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ENLARGING THE BALL OF CONVERGENCE OF SECANT-LIKE METHODS FOR NON-DIFFERENTIABLE OPERATORS

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Abstract. In this paper, we enlarge the ball of convergence of a uniparametric family of secant-like methods for solving non-differentiable operators equations in Banach spaces via using ω-condition and centered-like ω-condition meantime as well as some fine techniques such as the affine invariant form. Numerical examples are also provided.

1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is an operator defined on a convex subset Ω of a Banach space X with values in a Banach space Y.

In general, the solutions of these equations can be rarely be found in closed form. That is why most solution methods for these equations are usually iterative. If F is differentiable, the most known method is Newton's method [10]:

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n \geq 0. \end{cases} \tag{1.2}$$

To avoid some disadvantages of Newton's method, many Newton-like methods have been proposed, see [2, 14]. If F is not differentiable, we cannot use the derivative in the iterative methods. Then, we often use divided differences [11] instead of derivatives. The best known method of this type is the Secant method [1]:

$$\begin{cases} x_0,\ x_{-1} \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}F(x_n), \quad n \geq 0, \end{cases} \tag{1.3}$$

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where, $[u, v; F], u, v \in \Omega$ is a first order divided difference [11], which is a bounded linear operator from X to Y such that

$$[u, v; F](u - v) = F(u) - F(v). \tag{1.4}$$

In order to improve the Secant method in some way, Ref. [6] proposed a family of secant-like methods:

$$\begin{cases} x_0,\ x_{-1} \text{ given in } \Omega, \quad \lambda \in [0, 1], \\ y_n = \lambda x_n + (1 - \lambda)x_{n-1}, \qquad\qquad n \geq 0, \\ x_{n+1} = x_n - [y_n, x_n; F]^{-1} F(x_n), \end{cases} \tag{1.5}$$

which are considered as a combination of the Secant method and Newton's method, since (1.5) is reduced to the Secant method (1.3) if $\lambda = 0$ and, provided that F is differentiable, to Newton's method if $\lambda = 1$, since $y_n = x_n$ and $[y_n, x_n; F] = F'(x_n)$. Note that in [6], the authors show that the higher the value of $\lambda \in [0, 1]$ is, the higher the speed of convergence of (1.5) is, so that the speed of convergence is close to that of Newton's method (1.2) when λ is close to 1.

The study about convergence of iterative procedures is normally centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure. While the local analysis is based on the information around the solution, to find estimates of the radii of convergence ball.

A lot of works on the convergence of iterative methods including (1.2), (1.3) and (1.5) have been given, see [1–6,8–16]. In the case of local convergence, F is often supposed to be differentiable. Recently, Ref. [7] presents a new technique to give an analysis of local convergence for the secant-like methods (1.5) when F is non-differentiable. Two conditions are used to the local convergence analysis in [7]:

(A1) Let $x^\star$ be a solution of Eq. (1.1) and consider $\widehat{x} \in \Omega$ with $\|x^\star - \widehat{x}\| = \delta > 0$ so that the operator $[x^\star, \widehat{x}; F]^{-1}$ exists with $\|[x^\star, \widehat{x}; F]^{-1}\| \leq \gamma$;

(A2) $\|[x, y; F] - [u, v; F]\| \leq \omega(\|x - u\|, \|y - v\|), x, y, u, v \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function in both arguments.

Some other conditions are also needed to ensure the convergence of (1.5), see [7]. However, the main idea of [7] is the introduction of condition (A1). Note that, based on conditions (A1), (A2) and some other conditions, the convergence ball of (1.5) is given when F is non-differentiable.

In the present paper, we enlarge the ball of convergence of (1.5), provide tighter error bounds on the distances $\|x_n - x^\star\|$ and an at least as precise information on the location of the solution $x^\star$. We use the following new ideas: (1) give the results in affine invariant form; (2) use ω-condition and centered-like ω-condition meantime. Some other technique is also used in our analysis. These advantages are obtained under the same computational cost, since in practice the computation of function ω involves the computation of new functions ω_0 and $\overline{\omega}$ (see Section 2) as special cases.

The paper is organized as follows: Section 2 contains the local convergence analysis of method (1.5) under our new conditions. The numerical examples including favorable comparisons with earlier study [7] are presented in the concluding Section 3.

2. Improved local convergence analysis of method (1.5)

We present the local convergence of method (1.5) in this section. Denote $B(x, r)$ as a ball centered at x and with radius r. Firstly, we assume that there exists a first order of divided difference $[x, y; F] \in L(X, Y)$, for all pair of distinct points $x, y \in \Omega$, where $L(X, Y)$ denotes the space of bounded linear operators from X to Y, we suppose:

(C1) Let $x^\star$ be a solution of Eq. (1.1) and consider $\widehat{x} \in \Omega$ with $\|x^\star - \widehat{x}\| = \delta > 0$ so that the operator $[x^\star, \widehat{x}; F]^{-1}$ exists;

(C2) $\|[x^\star, \widehat{x}; F]^{-1}([x, y; F] - [x^\star, \widehat{x}; F])\| \leq \omega_0(\|x - x^\star\|, \|y - \widehat{x}\|), x, y \in \Omega$, where $\omega_0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function in both arguments;

(C3) $\|[x^\star, \widehat{x}; F]^{-1}([x, y; F] - [u, v; F])\| \leq \overline{\omega}(\|x - u\|, \|y - v\|), x, y, u, v \in \Omega_0 = \Omega \bigcap B(x^\star, r_0)$, where, $r_0 = \sup\{t \geq 0, \omega_0(t, \delta + t) < 1\} \in (0, +\infty)$ and $\overline{\omega} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function in both arguments.

(C4) The equation

$$\overline{\omega}(2(1 - \lambda)t, t) + \omega_0(t, \delta + t) - 1 = 0 \tag{2.1}$$

has at least one positive real root, the smallest positive root of (2.1) is denoted by $\overline{R}$;

(C5) $B(x^\star, \overline{R}) \subseteq \Omega$ and $\omega_0(\overline{R}, \delta + \overline{R}) < 1$.

Lemma 2.1. *Suppose the conditions* (C1)-(C5) *are satisfied. Then, for all* $x, y \in B(x^\star, \overline{R})$, $[x, y; F]^{-1}$ *exists and*

$$\|[x, y; F]^{-1}[x^\star, \widehat{x}; F]\| \leq \frac{1}{1 - \omega_0(\overline{R}, \delta + \overline{R})}. \tag{2.2}$$

Proof. In view of (C1)-(C5), for any $x, y \in B(x^\star, \overline{R})$, we have

$$\begin{aligned} \|I - [x^\star, \widehat{x}; F]^{-1}[x, y; F]\| &= \|[x^\star, \widehat{x}; F]^{-1}([x, y; F] - [x^\star, \widehat{x}; F])\| \\ &\leq \omega_0(\|x^\star - x\|, \|\widehat{x} - y\|) \\ &\leq \omega_0(\|x^\star - x\|, \|\widehat{x} - x^\star\| + \|x^\star - y\|) \\ &\leq \omega_0(\overline{R}, \delta + \overline{R}) < 1. \end{aligned} \tag{2.3}$$

By the Banach lemma on invertible operators [2], the operator $[x, y; F]^{-1}$ exists and (2.2) holds. □

Theorem 2.2. *Let* $F : D \subseteq X \to Y$ *be a nonlinear operator on a non-empty open convex domain* Ω *of a Banach space* X *with values in a Banach space*

Y. Suppose that conditions (C1)-(C5) *are satisfied. Then, fixed $\lambda \in [0,1)$, the sequence $\{x_n\}$ generated by method* (1.5)*, is well-defined, and converges to a solution $x^\star$ of Eq.* (1.1) *for all pair of distinct $x_{-1}, x_0 \in B(x^\star, \overline{R})$.*

Proof. First, from the density of real number, there must exist a real number $\overline{R}' \in (0, \overline{R})$ such that $x_{-1}, x_0 \in B(x^\star, \overline{R}')$, since $x_{-1}, x_0 \in B(x^\star, \overline{R})$. Second, as $\lambda \in [0,1)$, then, $y_0 \neq x_0$ and $y_0 \in B(x^\star, \overline{R}') \subseteq B(x^\star, \overline{R}) \subseteq \Omega$. Therefore, $[y_0, x_0; F]$ is well-defined and by Lemma 2.1, $[y_0, x_0; F]^{-1}$ exists. Moreover, using the similar analysis as that in Lemma 2.1, we deduce that

$$\|[y_0, x_0; F]^{-1}[x^\star, \widehat{x}; F]\| \leq \frac{1}{1 - \omega_0(\overline{R}', \delta + \overline{R}')}. \tag{2.4}$$

By the definition of $\overline{R}$ and $0 < \overline{R}' < \overline{R}$, we have

$$\overline{\omega}(2(1-\lambda)\overline{R}', \overline{R}') + \omega_0(\overline{R}', \delta + \overline{R}') < 1, \tag{2.5}$$

that is to say, we have

$$\frac{\overline{\omega}(2(1-\lambda)\overline{R}', \overline{R}')}{1 - \omega_0(\overline{R}', \delta + \overline{R}')} < 1. \tag{2.6}$$

Then, it follows

$$\begin{aligned}
&\|x_1 - x^\star\| \\
&= \|x_0 - x^\star - [y_0, x_0; F]^{-1}F(x_0) + [y_0, x_0; F]^{-1}F(x^\star)\| \\
&= |[y_0, x_0; F]^{-1}[x^\star, \widehat{x}; F][x^\star, \widehat{x}; F]^{-1}([y_0, x_0; F] - [x_0, x^\star; F])(x_0 - x^\star)\| \\
&\leq \tfrac{1}{1-\omega_0(\overline{R}', \delta + \overline{R}')}\overline{\omega}(\|y_0 - x_0\|, \|x_0 - x^\star\|)\|x_0 - x^\star\| \\
&= \tfrac{1}{1-\omega_0(\overline{R}', \delta + \overline{R}')}\overline{\omega}((1-\lambda)\|x_{-1} - x_0\|, \|x_0 - x^\star\|)\|x_0 - x^\star\| \\
&\leq \tfrac{1}{1-\omega_0(\overline{R}', \delta + \overline{R}')}\overline{\omega}((1-\lambda)(\|x_{-1} - x^\star\| + \|x^\star - x_0\|), \|x_0 - x^\star\|)\|x_0 - x^\star\| \\
&\leq \tfrac{\overline{\omega}(2(1-\lambda)\overline{R}', \overline{R}')}{1-\omega_0(\overline{R}', \delta + \overline{R}')}\|x_0 - x^\star\| \\
&\leq \tfrac{\overline{\omega}(2(1-\lambda)\overline{R}, \overline{R})}{1-\omega_0(\overline{R}, \delta + \overline{R})}\|x_0 - x^\star\| = \|x_0 - x^\star\| < \overline{R}',
\end{aligned} \tag{2.7}$$

which means $x_1 \in B(x^\star, \overline{R}')$.

By induction, for any integer $n \geq 0$, x_{n+1} is well-defined, and

$$\begin{aligned}
\|x_{n+1} - x^\star\| &\leq \frac{\overline{\omega}(2(1-\lambda)\overline{R}', \overline{R}')}{1 - \omega_0(\overline{R}', \delta + \overline{R}')}\|x_n - x^\star\| \\
&\leq \cdots \leq \Big(\frac{\overline{\omega}(2(1-\lambda)\overline{R}', \overline{R}')}{1 - \omega_0(\overline{R}', \delta + \overline{R}')}\Big)^{n+1}\|x_0 - x^\star\|,
\end{aligned} \tag{2.8}$$

which shows that $\{x_n\}$ converges to $x^\star$ linearly. $\square$

Remark 2.3. (a) The results are now given in affine invariant form. The advantages of affine invariant results over non-affine invariant results are well-known, see [4].

(b) If $\gamma = 1$, $\omega_0 = \omega = \overline{\omega}$ and $\Omega_0 = \Omega$, then the new results coincide with the ones of the paper of [7]. Otherwise, they constitute an improvement. Indeed, we have that

$$\omega_0(t,s) \le \gamma\omega(t,s). \tag{2.9}$$

The new equation (2.1) (i.e., $\overline{R}$) is more precise than the corresponding equation (7) (i.e., R) in [7], if $\omega_0 < \gamma\omega$ or $\overline{\omega} < \gamma\omega$. That is, we have

$$R \le \overline{R}. \tag{2.10}$$

Notice that the definition of function $\overline{\omega}$ depends on ω_0. This definition was not possible before, when the old condition is only used. We also have that

$$\overline{\omega}(t,s) \le \gamma\omega(t,s), \tag{2.11}$$

since $\Omega_0 \subseteq \Omega$. The new error bounds are also better, since we have

$$\|x_{n+1} - x^\star\| \le \frac{\overline{\omega}(2(1-\lambda)\overline{R}', \overline{R}')}{1 - \omega_0(\overline{R}', \delta + \overline{R}')}\|x_n - x^\star\|, \tag{2.12}$$

instead of the less precise in the paper [7] given by

$$\|x_{n+1} - x^\star\| \le \frac{\gamma\omega(2(1-\lambda)R, R)}{1 - \gamma\omega(R, \delta + R)}\|x_n - x^\star\|. \tag{2.13}$$

Concerning the uniqueness of the solution $x^\star$, we have the result:

Proposition 2.4. *Suppose that the hypotheses of Theorem 2.2 are satisfied. Moreover, suppose that there exists $\overline{R}_1 \ge \overline{R}$ such that*

$$\omega_0(0, \delta + \overline{R}_1) < 1 \tag{2.14}$$

or

$$\omega_0(\overline{R}_1, \delta) < 1, \tag{2.15}$$

then $x^\star$ is the only solution of equation $F(x) = 0$ in $\Omega_1 = \Omega \bigcap \overline{B}(x^\star, \overline{R}_1)$.

Proof. Let $y^\star \in \Omega_1$ with $F(y^\star) = 0$. Define operator $T = [x^\star, y^\star; F]$. Using (2.3) for $x = x^\star$, $y = y^\star$ and (2.14), we get that

$$\begin{aligned} \|I - [x^\star, \widehat{x}; F]^{-1}T\| &\le \omega_0(\|x^\star - x^\star\|, \|\widehat{x} - x^\star\| + \|x^\star - y^\star\|) \\ &\le \omega_0(0, \delta + \overline{R}_1) < 1, \end{aligned} \tag{2.16}$$

so T^{-1} exists. Then, from the identity $0 = F(x^\star) - F(y^\star) = T(x^\star - y^\star)$, we deduce that $x^\star = y^\star$. If (2.15) holds instead of (2.14), define operator $T_1 = [y^\star, x^\star; F]$, use (2.3) for $x = y^\star$, $y = x^\star$ and (2.15) to arrive at

$$\|I - [x^\star, \widehat{x}; F]^{-1}T_1\| \le \omega_0(\overline{R}_1, \delta) < 1, \tag{2.17}$$

so T_1^{-1} exists and from $0 = T_1(y^\star - x^\star)$, we conclude again that $x^\star = y^\star$. □

Remark 2.5. Uniqueness results were not given in [7]. But if they were conditions (2.14) and (2.15) would have looked like

$$\gamma\omega(0,\delta+R_0)<1 \tag{2.18}$$

or

$$\gamma\omega(R_0,\delta)<1 \tag{2.19}$$

for $R_0>R$. Then, again in view of (2.9), we would have

$$R_0\le\overline{R}_1. \tag{2.20}$$

That is the information on the uniqueness of the solution is at least as good with our approach as the one in [7].

3. Numerical examples

We present two examples in this section.

Example 3.1. Let $X=Y=\mathbb{R}^3$, $\Omega=(-1,1)^3$ and define $F=(F_1,F_2,F_3)$ on Ω by

$$F(x)=(e^{x_1}-1,\tfrac{e-1}{2}x_2^2+x_2,x_3+\tfrac{1}{10}|x_3|)^T, \tag{3.1}$$

where, $x=(x_1,x_2,x_3)$. Obviously, $x^\star=(0,0,0)$ is a solution of Eq. (1.1) and F is not differentiable at $x^\star$. We choose $\widehat{x}=(0,0,0.01)$. For $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)\in\mathbb{R}^3$, we choose $[u,v;F]\in L(X,Y)$ as follows [11]:

$$\begin{aligned}[u,v;F]_{ij}=\frac{1}{u_j-v_j}&(F_i(u_1,\dots,u_j,v_{j+1},\dots,v_3)\\ &-F_i(u_1,\dots,u_{j-1},v_j,\dots,v_3)),\ i,j=1,2,3.\end{aligned} \tag{3.2}$$

Let $\|u\|=\|u\|_\infty=\max\{|u_1|,|u_2|,|u_3|\}$. The corresponding norm on $A\in\mathbb{R}^3\times\mathbb{R}^3$ is $\|A\|=\max_{1\le i\le 3}\sum_{j=1}^3|a_{ij}|$. So, we have

$$[u,v;F]=\begin{pmatrix}\frac{e^{u_1}-e^{v_1}}{u_1-v_1} & 0 & 0\\ 0 & \frac{e-1}{2}(u_2+v_2)+1 & 0\\ 0 & 0 & 1+\frac{|u_3|-|v_3|}{10(u_3-v_3)}\end{pmatrix}. \tag{3.3}$$

Then, we have

$$\begin{aligned}&\delta=\|x^\star-\widehat{x}\|=0.01,\\ &\|L^{-1}\|=\|[x^\star,\widehat{x};F]^{-1}\|=\|\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{11}{10}\end{pmatrix}^{-1}\|=1,\end{aligned} \tag{3.4}$$

and for $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$, $u=(u_1,u_2,u_3)$, $v=(v_1,v_2,v_3)\in\Omega$, the following estimates hold:

$$\|L^{-1}([x,y;F]-[x^\star,\widehat{x};F])\| \tag{3.5}$$

$$
\begin{aligned}
&= \left\| \begin{pmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1} - 1 & 0 & 0 \\ 0 & \frac{e-1}{2}(x_2+y_2) & 0 \\ 0 & 0 & \frac{|x_3|-|y_3|}{11(x_3-y_3)} - \frac{1}{11} \end{pmatrix} \right\| \\
&\le \max\{\frac{e-1}{2}(|x_1|+|y_1|), \frac{e-1}{2}(|x_2|+|y_2|), \frac{2}{11}\} \\
&\le \frac{e-1}{2}(\|x-x^\star\| + \|y-\widehat{x}\|) + \frac{2}{11}
\end{aligned}
$$

and

(3.6)

$$
\begin{aligned}
&\|L^{-1}([x,y;F]-[u,v;F])\| \\
&= \left\| \begin{pmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1} - \frac{e^{u_1}-e^{v_1}}{u_1-v_1} & 0 & 0 \\ 0 & \frac{e-1}{2}(x_2-u_2+y_2-v_2) & 0 \\ 0 & 0 & \frac{|x_3|-|y_3|}{11(x_3-y_3)} - \frac{|u_3|-|v_3|}{11(u_3-v_3)} \end{pmatrix} \right\| \\
&\le \max\{\frac{e}{2}(|x_1-u_1|+|y_1-v_1|), \frac{e-1}{2}(|x_2-u_2|+|y_2-v_2|), \frac{2}{11}\} \\
&\le \frac{e}{2}(\|x-u\|+\|y-v\|) + \frac{2}{11}.
\end{aligned}
$$

Here, in (3.5) and (3.6), we use the following inequalities

$$
\begin{aligned}
|\frac{e^{x_1}-e^{y_1}}{x_1-y_1} - 1| &= |\int_0^1 (e^{tx_1+(1-t)y_1} - 1)dt| \\
&= |\int_0^1 (tx_1+(1-t)y_1 + \frac{(tx_1+(1-t)y_1)^2}{2!} + \cdots)dt| \\
&= |\int_0^1 (tx_1+(1-t)y_1)(1 + \frac{tx_1+(1-t)y_1}{2!} + \cdots)dt| \\
&\le \int_0^1 |tx_1+(1-t)y_1|(1+\frac{1}{2!}+\cdots)dt \\
&\le \frac{e-1}{2}(|x_1|+|y_1|), \quad x_1, y_1 \in \Omega,
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&|\frac{e^{x_1}-e^{y_1}}{x_1-y_1} - \frac{e^{u_1}-e^{v_1}}{u_1-v_1}| \\
&= |\int_0^1 (e^{tx_1+(1-t)y_1} - (e^{tu_1+(1-t)v_1})dt| \\
&= |\int_0^1 \int_0^1 (e^{s(tx_1+(1-t)y_1)+(1-s)(tu_1+(1-t)v_1)} \\
&\qquad (tx_1+(1-t)y_1-tu_1-(1-t)v_1))dsdt| \\
&\le \frac{e}{2}(|x_1-u_1|+|y_1-v_1|), \quad x_1, y_1, u_1, v_1 \in \Omega,
\end{aligned} \tag{3.8}
$$

and

$$||x_3| - |y_3|| \leq |x_3 - y_3|, \quad x_3, y_3 \in \Omega. \tag{3.9}$$

In view of (3.5), (3.6) and the definition of r_0 in $(C3)$, we can choose

$$\begin{aligned}\omega_0(t,s) &= \frac{e-1}{2}(t+s) + \frac{2}{11},\\ r_0 &= \frac{\frac{9}{11} - \frac{e-1}{2}\delta}{e-1} \approx 0.47116276,\\ \overline{\omega}(t,s) &= \frac{e}{2}(t+s) + \frac{2}{11},\end{aligned} \tag{3.10}$$

and Eq. (2.1) becomes

$$\begin{aligned}&\overline{\omega}((2(1-\lambda)t, t) + \omega_0(t, \delta + t) - 1\\ &= \frac{e}{2}(2(1-\lambda)+1)t + \frac{2}{11} + \frac{e-1}{2}(2t+\delta) + \frac{2}{11} - 1 = 0,\end{aligned} \tag{3.11}$$

which has the unique solution

$$\overline{R} = \frac{\frac{7}{11} - \frac{e-1}{2}\delta}{e(\frac{5}{2} - \lambda) - 1}. \tag{3.12}$$

Therefore, conditions (C1)-(C5) are satisfied and Theorem 2.2 applies.

Note that, if we use Theorem 2 in [7], we can choose

$$\delta = \|x^\star - \widehat{x}\| = 0.01,\ \gamma = \|[x^\star, \widehat{x}; F]^{-1}\| = 1,\ \omega(t,s) = \frac{e}{2}(t+s) + \frac{1}{5}, \tag{3.13}$$

since

(3.14)

$$\begin{aligned}&\|[x,y;F] - [u,v;F]\|\\ &= \| \begin{pmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1} - \frac{e^{u_1}-e^{v_1}}{u_1-v_1} & 0 & 0\\ 0 & \frac{e-1}{2}(x_2 - u_2 + y_2 - v_2) & 0\\ 0 & 0 & \frac{|x_3|-|y_3|}{10(x_3-y_3)} - \frac{|u_3|-|v_3|}{10(u_3-v_3)} \end{pmatrix} \|\\ &\leq \max\{\frac{e}{2}(|x_1-u_1| + |y_1-v_1|), \frac{e-1}{2}(|x_2-u_2| + |y_2-v_2|), \frac{1}{5}\}\\ &\leq \frac{e}{2}(\|x-u\| + \|y-v\|) + \frac{1}{5}, \quad x,,y,u,v \in \Omega.\end{aligned}$$

Then, Eq. (7) in [7] becomes

$$\frac{e}{2}(2(1-\lambda)t + t) + \frac{1}{5} + \frac{e}{2}(2t+\delta) + \frac{1}{5} - 1 = 0, \tag{3.15}$$

which has the unique solution

$$R = \frac{\frac{3}{5} - \frac{e}{2}\delta}{\frac{e}{2}(5 - 2\lambda)}. \tag{3.16}$$

It is easy to verify that

$$\overline{R} > R \tag{3.17}$$

TABLE 1. The comparison results of $\overline{R}$ and R for values of λ

λ	$\overline{R}$	R
0.0	0.108316809	0.086291066
0.2	0.119529030	0.093794637
0.4	0.133330498	0.102727459
0.6	0.150735198	0.113540876
0.8	0.173366063	0.126898626
0.9	0.187436606	0.13482979
0.99	0.202206759	0.142866003

holds for each $\lambda \in [0, 1)$. In Table 1, $\overline{R}$ and R are listed for some values of λ. From this table, we can see that the ball of convergence for secant-like methods (1.5) has been enlarged by using our new techniques.

Next we verify the results of uniqueness of the solution. Note that condition (2.14) (or (2.15)) becomes

$$\omega_0(0, \delta + \overline{R}_1) = \frac{e-1}{2}(\delta + \overline{R}_1) + \frac{2}{11} < 1, \tag{3.18}$$

that is

$$\overline{R}_1 < \frac{\frac{9}{11} - \frac{e-1}{2}\delta}{\frac{e-1}{2}} \approx 0.94232552. \tag{3.19}$$

So Proposition 2.4 applies and we can deduce that $x^\star$ is the only solution of Eq. (1.1) in $\Omega \bigcap \overline{B}(x^\star, \overline{R}_1)$ provided that we choose $\overline{R}_1$ such that $\overline{R}_1 \geq \overline{R}$ and (3.19) is satisfied. Note also that if we use condition (2.18) (or (2.19)), we can choose R_0 such that $R_0 > R$ and

$$R_0 < \frac{\frac{4}{5} - \frac{e}{2}\delta}{\frac{e}{2}} \approx 0.578607106. \tag{3.20}$$

Using (3.19) and (3.20), we see that (2.20) is true provided that we choose both the biggest value to satisfy (3.19) and (3.20).

Example 3.2. Let us consider the boundary value problem (BVP) of second order defined by the equation

$$\frac{d^2x(s)}{ds^2} + \mu(x(s)) = 0 \tag{3.21}$$

with the boundary conditions

$$x(0) = x(1) = 0, \tag{3.22}$$

where μ is a given function. Many problems in various disciplines can be brought in a form like BVP (3.21)-(3.22). As examples: in Physics they appear in connection to Newton's and numerous other laws; in Biology, they are related to population dynamics modes; in Chemistry they are related to the concentrations of different reagents during a reaction. It is worth noticing that BVP (3.21)-(3.22) is equivalent to solving the Fredholm integral equation [2, 6–8]:

$$x(s) = -\int_0^1 K(s,t)\mu(x(t))dt,$$

where the kernel K is the Green's function

$$K(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t \end{cases}$$

for each $(s,t) \in [0,1] \times [0,1]$. In order for us to apply our results, we shall discretize the BVP. Let p be a positive integer, set $q = \frac{1}{1+p}$ and $t_i = iq$, $i = 0, 1, \ldots, p+1$. We use the popular standard approximation for the second derivative given by

$$x'' \approx \frac{x_{i-1} - 2x_i + x_{i+1}}{q^2}, \ i = 1, 2, \ldots, p. \tag{3.23}$$

To obtain a scheme for determining number x_i and the approximate values $x(t_i)$ of the true solution at the points t_i we want

$$x_{i-1} - 2x_i + x_{i+1} + q^2\mu(x_i) = 0. \tag{3.24}$$

The unknowns are $x_1, x_2, \ldots, x_p$, since x_0 and x_{p+1} are determined by the boundary conditions (3.22). We usually simplify (3.24) by using the matrix and vector notation: $\mathbf{x}= (x_1, x_2, \ldots, x_i)^T$, $\mathbf{v}= (\mu(x_1), \mu(x_2), \ldots, \mu(x_p))^T$ and

$$M = \begin{pmatrix} -2 & 1 & 0 & . & . & . & 0 \\ 1 & -2 & 1 & . & . & . & 0 \\ 0 & 1 & -2 & . & . & . & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ . & . & . & . & . & . & -2 \end{pmatrix}.$$

Using (3.24) and the preceding notation we can equivalently write

$$F(\mathbf{x}) = M(\mathbf{x}) + q^2(\mathbf{v}), \tag{3.25}$$

where $F : \Omega \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}^p$ and $\Omega = (-1,1)^3$. Notice that if $\mu(\mathbf{x})$ is not linear in $\mathbf{x}$, equation

$$F(\mathbf{x}) = 0 \tag{3.26}$$

can be solved by algebraic methods only in special cases. That explains why we resort to iterative methods such as secant-like method (1.5) to solve equations like (3.26). Let us specialize function μ by

$$\mu((\mathbf{x}(s)) = e^{|(\mathbf{x}(s)|} - 1. \tag{3.27}$$

Then, operator F is not differentiable on Ω. Notice that the solution of BVP is $\mathbf{x}^*(s) = 0$ for each $s \in [0,1]$. Choose $p = 3$, $\mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{v} = (e^{|x_1|} - 1, e^{|x_2|} - 1, e^{|x_3|} - 1)^T$, $\mathbf{x}^* = (0,0,0)^T$, $\widehat{(\mathbf{x}} = (0,0,-0.01)^T$. Using the divided difference, norm and notation introduced in Example 3.1 and along the same lines, we get for $\alpha = 1.0050164$, $\delta = \|x^* - \widehat{x}\| = 0.01$, $\gamma = \|L^{-1}\| =$

TABLE 2. The comparison results of $\overline{R}$ and R for values of λ

λ	$\overline{R}$	R
0.0	0.169336765899	0.14368280109369
0.2	0.186865358007	0.15617695771053
0.4	0.208441842052	0.17105095368296
0.6	0.235651429556	0.18905631722854
0.8	0.271031324865	0.21129823690249
0.9	0.293028468158	0.22450437670889
0.99	0.31611934234	0.23788543227432

$$\|[x^*, \widehat{x}; F]^{-1}\| = \left\| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}^{-1} \right\| = \left\| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha^{-1} \end{pmatrix} \right\| = 1.$$

Moreover, as in (3.5), (3.6), (3.14), respectively, we can set

$$w_0(s,t) = \frac{e-1}{2}(s+t) + |\alpha^{-1}(1-\alpha)| = \frac{e-1}{2}(s+t) + 0.004991700279,$$

$$\bar{w}(s,t) = \frac{e}{2}(s+t) + 0.004991700279$$

and

$$w(s,t) = \frac{e}{2}(s+t) + |1-\alpha| = \frac{e}{2}(s+t) + 0.0050164.$$

Then, we get under our approach

$$\bar{R} = \frac{2(1-2|\alpha^{-1}(1-\alpha)|) + (1-e)\delta}{(5-2\lambda)e-2},$$

$$r_0 = \frac{2(1-|\alpha^{-1}(1-\alpha)|) + (1-e)\delta}{2(e-1)},$$

$$\bar{R}_1 = \frac{2(1-|\alpha^{-1}(1-\alpha)|) - (1-e)\delta}{e-1}$$

and under the approach in [7]

$$R = \frac{2(1-2|\alpha^{-1}(1-\alpha)|) - e\delta}{(5-2\lambda)e},$$

$$R_0 = \frac{2(1-|1-\alpha|) - e\delta}{e}.$$

Notice that a parameter like r_0 was not used in [7] but if it was (with w_0 replaced by w in the definition of $\bar{r}_0$), then

$$\bar{r}_0 = \frac{2(1-|1-\alpha|) - e\delta}{2e}.$$

Then, again we obtain results for each $\lambda \in [0,1]$, since $\bar{R} > R$. See also Table 2.

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