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L^p-ESTIMATES FOR THE $\bar{\partial}$-EQUATION WITH EXACT SUPPORT ON q-CONVEX INTERSECTIONS

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Abstract. We construct bounded linear integral operators that giving solutions to the $\bar{\partial}$-equation in L^p-spaces and with compact supports on a q-convex intersection ($q \geq 1$) with $\mathcal{C}^3$ boundary in Kähler manifolds, and we apply it to obtain a Hartogs-like extension theorems for $\bar{\partial}$-closed forms for some bidegree.

1. Introduction

The problem of solving $\bar{\partial}$ with exact support was initiated for differential forms by Andreotti and Hill in [5] and [6]. Later, it has been widely investigated (cf. [11], [12], [13] and [18]). More precisely, let $\Omega \subset\subset X$ be a relatively compact domain in a complex manifold X of complex dimension n and E be a holomorphic Hermitian vector bundle over X. If Ω is a bounded pseudoconvex domain in $\mathbb{C}^n$, Chen and Shaw proved in [11, Chapter 9] that for any $\bar{\partial}$-closed (r,s)-form f with L^2 (or $\mathcal{C}^\infty$) coefficients in $\mathbb{C}^n$ and compactly supported in $\overline{\Omega}$, there exists a form u in $L^2_{r,s-1}(\mathbb{C}^n)$ (or in $\mathcal{C}^\infty_{r,s-1}(\mathbb{C}^n)$) such that u is compactly supported in $\overline{\Omega}$ and $\bar{\partial}u = f$ in $\mathbb{C}^n$, for $0 \leq r \leq n$ and $1 \leq s \leq n-1$. When Ω is a bounded domain with Lipschitz boundary and satisfies a convexity condition called log δ-pseudoconvex in an n-dimensional Kähler manifold X, Brinkschulte proved in [9] that if f is a $\bar{\partial}$-closed E-valued (r,s)-form with $\mathcal{C}^\infty$-coefficients in X and with compact support in $\overline{\Omega}$, then there exists a $(r,s-1)$-form u with $\mathcal{C}^\infty$-coefficients in X and with compact support in $\overline{\Omega}$ such that $\bar{\partial}u = f$ in X, for $0 \leq r \leq n$ and $1 \leq s \leq n-1$. Moreover, she proved that the range of the $\bar{\partial}$-operator acting on the subspace of $(r,n-1)$-forms of class $\mathcal{C}^\infty$ and compactly supported in $\overline{\Omega}$ is closed.

Analogous results to those of [9] have been obtained by Sambou in [23] for $\mathbb{C}$-valued (r,s)-forms with compact support in $\overline{\Omega}$ when Ω is a completely strictly q-convex domain ($0 \leq q \leq n-1$) with smooth boundary in an n-dimensional complex manifold X for all $1 \leq s \leq q$. This domain is defined in the sense

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of Henkin by $\Omega = \{z \in U \mid \rho(z) < 0\}$ where ρ is a smooth function defined on an open neighborhood U of $\overline{\Omega}$ whose Levi form has at least $q+1$ positive eigenvalues everywhere (see e.g. [16]). In addition, he showed that the range of the $\bar{\partial}$-operator acting on the subspace of $\mathcal{C}^\infty$ $(r, s-1)$-forms with compact support in $\overline{\Omega}$ is closed for $1 \leq s \leq q+1$. Further, he proved that the $\bar{\partial}$-equation is solvable on such domain for extensible currents of bidegree $(n, n-s)$ for all s with $1 \leq n-q \leq s \leq n$. Furthermore, the case for strictly q-concave domains is dealt by the author in [24].

It worth also to mention that solving the $\bar{\partial}$-equation with prescribed support enables one to obtain $\bar{\partial}_b$-closed extensions of forms from boundaries of bounded domains (see e.g. [9], [10], [11] and [14]).

Solving $\bar{\partial}$ with compact support in L^p-spaces (or the so called the weak L^p $\bar{\partial}$-Cauchy problem) is formulated by giving a $\bar{\partial}$-closed form f in $L^p_{r,s}(X, E)$ with compact support in $\overline{\Omega}$, $0 \leq r \leq n$, $1 \leq s \leq n$ and $p \geq 1$, the problem is then to find a form u in $L^p_{r,s-1}(X, E)$ such that

$$
\begin{cases} \bar{\partial} u = f \text{ in the weak sense in } X, \\ \operatorname{supp} u \subset \overline{\Omega}. \end{cases} \tag{1}
$$

This problem was solved by Amar and Mongodi in [4] for $\mathbb{C}$-valued forms on Stein open domains of the form $\mathbb{D}^n \setminus \mathcal{Z}$ in $\mathbb{C}^n$, where $\mathbb{D}^n$ is a polydisc and $\mathcal{Z}$ is the zero locus of some holomorphic function, and by Amar in [3] for weakly p-regular domains in Stein manifolds. In [19], Laurent-Thiébaut gave some general cohomological and geometric conditions on X and Ω under which (1) can be solved. In particular, she solved (1) for $\mathbb{C}$-valued (r, s)-forms on completely q-convex domains in complex manifolds for all $1 \leq s \leq n-q$ (See Corollary 2.23 in [19]).

The plane of the paper is as follows. We first extend some complex analysis results of Amar [3] to E-valued currents on relatively compact domains in Kähler manifolds. Next, via a partition of unity, we globalize the local compact integral homotopy operators constructed in [20] for the $\bar{\partial}$-equation to get global ones for E-valued forms on a $\mathcal{C}^3$ q-convex intersection Ω in an n-dimensional Kähler manifold X. We moreover show that the L^p-$\bar{\partial}$-cohomology group $H^{r,s}_{L^p}(\Omega, E)$ is finite dimensional and the space $\bar{\partial}\big(L^p_{r,s-1}(\Omega, E)\big)$ is closed subspace of $L^p_{r,s}(\Omega, E)$ for all $q \leq s \leq n-1$. By using these integral homotopy formulas, we then solve the $\bar{\partial}$-equation with global L^p-estimates for $\bar{\partial}$-closed E-valued forms of type (n, s) with $1 \leq s \leq \min\{n-q, n-m\}$ and $1 \leq q, m \leq n-1$ (respectively of type $(0, s)$ with $m \leq s \leq n-q$ and $1 \leq q, m \leq n-1$) if E is Nakano semi-positive (respectively Nakano semi-negative) of type m on $\overline{\Omega}$ (see Theorem 3.4 below). This result generalizes some results of [17] to E-valued forms for some bidegree. Combining these results together with some arguments inspired from Laurent-Thiébaut [19], we have the following main theorem.

Theorem 1.1. *Let $\Omega \subset\subset X$ be a C^3 q-convex intersection $(1 \leq q \leq n-1)$ in an n-dimensional Kähler manifold X with $n \geq 2$ and E be a holomorphic Hermitian vector bundle over X such that $X \setminus \Omega$ is connected. Then the following assertions hold true.*

(i) *If E is Nakano semi-positive of type m $(1 \leq m \leq n-1)$ on $\overline{\Omega}$, then for any $\bar{\partial}$-closed form f in $L^{p'}_{0,s}(X, E^*)$ such that f is supported in $\overline{\Omega}$, there exists a form u in $L^{p'}_{0,s-1}(X, E^*)$ supported in $\overline{\Omega}$ such that $\bar{\partial} u = f$ in X, for $1 \leq s \leq \min\{n-q, n-m\}$.*

(ii) *If E is Nakano semi-negative of type m on $\overline{\Omega}$, then for any $\bar{\partial}$-closed form f in $L^{p'}_{n,s}(X, E^*)$ with compact in $\overline{\Omega}$, there exists a form u in $L^{p'}_{n,s-1}(X, E^*)$ with compact support in $\overline{\Omega}$ such that $\bar{\partial} u = f$ in X, for $m \leq s \leq n-q$.*

This result generalizes Corollary 2.23 of Laurent-Thiébaut [19] (which was obtained for $\mathbb{C}$-valued (r,s)-forms, $0 \leq r \leq n$, $1 \leq s \leq n-q$, on completely q-convex domains $(q \geq 1)$ in complex manifolds) to more general q-convex domains and to E^*-valued forms for some bidegree.

Remark 1.2. We note that $\max\{q, m\} = 1$ means that $m = q = 1$. The case $m = 1$ implies that E is Nakano-positive on $\overline{\Omega}$, then there is a Kähler metric on $\overline{\Omega}$, so the Kählerity assumption is automatically satisfied (see e.g. [1]) and $q = 1$ means that Ω is a strictly pseudoconvex domain with piecewise $\mathcal{C}^3$-boundary (see e.g. [21]). Therefore the assertion (i) in Theorem 1.1 for this case still valid if X is replaced by any complex manifold of complex dimension $n \geq 2$.

Actually, the Kählerity property of X and the positivity assumptions on E play a crucial role in the proof of our results, these conditions ensure the L^2-existence theorem for the $\bar{\partial}$-equation in our setting (see [2]).

In addition, by using these results, we prove a Hartogs-like theorem for $\bar{\partial}$-closed forms in L^p-setting with $p \geq 1$ (see Theorem 4.1 below). This result generalizes the result of Theorem 5 in [10] which was obtained for $\mathbb{C}$-valued forms with coefficients in the usual L^2-Sobolev spaces W^k, $k \geq 0$, on bounded pseudoconvex domains with Lipschitz boundaries in Stein manifolds. As a results, we finally solve the $\bar{\partial}$-equation for forms with $W^{1,p}$-coefficients on an annulus type domain between two strictly q-convex domains with smooth boundaries in a Kähler manifold. This result was also proved in [10] for forms with W^k-coefficients on an annulus domain between two pseudoconvex domains in a Stein manifold.

2. Preliminaries

In this section, we fix notations, definitions, and auxiliary results that will be used throughout the paper. Let X be a complex manifold of complex dimension n and E be a holomorphic Hermitian vector bundle of rank N over X. Let $\{U_j\}; j \in I$, be an open covering of X consisting of coordinates neighborhoods

U_j with holomorphic coordinates $z_j = (z_j^1, z_j^2, \dots, z_j^n)$ over which E is trivial, namely, $\pi^{-1}(U_j) = U_j \times \mathbb{C}^N$. The N-dimensional complex vector space $E_z = \pi^{-1}(z)$; $z \in X$, is called the fiber of E over z. Let h be a Hermitian metric along the fibers of E that defined by a system of Hermitian matrix-valued positive C^∞ functions $\{h_j\}$; $h_j = (h_{j\mu\bar{\eta}})$. Denote by $(h_j^{\mu\bar{\eta}})$ the inverse matrix of $(h_{j\mu\bar{\eta}})$. Let $\theta = \{\theta_j\}$; $\theta_j = (\theta_{j\mu}^{\nu})$, $\theta_{j\mu}^{\nu} = \sum_{\alpha=1}^{n}\sum_{\eta=1}^{N} h_j^{\nu\bar{\eta}} \frac{\partial h_{j\mu\bar{\eta}}}{\partial z_j^\alpha} dz_j^\alpha$ and $\Theta = \{\Theta_j\}$; $\Theta_j = (\Theta_{j\mu}^{\nu})$, $\Theta_{j\mu}^{\nu} = \sqrt{-1}\bar{\partial}\partial \log h_j = \sqrt{-1}\sum_{\alpha,\beta=1}^{n} \Theta_{j\mu\alpha\bar{\beta}}^{\nu} dz_j^\alpha \wedge d\bar{z}_j^\beta$, where $\Theta_{j\mu\alpha\bar{\beta}}^{\nu} = -\frac{\partial}{\partial \bar{z}_j^\beta}(h_j^{\nu\bar{\eta}} \frac{\partial h_{j\mu\bar{\eta}}}{\partial z_j^\alpha})$, $1 \le \mu,\, \nu \le N$, be the connection and curvature forms associated to the metric h. The curvature matrix is given by $H_{j\bar{\eta}\bar{\beta},\nu\alpha} = \sum_{\mu=1}^{N} h_{j\mu\bar{\eta}} \Theta_{j\nu\alpha\bar{\beta}}^{\mu}$.

Definition 2.1 (see e.g. [2])**.** Let E be a holomorphic Hermitian vector bundle of rank N over a complex manifold X of complex dimension n.

(a) E is said to be Nakano m-positive (respectively m-negative), at $x \in U_j$, if there exists an $(n-m+1)$-dimensional subspace S_x of the holomorphic tangent $T_x(X)$ such that the Hermitian form

$$\sum H_{j\bar{\eta}\bar{\beta},\nu\alpha}(x)\zeta_\alpha^\nu \bar{\zeta}_\beta^\eta \tag{2}$$

is positive (respectively negative) definite for any $\zeta = (\zeta_\alpha^\nu) \in S_x \otimes E_x; \zeta \neq 0$.

(b) E is said to be Nakano semi-positive (respectively semi-negative), at $x \in U_j$, if the Hermitian form (2) is positive (respectively negative) semi-definite for any $\zeta = (\zeta_\alpha^\nu) \in T_x(X) \otimes E_x$.

(c) E is said to be Nakano semi-positive (respectively semi-negative) of type m if E is both Nakano semi-positive and Nakano m-positive (respectively Nakano semi-negative and Nakano m-negative) at x.

For all $0 \le r, s \le n$, we denote by $\Lambda^{r,s}(X,E)$ the space of E-valued forms of bidegree (r,s) and of class C^∞ on X with the topology of uniform convergence of forms and all their derivatives on compact subsets of X and by $\mathcal{D}^{r,s}(X,E)$ the subspace of $\Lambda^{r,s}(X,E)$ consisting of forms with compact supports in X. The associated cohomologies groups are denoted by $H^{r,s}(X,E)$ and $H_c^{r,s}(X,E)$ respectively. Let K be a compact subset of X and $\mathcal{D}_K^{r,s}(X,E)$ the closed subspace of $\Lambda^{r,s}(X,E)$ of forms with supports in K endowed with the induced topology. Let $\{K_i\}_{i\in\mathbb{N}}$ be an increasing sequence of compact subsets of X such that $K_i \subset K_{i+1}^\circ$ and $\bigcup_{i\in\mathbb{N}} K_i = X$. Then $\mathcal{D}^{r,s}(X,E) = \bigcup_{i=1}^{\infty} \mathcal{D}_{K_i}^{r,s}(X,E)$. We put on $\mathcal{D}^{r,s}(X,E)$ the strict inductive limit topology defined by the spaces $\mathcal{D}_{K_i}^{r,s}(X,E)$. If $\varphi \in \Lambda^{r,s}(X,E)$, then $\bar{\partial}\varphi = \{\bar{\partial}\varphi_j\}$, where $\bar{\partial}\varphi_j = (\bar{\partial}\varphi_j^1, \bar{\partial}\varphi_j^2, \dots, \bar{\partial}\varphi_j^N)$. Let ds^2 be a Kähler metric on X defined by

$$ds^2 = \sum_{\alpha,\beta=1}^{n} g_{j\alpha\bar{\beta}}\, dz_j^\alpha d\bar{z}_j^\beta,$$

where $g_{j\alpha\bar\beta}$ is a C^∞-section of $T^\star(X)\otimes\bar T^\star(X)$ on U_j. For $\varphi,\psi\in\Lambda^{r,s}(M,E)$, we define a local inner product, at $z\in U_j$, by

$$\sum_{\nu,\mu=1}^{N} h_{j\nu\bar\mu}\varphi_j^\nu(z)\wedge\star\overline{\psi_j^\mu(z)} = a(\varphi(z),\psi(z))dv,$$

where the Hodge star operator $\star$ and the volume element dv are defined by ds^2 and $a(\varphi,\psi)$ is a function on X independent of j. For $\varphi,\psi\in\mathcal{D}^{r,s}(X,E)$, a global inner product is then defined by $(\varphi,\psi)=\int_X a(\varphi,\psi)dv$. Let $L^2_{r,s}(X,E)$ be the Hilbert space obtained by completing the space $\mathcal{D}^{r,s}(X,E)$ under the norm $\|\varphi\|^2=(\varphi,\varphi)$.

We now extend the complex analysis results obtained in [3] to domains in complex manifolds. Let $\Omega\subset\subset X$ be a bounded domain with smooth boundary in a Kähler manifold X of complex dimension n, E be a holomorphic Hermitian vector bundle of rank N over X, and E^* be the dual vector bundle of E. The space of $E^\star$-valued currents of bidegree $(n-r,n-s)$ (or bidimension (r,s)) denoted by $\mathcal{D}^{n-r,n-s}_{\text{cur}}(\Omega,E^\star)$ is the topological dual to the space $\mathcal{D}^{r,s}(\Omega,E)$. The $\bar\partial$-operator is defined from $\mathcal{D}^{n-r,n-s}_{\text{cur}}(\Omega,E^\star)$ into $\mathcal{D}^{n-r,n-s+1}_{\text{cur}}(\Omega,E^\star)$ as the transpose of the original $\bar\partial$-operator from $\mathcal{D}^{r,s}(\Omega,E)$ into $\mathcal{D}^{r,s+1}(\Omega,E)$. The topological dual to the space $\Lambda^{r,s}(\Omega,E)$ denoted by $\Lambda^{n-r,n-s}_{c,\text{cur}}(\Omega,E^\star)$ is the space of $E^\star$-valued currents of of bidegree $(n-r,n-s)$ with compact supports in Ω. The restriction of the $\bar\partial$-operator to $\Lambda^{n-r,n-s}_{c,\text{cur}}(\Omega,E^\star)$ gives unbounded operator $\bar\partial:\Lambda^{n-r,n-s}_{c,\text{cur}}(\Omega,E^\star)\to\Lambda^{n-r,n-s+1}_{c,\text{cur}}(\Omega,E^\star)$. For further details on duality for complexes of topological vector spaces, we refer to [7] and the references therein.

Let $A_{r,s}(\Omega,E)$ be a topological space of E-valued (r,s)-forms on Ω and $A'_{r,s}(\Omega,E)$ be its dual. Assume that the injections

$$\mathcal{D}^{r,s}(\Omega,E)\hookrightarrow A_{r,s}(\Omega,E)\hookrightarrow\mathcal{D}^{n-r,n-s}_{\text{cur}}(\Omega,E)$$

being continuous. Then $B_{n-r,n-s}(\Omega,E^\star)=A'_{r,s}(\Omega,E)$ still a space of currents and asking that the duality pairing $\langle\phi,\psi\rangle=\int_\Omega\phi\wedge\psi$ be $\bar\partial$ compatible with currents, i.e., $\forall\ \phi\in\mathcal{D}^{r,s}_{\text{cur}}(\Omega,E)$ and $\psi\in\mathcal{D}^{n-r,n-s-1}(\Omega,E^\star)$,

$$\langle\bar\partial\phi,\psi\rangle=(-1)^{r+s+1}\langle\phi,\bar\partial\psi\rangle.$$

For $0\le r\le n$, $1\le s\le n$, we recall that the equation $\bar\partial g=f$ is solvable in $A_{r,s}(\Omega,E)$ if for any $\bar\partial$-closed form f in $A_{r,s}(\Omega,E)$ there exists a form g in $A_{r,s-1}(\Omega,E)$ such that $\bar\partial g=f$ in Ω. Suppose now that the $\bar\partial$-equation is solvable in $A_{r,s}(\Omega,E)$ and $A_{r,s+1}(\Omega,E)$ for all $1\le s\le n-1$. Let u be a $\bar\partial$-closed form in $B_{n-r,n-s}(\Omega,E^\star)$ and consider the form

$$\mathcal{L}_u(\eta)=\langle g,u\rangle\quad\forall\ \eta\in A_{r,s+1}(\Omega,E),\ \bar\partial\eta=0,\tag{3}$$

with $\bar\partial g=\eta$, which exists by hypothesis. Denote by $\mathcal{H}_r(\Omega,E)$, the space of E-valued $\bar\partial$-closed $(r,0)$-forms on Ω. Then we have:

Lemma 2.2. *The form $\mathcal{L}_u$ defined by* (3)*, with $\langle g, u\rangle = 0$ for $s = 0$ and $g \in \mathcal{H}_r(\Omega, E)$, is well defined and linear.*

Proof. In order to prove that the form $\mathcal{L}_u$ is well defined, we have to show that $\langle g, u\rangle = \langle h, u\rangle$ if $g, h \in A_{r,s}(\Omega, E)$ with $\bar\partial g = \bar\partial h$.

We consider first the case when $1 \le s \le n$. Let $g, h \in A_{r,s}(\Omega, E)$ be such that $\bar\partial g = \bar\partial h = \eta$, then the difference $g - h$ is a $\bar\partial$-closed form in $A_{r,s}(\Omega, E)$. By hypothesis, there exists $\phi \in A_{r,s-1}(\Omega, E)$ such that $\bar\partial\phi = g - h$. Hence,

$$\langle g - h, u\rangle = \langle \bar\partial\phi, u\rangle = (-1)^{r+s}\langle \phi, \bar\partial u\rangle = 0.$$

Thus $\mathcal{L}_u$ is also well defined for all $s \ge 1$.

For $s = 0$, we have $\bar\partial u = 0$ (because u is an $(n - r, n)$-form). Again, let $g, h \in A_{r,0}(\Omega, E)$ with $\bar\partial g = \bar\partial h$, hence $g - h$ is a $\bar\partial$-closed $(r, 0)$-form, i.e., $g - h \in \mathcal{H}_r(\Omega, E)$. Since, by hypothesis, $u \perp \mathcal{H}_r(\Omega, E)$, we have $\langle g - h, u\rangle = 0$. Then $\mathcal{L}_u$ is also well defined in this case.

Next, we show that the form $\mathcal{L}_u$ is linear, let η_1 and η_2 be in $A_{r,s+1}(\Omega, E)$ such that $\bar\partial\eta_1 = \bar\partial\eta_2 = 0$ and put $\eta = \eta_1 + \eta_2$, then $\bar\partial\eta = 0$ and so there are g , g_1 and g_2 in $A_{r,s}(\Omega, E)$ such that $\bar\partial g = \eta$, $\bar\partial g_1 = \eta_1$ and $\bar\partial g_2 = \eta_2$. Thus $\bar\partial(g - g_1 - g_2) = 0$ and hence there is a form h in $A_{r,s-1}(\Omega, E)$ such that $g = g_1 + g_2 + \bar\partial h$, therefore

$$\mathcal{L}_u(\eta) = \langle g, u\rangle = \langle g_1 + g_2 + \bar\partial h, u\rangle = \langle g_1, u\rangle + \langle g_2, u\rangle + \langle \bar\partial h, u\rangle = \mathcal{L}_u(\eta_1) + \mathcal{L}_u(\eta_2),$$

where $\langle \bar\partial h, u\rangle = (-1)^{r+s}\langle h, \bar\partial u\rangle = 0$. Similarly for $\lambda\eta$; $\lambda \in \mathbb{C}$. The proof is complete. □

Following [3], the equation $\bar\partial u = \eta$ is continuously solvable in $A_{r,s+1}(\Omega, E)$ if it is solvable in $A_{r,s}(\Omega, E)$ and $A_{r,s+1}(\Omega, E)$ and, moreover, if the form $\mathcal{L}_u(\eta)$ is continuously linear on the subspace of all $\bar\partial$-closed forms η in $A_{r,s+1}(\Omega, E)$.

Theorem 2.3. *If the $\bar\partial$-equation is continuously solvable in $A_{r,s+1}(\Omega, E)$, then it is solvable in $B_{n-r,n-s}(\Omega, E^\star)$, that is, for any $f \in B_{n-r,n-s}(\Omega, E^\star)$ with $\bar\partial f = 0$ if $1 \le s \le n - 1$ and $\langle f, g\rangle = 0$ for all $g \in \mathcal{H}_r(\Omega, E)$ if $s = 0$, there exists $\omega \in B_{n-r,n-s-1}(\Omega, E^\star)$ such that $\bar\partial\omega = f$.*

Proof. Let $f \in B_{n-r,n-s}(\Omega, E^\star)$ with $\bar\partial f = 0$ for $1 \le s \le n - 1$ and $\langle f, g\rangle = 0$ for $s = 0$ and $g \in \mathcal{H}_r(\Omega, E)$. Consider the form $\mathcal{L}_f$ on the subspace of all η in $A_{r,s+1}(\Omega, E)$ with $\bar\partial\eta = 0$, which exists by hypothesis on $A_{r,s+1}(\Omega, E)$ and is continuous by assumption.

By the Hahn-Banach extension theorem, it can be extended to the whole $A_{r,s+1}(\Omega, E)$. By duality, the extended form can be represented by a current $\omega \in B_{n-r,n-s-1}(\Omega, E^\star)$. Then we have

$$\langle \eta, \omega\rangle = \langle \bar\partial g, \omega\rangle = \mathcal{L}_f(g) = \langle g, f\rangle.$$

But $\langle \bar\partial g, \omega\rangle = (-1)^{r+s+1}\langle g, \bar\partial\omega\rangle$ and hence $\langle g, f\rangle = (-1)^{r+s+1}\langle g, \bar\partial\omega\rangle$ for all $g \in \mathcal{D}^{r,s}(\Omega, E)$. This means that $\bar\partial\omega = f$. The proof is complete. □

3. Solving $\bar\partial$ with exact support in L^p

Let $\Omega \subset\subset X$ be a relatively compact domain with smooth boundary in a Kähler manifold X of complex dimension n and E be holomorphic Hermitian vector bundle of rank N over X. Let $\{U_{j_\nu}\}$ be a finite elements of the covering $\{U_j\}$ such that $\cup_\nu U_{j_\nu}$ cover $\overline{\Omega}$ and $\{\chi_{j_\nu}\}$ be a partition of unity subordinate to $\{U_{j_\nu}\}$. Then every E-valued (r,s)-form f can be identified with a system $\{f_{j_\nu}\}$ of vectors $f_{j_\nu} = (f^1_{j_\nu}, f^2_{j_\nu}, \ldots, f^N_{j_\nu})$ of differential forms $f^\mu_{j_\nu}$ on $U_{j_\nu} \cap \Omega$. For $1 \le p \le \infty$, we denote by $L^p_{r,s}(\Omega, E)$ the Banach space of E-valued forms f of bidegree (r,s) on Ω for which $\|f\|_{L^p_{r,s}(\Omega,E)} < \infty$. The norm $\|f\|_{L^p_{r,s}(\Omega,E)}$ is defined by means of a partition of unity in the following way: On each U_{j_ν}, we can choose an orthonormal basis $\omega^1, \ldots, \omega^N$ for the fibers E_z for every $z \in U_{j_\nu}$. In such a basis, the $L^p(\Omega,E)$-norm is defined by $\|f\|_{L^p_{r,s}(\Omega,E)} = \sum_{\mu=1}^N \sum_{j_\nu} \|\chi_{j_\nu} f^\mu_{j_\nu}\|_{L^p(U_{j_\nu}\cap\Omega)}$, where $\|\chi_{j_\nu} f^\mu_{j\nu}\|_{L^\infty_{r,s}(U_{j_\nu}\cap\Omega)} = \operatorname{ess\,sup}_{U_{j_\nu}\cap\Omega} |\chi_{j_\nu} f^\mu|$. This norm depends on the choice of the coverings and their local coordinates, however, as $\overline{\Omega}$ is compact, different choices give equivalent norms. The associated $\bar\partial$-cohomology group is denoted by $H^{r,s}_{L^p}(\Omega, E)$.

For $p \ge 1$, we denote by $L^{p,\mathrm{loc}}_{r,s}(\Omega,E)$ the subspace of $\mathcal{D}^{r,s}_{\mathrm{cur}}(\Omega,E)$ consisting of E-valued (r,s)-currents with coefficients in $L^{p,\mathrm{loc}}(\Omega)$ and endowed with the topology of L^p-convergence on compact subsets of Ω. Taking the restriction to $L^{p,\mathrm{loc}}_{r,s}(\Omega)$ of the $\bar\partial$-operator in the sense of distributions we get an unbounded operator whose domain of definition is the set of forms f with $L^{p,\mathrm{loc}}$-coefficients such that $\bar\partial f$ has also $L^{p,\mathrm{loc}}$-coefficients, moreover, since $\bar\partial\circ\bar\partial = 0$, we get a complex of unbounded operators $\big(L^{p,\mathrm{loc}}_{r,s}(\Omega,E), \bar\partial\big)$. The associated $\bar\partial$-cohomology group is denoted by $H^{r,s}_{L^{p,\mathrm{loc}}}(\Omega,E)$. By $L^{p,c}_{r,s}(\Omega,E)$, we denote the subspace of $L^{p,\mathrm{loc}}_{r,s}(\Omega,E)$ consisting of forms with compact supports in Ω. We also consider the subcomplex $\big(L^{p,c}_{r,s}(\Omega,E), \bar\partial\big)$ of the previous one consisting of forms with compact supports. For all $k \ge 1$ and $1 \le p \le \infty$, the L^p-Sobolev spaces $W^{k,p}_{r,s}(\Omega,E)$ and their norms are defined in similar manner. Finally, for $1 < p < \infty$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, $L^{p'}_{n-r,n-s}(\Omega, E^\star)$ is the dual space of $L^p_{r,s}(\Omega,E)$ with respect to the duality pairing $\langle f, g\rangle = \int_\Omega f \wedge g$. Using a partition of unity, as in [3], we have the following duality theorem.

Theorem 3.1. *For any p with $1 \le p < \infty$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, $L^{p',c}_{n-r,n-s}(\Omega, E^\star)$ is the dual space of $L^{p,\mathrm{loc}}_{r,s}(\Omega,E)$ with respect to the duality pairing $\langle f, g\rangle = \int_\Omega f \wedge g$.*

For $0 \le r \le n$ and $1 \le s \le n-1$, let $\bar\partial_c : L^p_{r,s}(\Omega,E) \to L^p_{r,s+1}(\Omega,E)$ be the minimal closed extension of $\bar\partial|_{\mathcal{D}^{r,s}(\Omega,E)}$. The domain of $\bar\partial_c$ denoted $\mathrm{Dom}(\bar\partial_c)$ consists of those forms f in $L^p_{r,s}(\Omega,E)$ for which there exist a sequence $\{f_i\}$ of elements f_i in $\mathcal{D}^{r,s}(\Omega,E)$ and a form g in $L^p_{r,s+1}(\Omega,E)$ such that $f_i \to f$ and $\bar\partial_c f_i \to g$ in the $L^p(\Omega,E)$-norm. We then set $\bar\partial_c f = g$.

We consider also $\bar\partial_s : L^p_{r,s}(\Omega, E) \to L^p_{r,s+1}(\Omega, E)$ the minimal closed extension of $\bar\partial|_{\Lambda^{r,s}(X,E)|_\Omega}$, it is also a closed operator and $\mathrm{Dom}(\bar\partial_s)$ consists of those forms f in $L^p_{r,s}(\Omega, E)$ for which there exist a sequence $\{f_i\}$ of elements f_i in $\Lambda^{r,s}(\Omega, E)$ and a form $g \in L^p_{r,s+1}(\Omega, E)$ such that $f_i \to f$ and $\bar\partial_s f_i \to g$ in the $L^p(\Omega, E)$-norm. We then set $\bar\partial_s f = g$.

The operator $\bar\partial$ extends to $L^p_{r,s}(\Omega, E)$, in the sense of distributions, so we can consider the operators $\bar\partial_{\tilde c} : L^p_{r,s}(\Omega, E) \to L^p_{r,s+1}(\Omega, E)$ and $\bar\partial : L^p_{r,s}(\Omega, E) \to L^p_{r,s+1}(\Omega, E)$ which coincides with the original $\bar\partial$ such that

$$\mathrm{Dom}(\bar\partial_{\tilde c}) = \{f \in L^p_{r,s}(X, E), \mathrm{supp}\, f \subset \overline{\Omega},\ \bar\partial f \in L^p_{r,s+1}(X, E)\}, \quad \text{and}$$
$$\mathrm{Dom}(\bar\partial) = \{f \in L^p_{r,s}(\Omega, E),\ \bar\partial f \in L^p_{r,s+1}(\Omega, E)\}.$$

We refer to [15, Chapter 4] for more details on maximal (minimal) closed extensions of differential operators.

Definition 3.2 (see e.g. [17])**.** A bounded domain D in a complex manifold X of complex dimension n is called a $\mathcal{C}^d$ $(d \geq 2)$ q-convex intersection $(q \geq 1)$ in the sense of Grauert if there exist a bounded neighborhood U of $\overline{D}$ and a finite number of real-valued $\mathcal{C}^d$ functions $\rho_1(z), \ldots, \rho_b(z)$, where $n \geq b + 2$, defined on U such that

$$D = \{z \in U \mid \rho_1(z) < 0, \ldots, \rho_b(z) < 0\}$$

and the following conditions are fulfilled:

(1) For $1 \leq i_1 < i_2 < \cdots < i_\ell \leq b$ the 1-forms $d\rho_{i_1}, \ldots, d\rho_{i_\ell}$ are $\mathbb{R}$-linearly independent on the set $\bigcap_{j=1}^{\ell} \{\rho_{i_j}(z) \leq 0\}$.

(2) For $1 \leq i_1 < i_2 < \cdots < i_\ell \leq b$ and every $z \in \bigcap_{j=1}^{\ell} \{\rho_{i_j}(z) \leq 0\}$, if we set $I = (i_1, \ldots, i_\ell)$, there exists a linear subspace T^I_z of X of complex dimension at least $n - q + 1$ such that for $i \in I$ the Levi forms L_{ρ_i} restricted on T^I_z are positive definite.

Theorem 3.3. *Let $\Omega \subset\subset X$ be a $\mathcal{C}^3$ q-convex intersection $(q \geq 1)$ in an n-dimensional complex manifold X and E be a holomorphic Hermitian vector bundle of rank N over X. Then for any form f in $L^p_{r,s}(\Omega, E) \cap \mathrm{Ker}(\bar\partial)$, $1 \leq p \leq \infty$, $q \leq s \leq n-1$, there exist bounded linear operators $\widetilde{T}_s$ from $L^p_{r,s}(\Omega, E)$ into $L^p_{r,s-1}(\Omega, E)$ and compact linear operators $\widetilde{K}_s$ from $L^p_{r,s}(\Omega, E)$ into itself such that*

$$f = \bar\partial \widetilde{T}_s f + \widetilde{K}_s f \quad \text{in } \Omega. \tag{4}$$

Furthermore, for all s with $q \leq s \leq n-1$, the L^p-$\bar\partial$-cohomology group $H^{r,s}_{L^p}(\Omega, E)$ is finite dimensional and the space $\bar\partial\big(L^p_{r,s-1}(\Omega, E)\big)$ is closed subspace of $L^p_{r,s}(\Omega, E)$.

Proof. Let $\Omega \subset\subset U \subset\subset \mathbb{C}^n$ be a C^3 q-convex intersection with the defining functions $\{\rho_i\}_{i=1}^b$ and U as in Definition 3.2. Set

$$\Omega_I = \{z \in U \mid \rho_i(z) < 0,\ i \in I\} \quad \text{and} \quad S_I = \{z \in U \mid \rho_i(z) = 0,\ i \in I\}.$$

For each $\xi \in S_I$ there exists a smoothly bounded strictly pseudoconvex domain D^* defined by $D^* = \{z \in U; \rho_*(z) < 0\}$ such that ∂D^* intersects real transversely $\{z \in U; \rho_{i_1}(z) < 0\}, \dots, \{z \in U; \rho_{i_\ell}(z) < 0\}$ and $\xi \in D^*$.

Denote by I_* the multi-index $(i_1 \dots, i_\ell, *)$, where $I = (i_1 \dots, i_\ell)$, $1 \le i_1 < \dots < i_\ell < b$, and define

$$\Omega_{I_*} = \{z \in U;\ \rho_j(z) < 0,\ j \in I_*\}.$$

The domain Ω_{I_*} is still q-convex and is called a local q-convex intersection. Since Ω_I is q-convex intersection, for every $z \in \Omega_I$ there is then an $(n-q+1)$-linear vector subspace T_z^I of $\mathbb{C}^n$ such that the Levi forms L_{ρ_i} are positive definite on T_z^I for all $i \in I$. Therefore, by means of generalized Berndtsson-Andersson formula with multiple weights, Lan Ma and Vassiliadou proved in [20] that if $f \in \mathcal{C}^1_{r,s}(\overline{\Omega_{I_*}})$ with $\bar\partial f \in \mathcal{C}^1_{r,s+1}(\overline{\Omega_{I_*}})$, $0 \le r \le n$, $q \le s \le n-1$, there exist local kernels $K_s^\varepsilon(\zeta, z)$ $(\varepsilon > 0)$ of bidegree (r,s) in z and of bidegree $(n-r, n-s-1)$ in ζ such that the map

$$f \longmapsto \int_{\zeta \in \Omega_{I_*}} f(\zeta) \wedge K^\varepsilon_{s-1}(\zeta, z)$$

defines a bounded linear operator $T_s : \mathcal{C}^1_{r,s}(\overline{\Omega_{I_*}}) \to \mathcal{C}^1_{r,s-1}(\overline{\Omega_{I_*}})$, the map

$$f \longmapsto \int_{\zeta \in \partial\Omega_{I_*}} f(\zeta) \wedge K^\varepsilon_{s}(\zeta, z)$$

defines a compact linear operator $K_s : \mathcal{C}^1_{r,s}(\overline{\Omega_{I_*}}) \to \mathcal{C}^1_{r,s}(\overline{\Omega_{I_*}})$ and the homotopy formula

$$f = \bar\partial T_s f + T_{s+1}\bar\partial f + K_s f$$

holds on Ω_{I_*} for every f in $\mathcal{C}^1_{r,s}(\overline{\Omega_{I_*}})$ with $\bar\partial f$ in $\mathcal{C}^1_{r,s+1}(\overline{\Omega_{I_*}})$.

Now we extend these operators to E-valued (r,s)-forms defined on q-convex intersections in complex manifolds. Let $\Omega \subset\subset X$ be a $\mathcal{C}^3$ q-convex intersection $(q \ge 1)$ in an n-dimensional complex manifold X and E be a holomorphic Hermitian vector bundle over X. Cover $\overline{\Omega}$ by a finite number of open sets $V_1, V_2, \dots, V_m$ such that $\overline{\Omega} \subseteq V_1 \cup \dots \cup V_m$ and for every $1 \le j \le m$ the intersection $V_j \cap \Omega$ is a local q-convex intersection, moreover, we may assume that E is trivial over some coordinates neighborhoods $z_j = (z_j^1, z_j^2, \dots, z_j^n)$ of each $\overline{V_j \cap \Omega}$. Then, for every $f \in \mathcal{C}^1_{r,s}(\overline{\Omega \cap V_j}, E)$, $q \le s \le n-1$, with $\bar\partial f = 0$, there exist bounded linear operators

$$T_s^j : \mathcal{C}^1_{r,s}(\overline{\Omega \cap V_j}, E) \longrightarrow \mathcal{C}^1_{r,s-1}(\overline{\Omega \cap V_j}, E)$$

and compact operators

$$K_s^j : \mathcal{C}^1_{r,s}(\overline{\Omega \cap V_j}) \to \mathcal{C}^1_{r,s}(\overline{\Omega \cap V_j})$$

such that the homotopy formulas

$$f = \bar{\partial} T_s^j f + K_s^j f$$

hold on $\Omega \cap V_j$ for all $f \in \mathcal{C}^1_{r,s}(\overline{\Omega \cap V_j}) \cap \mathrm{Ker}(\bar{\partial})$.

Choose a $\mathcal{C}^\infty$ partition of unity $\{\chi_j\}$ subordinate to the covering $\{V_j\}$ and define

$$\widetilde{T}_s f = \sum_{j=1}^m \chi_j T_s^j f \qquad \text{and} \qquad \widetilde{K}_s f = \sum_{j=1}^m \chi_j K_s^j f$$

for $f \in \mathcal{C}^1_{r,s}(\overline{\Omega}, E) \cap \mathrm{Ker}(\bar{\partial})$, $q \le s \le n-1$.

We then have

$$f = \bar{\partial} \widetilde{T}_s f + \widetilde{K}_s f, \quad f \in \mathcal{C}^1_{r,s}(\overline{\Omega}, E) \cap \mathrm{Ker}(\bar{\partial}), \quad q \le s \le n-1. \tag{5}$$

By using the L^p-estimates proved in [20] and the mollification method of Friedrichs (see e.g. [11]), the formula (5) extends to forms in $L^p_{r,s}(\Omega, E) \cap \mathrm{Ker}(\bar{\partial})$ for all $1 \le p \le \infty$ and $q \le s \le n-1$. This proves (4). As the operators $\widetilde{K}_s$ are compact operators from $L^p_{r,s}(\Omega, E)$ into itself, the operator $Id - \widetilde{K}_s$ is a Fredholm operator maps $L^p_{r,s}(\Omega, E) \cap \mathrm{Ker}(\bar{\partial})$ into itself whose range is contained in $\bar{\partial}(L^p_{r,s-1}(\Omega, E))$ by the formula (4) and hence the dimension of the cohomology group $H^{r,s}_{L^p}(\Omega, E)$ is smaller than the codimension of the range of $Id - \widetilde{K}_s$ which is finite. Therefore, the open mapping theorem implies that $\bar{\partial}(L^p_{r,s-1}(\Omega, E))$ is a closed subspace of $L^p_{r,s}(\Omega, E)$. The proof is complete. □

By using a partition of unity and the L^p-estimates obtained in [2, Theorem 0.1], we have the following L^p-existence theorem.

Theorem 3.4. *Let $\Omega \subset\subset X$ be a C^3 q-convex intersection $(q \ge 1)$ in a Kähler manifold X of complex dimension n and E be a holomorphic Hermitian vector bundle of rank N over X. Then*

(i) *If E is Nakano semi-positive of type m on $\overline{\Omega}$, then for any $\bar{\partial}$-closed form f in $L^1_{n,s}(\Omega, E)$ there exists a form g in $L^1_{n,s-1}(\Omega, E)$ such that $\bar{\partial} g = f$ for all s so that* $\max\{q, m\} \le s \le n-1$. *Moreover, if f is in $L^p_{n,s}(\Omega, E)$, $1 \le p \le \infty$, then g is in $L^p_{n,s-1}(\Omega, E)$ and there is a constant $C_s > 0$ (independent of f and p) such that*

$$\|g\|_{L^p_{n,s-1}(\Omega,E)} \le C_s \|f\|_{L^p_{n,s}(\Omega,E)}, \quad 1 \le p \le \infty.$$

(ii) *If E is Nakano semi-negative of type m on $\overline{\Omega}$, the assertion* (i) *holds for E-valued $(0, s)$-forms with L^p-coefficients, for all $q \le s \le n-m$, $1 \le q, m \le n-1$ and $n \ge 2$.*

Since the q-convexity is stable with respect to small $\mathcal{C}^3$ perturbations, we may assume that the defining functions ρ_i of Ω are Morse functions (i.e., all critical points of ρ_i are non-degenerate and if ζ_1 and ζ_2 are two different critical points of ρ_i, then $\rho_i(\zeta_1) = \rho_i(\zeta_2)$). Then we can approximate Ω from inside by a sequence of $\mathcal{C}^3$ q-convex intersections $\{\Omega_k\}$ such that $\Omega_k \subset\subset \Omega_{k+1} \subset\subset \Omega$ and $\Omega = \bigcup_k \Omega_k$. This approach is known as Grauert's bumping method where each Ω_{k+1} is obtained from Ω_k by an appropriate small bump (see e.g. [16] for the q-convex (q-concave) domains or [22] for q-convex intersections). Then, as in [19, Theorem 2.10], the next theorem follows immediately from Theorems 3.3 and 3.4.

Theorem 3.5. *Let Ω, X and E be given as in Theorem 3.4. Then we have the following assertions.*

(i) *If E is Nakano semi-positive of type m on $\overline{\Omega}$, then for all s so that $\max\{q,m\} \le s \le n-1$, we have*

$$H^{n,s}_{L^p}(\Omega, E) \sim H^{n,s}_{L^{p,\mathrm{loc}}}(\Omega, E).$$

(ii) *If E is Nakano semi-negative of type m on $\overline{\Omega}$, then for all s so that $q \le s \le n-m$, $1 \le q, m \le n-1$, $n \ge 2$, we have*

$$H^{0,s}_{L^p}(\Omega, E) \sim H^{0,s}_{L^{p,\mathrm{loc}}}(\Omega, E).$$

We note that since every smooth domain in the complex plane is strictly pseudoconvex, the assertions (i) in Theorems 3.4 and 3.5 are still valid when $n = 1$ and E is the trivial line bundle with the flat metric with $q = s = m = 1$.

Following [19], we recall that for any two real numbers p and p' so that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$ and any $r \in \mathbb{N}$ with $0 \le r \le n$, the complexes $(L^p_{r,\bullet}(\Omega, E), \bar\partial)$ and $(L^{p'}_{n-r,\bullet}(\Omega, E^*), \bar\partial_c)$ are dual complexes. Moreover, we recall the following abstract result on duality.

Proposition 3.6. *Let $(E^\bullet, d)$ and $(E'_\bullet, d')$ be two dual complexes of reflexive Banach spaces with densely defined unbounded operators. Assume that $H_s(E'_\bullet)$ is Hausdorff and $H_{s+1}(E'_\bullet) = 0$, then $H^{s+1}(E^\bullet) = 0$.*

Let $p, p' > 1$ be real numbers with $\frac{1}{p} + \frac{1}{p'} = 1$. It follows from Theorem 3.3 and Theorem 3.4(i) that the cohomology group $H^{n,s}_{L^{p'}}(\Omega, E)$ is Hausdorff for all s such that $q \le s \le n-1$ and $H^{n,s}_{L^{p'}}(\Omega, E) = 0$ for all s such that $\max\{q,m\} \le s \le n-1$. Moreover, by Theorem 3.3 and Theorem 3.4(ii), we get that $H^{0,s}_{L^{p'}}(\Omega, E)$ is Hausdorff for all $q \le s \le n-1$ and $H^{0,s}_{L^{p'}}(\Omega, E) = 0$ for all $q \le s \le n-m$, where $1 \le q, m \le n-1$ and $n \ge 2$.

End proof of Theorem 1.1. On applying Proposition 3.6 to the complex $(E^\bullet, d)$ with, for fixed r so that $0 \le r \le n$, $E^s = L^{p'}_{r,s}(\Omega, E^*)$ if $0 \le s \le n$ and $E^s = \{0\}$ if $s < 0$ or $s > n$, and $d = \bar\partial_{\tilde c}$, we deduce that the cohomological hypotheses of Theorem 2.20 in [19] are satisfied in the current situations. This implies L^p-solvability for the $\bar\partial$-equation with exact support on a q-convex intersection in a complex manifold, and this completes the proof of Theorem 1.1. □

4. $\bar\partial$-closed extensions of forms in L^p

As an application of Theorem 1.1, we obtain a Hartogs-like extension theorem for $\bar\partial$-closed forms.

Theorem 4.1. *Let $\Omega \subset\subset X$ be a C^3 q-convex intersection $(q \geq 1)$ in a Kähler manifold X of complex dimension $n \geq 3$ such that $X \setminus \Omega$ is connected. Let E be a holomorphic Hermitian vector bundle of rank N over X.*

(1) *If E is Nakano semi-positive of type m on $\overline{\Omega}$, then for every $\bar\partial$-closed form f in $W^{1,p'}_{0,s}(X\setminus\Omega, E^*)$, $1 \leq s \leq \min\{n-q, n-m\}$, $2 \leq q, m \leq n-1$, there exists a form F in $L^{p'}_{0,s}(X, E^*)$ such that $F|_{X\setminus\Omega} = f$ and $\bar\partial F = 0$ in X in the distribution sense.*

For $s = n-1$, if we assume furthermore that the restriction of f to $\partial\Omega$ satisfies the moment condition

$$\int_{\partial\Omega} f \wedge \phi = 0, \quad \forall\ \phi \in L^p_{n,0}(\Omega, E) \cap \operatorname{Ker}(\bar\partial),$$

then the same statement holds.

(2) *If E is Nakano semi-negative of type m on $\overline{\Omega}$, then statement* (1) *holds for all $\bar\partial$-closed form f in $W^{1,p'}_{n,s}(X \setminus \Omega, E^*)$ for $m \leq s \leq n-q$ and $2 \leq q, m \leq n-1$.*

For $s = n-1$, the same statement holds true if we assume furthermore that f satisfies the moment condition

$$\int_{\partial\Omega} f \wedge \phi = 0, \quad \forall\ \phi \in L^p_{n,0}(\Omega, E) \cap \operatorname{Ker}(\bar\partial).$$

Proof. We consider the assertion in (1), i.e., the case when E is Nakano semi-positive of type m on $\overline{\Omega}$, as the defining functions ρ_i of Ω are of class $\mathcal{C}^3$, there is a bounded extension operator of $W^{k,p'}_{0,s}(X \setminus \Omega, E^*)$ into $W^{k,p'}_{0,s}(X, E^*)$ for all $k \geq 0$ and $1 \leq p' < \infty$ (see e.g. [8, Theorem 9.7]). Let $\widetilde{f} \in W^{1,p'}_{0,s}(X, E^*)$ be the extension of f such that $\widetilde{f}|_{X\setminus\Omega} = f$. Then $\bar\partial\widetilde{f}$ is in $L^{p'}_{0,s+1}(X, E^*)$ and is compactly supported in $\overline{\Omega}$. In view of Theorem 1.1, there exists a form g in $L^{p'}_{0,s}(X, E^*)$ with compact support in $\overline{\Omega}$ such that $\bar\partial g = \bar\partial\widetilde{f}$ in the distribution sense in X. Set $F = \widetilde{f} - g$, we have $\bar\partial F = 0$ in X, $F|_{X\setminus\Omega} = f$ and F is compactly supported in $\overline{\Omega}$. Thus the form $F \in L^{p'}_{0,s}(X, E^*)$ is the desired $\bar\partial$-closed extension of f to X. The assertion in (2) follows on using similar arguments. This completes the proof. □

Corollary 4.2. *Let Ω_1 and Ω_2 be two strictly q-convex and q^*-convex intersections with smooth $\mathcal{C}^\infty$ boundaries in an $n \geq 3$-dimensional Kähler manifold X, respectively, such that $\overline{\Omega}_2 \subset \Omega_1 \subset\subset X$. Assume that $H^{r,s}_{L^p}(X) = 0$. Then for any $\bar\partial$-closed form f in $W^{1,p}_{r,s}(\Omega_1 \setminus \overline{\Omega}_2)$ there exists a form u in*

$W^{1,p}_{r,s-1}(\Omega_1 \setminus \overline{\Omega}_2) \cap W^{\frac{1}{2},p}_{r,s}(\Omega_1 \setminus \overline{\Omega}_2)$ *such that* $\bar\partial u = f$ *in* $\Omega_1 \setminus \overline{\Omega}_2$*, where* $r \geq 0$*,* $q^* \leq s \leq n - q - 1$*.*

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