm ix_2 = 0$. The *absolute point*, $(0 : 0 : 0 : 1)$, is the intersection of the absolute lines.

For $x_0 \neq 0$, we have the affine coordinates by $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, $z = \frac{x_3}{x_0}$. The group of motions of $\mathbb{I}^3$ is given by

$$(x, y, z) \longmapsto (x', y', z') : \begin{cases} x' = a_1 + x\cos\phi - y\sin\phi, \\ y' = a_2 + x\sin\phi + y\cos\phi, \\ z' = a_3 + a_4 x + a_5 y + z, \end{cases} \tag{2.1}$$

where $a_1, \ldots, a_5, \phi \in \mathbb{R}$. The *isotropic metric* that is an invariant of (2.1) is induced by the absolute figure, namely $ds^2 = dx^2 + dy^2$.

There are two types of the lines and the planes in $\mathbb{I}^3$ arising from its absolute figure: The lines parallel (resp. non-parallel) to z-direction are said to be *isotropic* (resp. *non-isotropic*). A plane is said to be *isotropic* if it involves an isotropic line. Otherwise it is called *non-isotropic plane* or *Euclidean plane*. For example the equations $ax + by + cz = 0$ $(a, b, c \in \mathbb{R},\ c \neq 0)$ and $ax + by = 0$ determine a non-isotropic plane and an isotropic plane, respectively.

We restrict our framework to regular surfaces whose the tangent plane at each point is non-isotropic, namely *admissible surfaces*.

Let M be a regular admissible surface in $\mathbb{I}^3$ locally parameterized by

$$r(u, v) = (x(u, v), y(u, v), z(u, v))$$

for a coordinate pair (u, v). The components E, F, G of the first fundamental form of M in $\mathbb{I}^3$ are computed by the induced metric from $\mathbb{I}^3$. The unit normal vector of M is the unit vector parallel to the z-direction. The components of the second fundamental form II of M are given by

$$l = \frac{\det(r_{uu}, r_u, r_v)}{\sqrt{EG - F^2}},\ m = \frac{\det(r_{uv}, r_u, r_v)}{\sqrt{EG - F^2}},\ n = \frac{\det(r_{vv}, r_u, r_v)}{\sqrt{EG - F^2}}.$$

Accordingly, the *isotropic Gaussian* (or *relative*) and *mean curvature* of M are respectively defined by

$$K = \frac{ln - m^2}{EG - F^2},\ H = \frac{En - 2Fm + Gl}{2(EG - F^2)}.$$

A surface in $\mathbb{I}^3$ is said to be *isotropic minimal* (resp. *flat*) if H (resp. K) vanishes identically. Further, it is said to have constant isotropic mean (resp. Gaussian) curvature if H (resp. K) is a constant function on whole surface.

3. Proof of Theorem 1.1

A factorable surface of type 2 in $\mathbb{I}^3$ can be locally expressed by either

$$\Phi_2 : r(x,z) = (x, f(x)g(z), z) \text{ or } \Phi_3 : r(y,z) = (f(y)g(z), y, z).$$

All over this paper, all calculations shall be done for the surface Φ_3. Its first fundamental form in $\mathbb{I}^3$ turns to

$$ds^2 = \left(1 + (f'g)^2\right) dy^2 + 2(fgf'g')\, dydz + (fg')^2 dz^2,$$

where $f' = \frac{df}{dy}$, $g' = \frac{dg}{dz}$. Note that g' must be nonzero to obtain a regular admissible surface. By a calculation for the second fundamental form of Φ_3 we have

$$II = \left(\frac{f''g}{fg'}\right) dy^2 + 2\left(\frac{f'}{f}\right) dydz + \left(\frac{g''}{g'}\right) dz^2, \; g' \neq 0.$$

Therefore, the isotropic mean curvature H of Φ_3 becomes

$$H = \frac{\left((f'g)^2 + 1\right) g'' + \left(ff'' - 2(f')^2\right) g (g')^2}{2f^2 (g')^3}. \tag{3.1}$$

Let us assume that $H = H_0 = const$. First we distinguish the case in which Φ_3 is isotropic minimal:

Case A: $H_0 = 0$. (3.1) reduces to

$$\left((f'g)^2 + 1\right) g'' + \left(ff'' - 2(f')^2\right) g (g')^2 = 0. \tag{3.2}$$

We have three cases in order to solve (3.2):

Case A.1. $f = f_0 \neq 0 \in \mathbb{R}$. (3.2) immediately implies $g = c_1 z + c_2$, $c_1, c_2 \in \mathbb{R}$, and thus we deduce that Φ_3 is a non-isotropic plane. This gives the statement (i.1) of Theorem 1.1.

Case A.2. $f = c_1 y + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. (3.2) turns to

$$\frac{g''}{g'} = \frac{2c_1^2 g g'}{1 + (c_1 g)^2}.$$

By solving this one, we obtain

$$g = \frac{1}{c_1} \tan(c_2 z + c_3), \; c_2, c_3 \in \mathbb{R}, \; c_2 \neq 0,$$

which proves the statement (i.2) of Theorem 1.1.

Case A.3. $f'' \neq 0$. By dividing (3.2) with $g(g')^2$ one can be rewritten as

$$\left((f'g)^2 + 1\right) \frac{g''}{g(g')^2} + ff'' - 2(f')^2 = 0. \tag{3.3}$$

Taking partial derivative of (3.3) with respect to z and after dividing with $(f')^2$, we get

$$2\frac{g''}{g'}+\left(\frac{1}{(f')^2}+g^2\right)\left(\frac{g''}{g\,(g')^2}\right)'=0. \tag{3.4}$$

By taking partial derivative of (3.4) with respect to y, we find $g''=c_1 g\,(g')^2$, $c_1\in\mathbb{R}$. We have two cases:

Case A.3.1. $c_1=0$. (3.3) reduces to

$$ff''-2\,(f')^2=0. \tag{3.5}$$

By solving (3.5) we derive

$$f=-\frac{1}{c_2 y+c_3},\ c_2,c_3\in\mathbb{R},\ c_2\neq 0.$$

This implies the statement (i.3) of Theorem 1.1.

Case A.3.2. $c_1\neq 0$. (3.4) immediately leads to the contradiction $2c_1 g g'=0$.

Case B: $H_0\neq 0$. We have cases:

Case B.1. $f=f_0\neq 0\in\mathbb{R}$. Then (3.1) follows

$$2H_0 f_0^2=\frac{g''}{(g')^3}. \tag{3.6}$$

Solving it gives $g(z)=\pm\frac{1}{2H_0 f_0^2}\sqrt{-4H_0 f_0^2 z+c_1}+c_2$, $c_1,c_2\in\mathbb{R}$. This is the proof of the statement (ii) of Theorem 1.1.

Case B.2. $f=c_1 y+c_2$, $c_1,c_2\in\mathbb{R}$, $c_1\neq 0$. By considering this one into (3.1) we conclude

$$2\,(c_1 y+c_2)^2 H_0=\left(1+c_1^2 g^2\right)\frac{g''}{(g')^3}-2c_1^2\frac{g}{g'}. \tag{3.7}$$

The left side in (3.7) is a function of y while other side is either a constant or a function of z. This is not possible.

Case B.3. $f''\neq 0$. By multiplying both side of (3.1) with $2f^2\frac{g'}{g}$ one can be rearranged as

$$2H_0 f^2\frac{g'}{g}=\left((f'g)^2+1\right)\frac{g''}{g\,(g')^2}+ff''-2\,(f')^2. \tag{3.8}$$

Taking partial derivative of (3.8) with respect to z and thereafter dividing with $(f')^2$ yields

$$2H_0\left(\frac{f}{f'}\right)^2\left(\frac{g'}{g}\right)'=2\frac{g''}{g'}+\left(\frac{1}{(f')^2}+g^2\right)\left(\frac{g''}{g\,(g')^2}\right)'. \tag{3.9}$$

It is obvious in (3.9) that $g''\neq 0$. To solve (3.9) we have two cases:

Case B.3.1. $g'' = c_1 g (g')^2$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$. This implies that

$$g' = e^{\frac{c_1}{2} g^2 + c_2}, \ c_2 \in \mathbb{R}. \tag{3.10}$$

Substituting (3.10) into (3.9) gives an equation in the following form:

$$\left(c_1 e^{\frac{-c_1}{2} g^2 - c_2} \right) g^3 - \left(c_1 H_0 \left(\frac{f}{f'} \right)^2 \right) g^2 + H_0 \left(\frac{f}{f'} \right)^2 = 0,$$

where all coefficients with respect to g must be zero and this is a contradiction.

Case B.3.2. $\left(\frac{g''}{g(g')^2} \right)' \neq 0$. By dividing (3.9) with $\left(\frac{g''}{g(g')^2} \right)'$, it turns to the following form:

$$A_1(y) B_1(z) = A_2(y) + B_2(z), \tag{3.11}$$

where

$$\begin{cases} A_1(y) = 2H_0 \left(\dfrac{f}{f'} \right)^2, & A_2(y) = \dfrac{1}{(f')^2}, \\ B_1(z) = \dfrac{\left(\frac{g'}{g} \right)'}{\left(\dfrac{g''}{g(g')^2} \right)'}, & B_2(z) = 2\dfrac{g''}{g'} + g^2. \end{cases}$$

The fact that all terms in (3.11) must be constant for every y and z yields $A_2(y) = \frac{1}{(f')^2} = const.$, which contradicts with the assumption of Case B.3.

4. Proof of Theorem 1.2

By a calculation for a factorable graph of type 2 in $\mathbb{I}^3$, the isotropic Gaussian curvature turns to

$$K = \frac{fgf''g'' - (f'g')^2}{(fg')^4}. \tag{4.1}$$

Let us assume that $K = K_0 = const$. We have cases:

Case A: $K_0 = 0$. (4.1) reduces to

$$fgf''g'' - (f'g')^2 = 0. \tag{4.2}$$

f or g constants are solutions for (4.2) and by regularity we have the statement (i.1) of Theorem 1.2. Suppose that f, g are non-constants. Then (4.2) yields $f''g'' \neq 0$. Thereby (4.2) can be arranged as

$$\frac{ff''}{(f')^2} = \frac{(g')^2}{gg''}. \tag{4.3}$$

Both sides of (4.3) are equal to same nonzero constant, namely

$$ff'' - c_1 (f')^2 = 0 \text{ and } gg'' - \frac{1}{c_1} (g')^2 = 0. \tag{4.4}$$

If $c_1 = 1$ in (4.4), then by solving it we obtain

$$f(y) = c_2 e^{c_3 y} \text{ and } g(z) = c_4 e^{c_5 z},\ c_2, \ldots, c_5 \in \mathbb{R}.$$

This gives the statement (i.2) of Theorem 1.2. Otherwise, i.e., $c_1 \neq 1$ in (4.4), we derive

$$f(y) = ((1 - c_1)(c_6 y + c_7))^{\frac{1}{1-c_1}} \text{ and } g(z) = \left(\left(\frac{c_1 - 1}{c_1} \right) (c_8 z + c_9) \right)^{\frac{c_1}{c_1 - 1}},$$

where $c_6, \ldots, c_9 \in \mathbb{R}$. This completes the proof of the statement (i) of Theorem 1.2.

Case B : $K_0 \neq 0$. (4.1) can be rewritten as

$$K_0 (g')^2 = \frac{f''}{f^3} \left(\frac{gg''}{(g')^2} \right) - \left(\frac{f'}{f^2} \right)^2. \tag{4.5}$$

Taking partial derivative of (4.5) with respect to z leads to

$$2K_0 g' g'' = \frac{f''}{f^3} \left(\frac{gg''}{(g')^2} \right)'. \tag{4.6}$$

We have two cases for (4.6):

Case B.1. The situaton that $g'' = 0$, $g(z) = c_1 z + c_2$, $c_1, c_2 \in \mathbb{R}$, is a solution for (4.6). Hence, from (4.5), we deduce

$$K_0 (c_1)^2 = - \left(\frac{f'}{f^2} \right)^2,$$

which implies that K_0 is negative and

$$f(y) = \frac{1}{\pm c_1 \sqrt{-K_0} y + c_3}.$$

This proves the statement (ii.1) of Theorem 1.2.

Case B.2. $g'' \neq 0$. (4.6) immediately implies

$$f'' = c_1 f^3,\ c_1 \in \mathbb{R},\ c_1 \neq 0. \tag{4.7}$$

Considering (4.7) into (4.5) yields to

$$f' = c_2 f^2,\ c_2 \in \mathbb{R},\ c_2 \neq 0. \tag{4.8}$$

It follows from (4.7) and (4.8) that $c_1 = 2c_2^2$ and

$$f(y) = -\frac{1}{c_2 y + c_3}$$

for some constant c_3. Nevertheless, by substituting (4.7) and (4.8) into (4.5), we conclude

$$\frac{K_0}{c_2^2}r^3 + r = 2g\dot{r}, \tag{4.9}$$

where $r = g'$ and $\dot{r} = \frac{dr}{dg} = \frac{g''}{g'}$. After solving (4.9) we obtain

$$r = \pm\left(c_4^2 g^{-1} - \frac{K_0}{c_2^2}\right)^{-1/2}, \; c_4 \in \mathbb{R}, \; c_4 \neq 0,$$

or

$$z = \pm\int\left(c_4^2 g^{-1} - \frac{K_0}{c_2^2}\right)^{1/2} dg,$$

which proves the statement (ii.2) of Theorem 1.2.

5. Proof of Theorem 1.3

Assume that a factorable surface of type 2 in $\mathbb{I}^3$ fulfills the condition $H + \lambda K = 0$, $\lambda HK \neq 0$, $\lambda \in \mathbb{R}$. Then (3.1) and (4.1) give rise to

$$\left(1 + (f'g)^2\right) f^2 g' g'' + \left(ff'' - 2(f')^2\right) f^2 g (g')^3 + 2\lambda\left(ff''gg'' - (f'g')^2\right) = 0. \tag{5.1}$$

Due to $K \neq 0$, f must be a non-constant function and therefore dividing (5.1) with $(ff')^2$ leads to

$$\left(\frac{1}{(f')^2} + g^2\right) g'g'' + \left(\frac{ff''}{(f')^2} - 2\right) g (g')^3 + 2\lambda\left[\left(\frac{f''}{f(f')^2}\right) gg'' - \frac{(g')^2}{f^2}\right] = 0. \tag{5.2}$$

If $g'' = 0$, namely $g = c_1 z + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$, then (5.2) reduces to the following polynomial equation on z:

$$c_1^2\left(\frac{ff''}{(f')^2} - 2\right) z + c_1 c_2\left(\frac{ff''}{(f')^2} - 2\right) - \frac{2\lambda}{f^2} = 0. \tag{5.3}$$

All coefficients in (5.3) must be zero and this fact yields the contradiction $\lambda = 0$. Then $g'' \neq 0$ and, by dividing (5.2) with the product $g'g''$, we get

$$\frac{1}{(f')^2} + g^2 - 2\frac{g(g')^2}{g''} + \left(\frac{ff''}{(f')^2}\right)\frac{g(g')^2}{g''} + 2\lambda\left[\left(\frac{f''}{f(f')^2}\right)\frac{g}{g'} - \left(\frac{1}{f^2}\right)\frac{g'}{g''}\right] = 0. \tag{5.4}$$

Putting $p = f'$, $\dot{p} = \frac{dp}{df} = \frac{f''}{f'}$ and $r = g'$, $\dot{r} = \frac{dr}{dg} = \frac{g''}{g'}$, (5.4) turns to

$$\frac{1}{p^2} + g^2 - 2\frac{gr}{\dot{r}} + \left(\frac{f\dot{p}}{p}\right)\frac{gr}{\dot{r}} + 2\lambda\left[\left(\frac{\dot{p}}{fp}\right)\frac{g}{r} - \left(\frac{1}{f^2}\right)\frac{1}{\dot{r}}\right] = 0. \tag{5.5}$$

Taking partial derivatives of (5.5) with respect to f and g implies an equation in the following form:

$$A_1(f)\,B_1(g) + 2\lambda\,(A_2(f)\,B_2(g) - A_3(f)\,B_3(g)) = 0, \tag{5.6}$$

where

$$\begin{cases} A_1(f) = \dfrac{d}{df}\left(\dfrac{f\dot{p}}{p}\right), & A_2(f) = \dfrac{d}{df}\left(\dfrac{\dot{p}}{fp}\right), \quad A_3(f) = \dfrac{d}{df}\left(\dfrac{1}{f^2}\right), \\ B_1(g) = \dfrac{d}{dg}\left(\dfrac{gr}{\dot{r}}\right), & B_2(g) = \dfrac{d}{dg}\left(\dfrac{g}{r}\right), \quad B_3(g) = \dfrac{d}{dg}\left(\dfrac{1}{\dot{r}}\right). \end{cases} \tag{5.7}$$

If $B_2 = 0$, i.e., $r = c_1 g$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$, then (5.5) yields the following polynomial equation g:

$$\left(\frac{f\dot{p}}{p} - 1\right)g^2 + \frac{2\lambda}{c_1 f^2}\left(\frac{f\dot{p}}{p} - 1\right) + \frac{1}{p^2} = 0. \tag{5.8}$$

The fact that the coefficient of the term g^2 in (5.8) must vanish leads to the contradiction $\frac{1}{p^2} = 0$ and so we deduce $B_2 \neq 0$. Nevertheless, due to $A_3 \neq 0$, (5.6) can be divided by the product A_3B_2 as follows:

$$\underbrace{\left(\frac{A_1(f)}{A_3(f)}\right)}_{A_4(f)}\underbrace{\left(\frac{B_1(g)}{B_2(g)}\right)}_{B_4(g)} + 2\lambda\left(\underbrace{\frac{A_2(f)}{A_3(f)}}_{A_5(f)} - \underbrace{\frac{B_3(g)}{B_2(g)}}_{B_5(g)}\right) = 0, \tag{5.9}$$

where the terms A_4, A_5, B_4, B_5 must be constant for every f and g. Since $A_4 = c_1$ and $A_5 = c_2$, by (5.7), we derive

$$\frac{f\dot{p}}{p} = \frac{c_1}{f^2} + c_3 \tag{5.10}$$

and

$$\frac{\dot{p}}{fp} = \frac{c_2}{f^2} + c_4, \ c_1, \ldots, c_4 \in \mathbb{R}. \tag{5.11}$$

After equalizing (5.10) and (5.11), we find

$$\frac{\dot{p}}{p} = \frac{c_2}{f}, \ c_2 = c_3, \tag{5.12}$$

where c_2 must be non-vanishing. Otherwise, considering the situation that $\dot{p} = 0$, $p(f) = c_5 \in \mathbb{R}$, $c_5 \neq 0$, into (5.5) gives

$$\frac{1}{c_5^2} + g^2 - 2\frac{gr}{\dot{r}} - \left(\frac{2\lambda}{\dot{r}}\right)\frac{1}{f^2} = 0. \tag{5.13}$$

The coefficient of the term $\frac{1}{f^2}$ in (5.13) cannot vanish and this leads to a contradiction. So, by (5.12), we derive $A_1 = 0$ and (5.9) reduces to

$$c_2 B_2(g) - B_3(g) = 0. \tag{5.14}$$

An integration of (5.14) yields

$$c_2 \frac{g}{r} - \frac{1}{\dot{r}} = c_6, \ c_6 \in \mathbb{R}. \tag{5.15}$$

Substituting (5.12) and (5.15) into (5.5) leads to

$$\frac{1}{p^2} + \frac{2\lambda c_6}{f^2} + g^2 + (c_2 - 2) \frac{gr}{\dot{r}} = 0. \tag{5.16}$$

By revisiting (5.12), we obtain $p = c_7 f^{c_2}$, $c_7 \in \mathbb{R}$, $c_7 \neq 0$ and considering this one into (5.16)

$$\frac{1}{c_7^2 f^{2c_2}} + \frac{2\lambda c_6}{f^2} + g^2 + (c_2 - 2) \frac{gr}{\dot{r}} = 0. \tag{5.17}$$

Due to the fact that f is an independent variable in (5.17), we conclude

$$c_2 = 1 \text{ and } \frac{1}{c_7^2} + 2\lambda c_6 = 0. \tag{5.18}$$

Thereby, (5.17) reduces to

$$g^2 - \frac{gr}{\dot{r}} = 0. \tag{5.19}$$

Comparing (5.19) with (5.15) yields $c_6 = 0$ which contradicts with (5.18).

6. Some examples

We illustrate several examples relating to the factorable surfaces of type 2 in $\mathbb{I}^3$ with K, H constants.

Example 6.1. Consider the factorable surfaces of type 2 in $\mathbb{I}^3$ given by

(1) $\Phi_3 : x(y,z) = y \tan z$, $(y,z) \in \left[0, \frac{\pi}{3}\right]$, (isotropic minimal),
(2) $\Phi_3 : x(y,z) = -\sqrt{z}$, $(y,z) \in [0, 2\pi]$, $(H = -1)$,
(3) $\Phi_3 : x(y,z) = -\frac{y^2}{4z}$, $(y,z) \in [1, 1.4] \times [1, 2\pi]$, (isotropic flat),
(4) $\Phi_3 : x(y,z) = \frac{z}{y}$, $(y,z) \in [1, \pi] \times [1, 2\pi]$, $(K = -1)$.

These surfaces can be respectively drawn by as in Figs. 1-4.

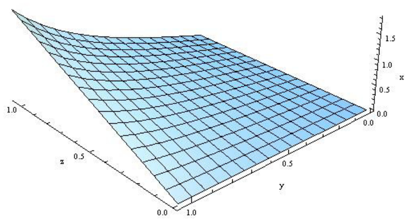

FIGURE 1. An isotropic minimal factorable surface of type 2.

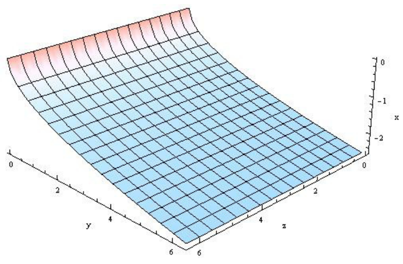

FIGURE 2. A factorable surface of type 2 with $H = -1$.

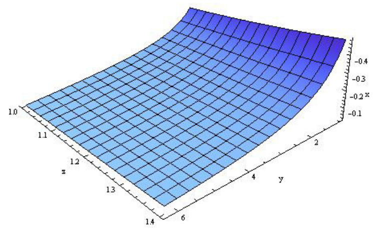

FIGURE 3. An isotropic flat factorable surface of type 2.

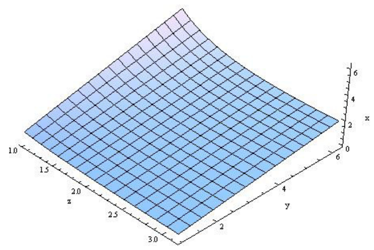

FIGURE 4. A factorable surface of type 2 with $K = -1$.

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