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VANISHING OF PROJECTIVE VECTOR FIELDS ON COMPACT FINSLER MANIFOLDS

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Abstract. In this paper, we give characteristic differential equations of a kind of projective vector fields on Finsler manifolds. Using these equations, we prove the vanishing theorem of projective vector fields on any compact Finsler manifold with the negative mean Ricci curvature, which is defined in [10]. This result involves the vanishing theorem of Killing vector fields in the Riemannian case and the work of [1, 14].

1. Introduction

Projective vector fields are a class of important vector fields on differential manifolds. They include some important concepts such as Killing vector fields, affine vector fields, etc. All those fields describe some symmetries of the space. On the other hand, a projective field is related to a projective transformation, which preserves the geodesics in set-theoretic sense. It is also related to the Hilbert's forth problem [11].

In Riemann geometry, a projective flat manifold must be a space form with constant curvature by the Beltrami's theorem. This fact somehow restricts applications of projective fields. However, conformal fields play much more important roles in Riemann geometry [9]. Even the Killing fields are concerned more than projective fields [8, 15, 16]. However, in general Finsler spaces, projective transformations perform much better than conformal transformations. There are lots of work on projective flat metrics in Finsler geometry [4, 13]. To look into the symmetries and the properties of connections, R. L. Lovas considered the equivalent characterization equations of affine and projective fields on Finsler manifolds using the Lie algebra [5]. H. J. Tian also discussed a vanishing theorem of projective fields [14]. However, H. Akbar-Zadeh had already studied this topic in [1], which included Tian's work. We can access their result in another approach. See Theorem 5.3 below.

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In this article, we consider the Finsler projective fields and give two equivalent differential equations about the Chern connection of some projective vector fields. From the convenience of analysis, we think the Chern connection is better than the Yano's derivative, which was used in [5].

Theorem 1.1. *Let (M,F) be a Finsler manifold and let V be a vector field on M. The following statements are equivalent*:

i) *There are a 0-homogeneous 1-form ψ and a (-1)-homogeneous 2-form τ such that*

$$V^i_{|j|k} - V^p R_j{}^i{}_{kp} + y^s V^r_{|s} P_j{}^i{}_{kr} = \psi_j \delta^i_k + \psi_k \delta^i_j + \tau_{jk} y^i, \tag{1}$$

ii) *there are a 0-homogeneous 1-form ψ and a (-1)-homogeneous 2-form τ such that*

$$\begin{aligned} V_{i|j|k} + V_{j|i|k} = {} & 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik} + \tau_{ik} y_j + \tau_{jk} y_i \\ & - 2V^p C_{ijs} R^s{}_{kp} - 2y^s V^r_{|s} (C_{ijr|k} - 2y^p P_k{}^l{}_{pr} C_{ijl}), \end{aligned} \tag{2}$$

where "$|$" denotes the horizontal covariant derivative with respect to the Chern connection. Either one of the above implies that

iii) *V is a projective field on M.*

We can prove vanishing theorems of projective fields by applying the Bochner technique on Finsler manifolds to the equations in Theorem 1.1. The most interesting result is the following.

Theorem 1.2. *Let (M,F) be a compact Finsler manifold with non-positive mean Ricci curvature $\widetilde{Ricci} \leq 0$. Then every projective field V on M is almost parallel, that is $\nabla V(y) = \lambda(x,y)y$ for some scalar function $\lambda(x,y)$ on TM, and $\widetilde{Ricci}(V,V) = 0$. Furthermore, if the mean Ricci curvature is negative, then there is no any nontrivial projective field.*

This conclusion improves the result in [1, 14] and can be considered as the extension of the similar theorem about Killing fields in [10].

2. Finsler manifolds and some concepts

In this section, we will shortly introduce the concepts we need on a general Finsler manifold.

Let (M,F) be a Finsler manifold, with $F = F(x,y)$ being the Finsler metric. F is actually a non-negative function continuously defined on TM and smoothly defined on $T_0M := TM \setminus \{0\}$. We call F a Riemannian metric if $F = \sqrt{g_{ij}(x)y^i y^j}$, where the *fundamental tensor* locally expressed by $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is independent of the tangent coordinates y. The *Cartan curvature* $C = C_{ijk} dx^i \otimes dx^j \otimes dx^k$ is locally given by $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}$, which vanishes if and only if the metric is Riemannian. Moreover, the *mean Cartan curvature* $I = I_k dx^k$ is locally defined by $I_k = g^{ij} C_{ijk}$, which also vanishes if and only if the metric is Riemannian. A Landsberg curvature $L = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ is locally related to the Cartan tensor by $L_{ijk} = C_{ijk|0}$ with "$|$" denoting the horizontal covariant derivative with respect to the Chern connection. It is obvious to see $L_{ijk}y^i = L_{ijk}y^j = L_{ijk}y^k = 0$.

The *spray* G is a special vector field defined on the punched tangent bundle T_0M. It is locally described by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}, \tag{3}$$

where the spray coefficients G^i satisfy $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$ for any $\lambda > 0$. If the spray is induced from a Finsler metric, then it is related to the *Chern connection* as

$$G^i = \Gamma^i_{jk} y^j y^k, \quad \text{and} \quad (G^i)_{y^j y^k} = \Gamma^i_{jk} + L^i_{jk}.$$

The Chern connection is the unique affine connection on a Finsler manifold which is torsion free and almost metric compatible. The *Christoffel symbol of Chern connection* is locally defined by

$$\Gamma^l_{ij} = \frac{1}{2} g^{lk} (\frac{\delta g_{ik}}{\delta x^j} + \frac{\delta g_{jk}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^k}),$$

where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$ is called the horizontal derivative. We call N^i_j the nonlinear connection coefficients, which can be obtained from the spray coefficients by

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{4}$$

[2] provides the definition of *Chern-Riemannian curvature tensor* locally by

$$R^{\ i}_{j\ kl} := \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk}. \tag{5}$$

One can compute directly to get that

$$R_{ijkl} + R_{jikl} = -2C_{ijm}R^m_{\ \ kl}, \tag{6}$$

and

$$\begin{aligned} R_{klij} - R_{jikl} = & - C_{klm}R^m_{\ \ ji} + C_{jim}R^m_{\ \ kl} - C_{kim}R^m_{\ \ lj} - C_{ljm}R^m_{\ \ ki} \\ & - C_{ilm}R^m_{\ \ jk} - C_{jkm}R^m_{\ \ il}. \end{aligned} \tag{7}$$

The *flag curvature* is an analog of the sectional curvature in Riemann geometry, which is defined by

$$K(\Pi_y) := \frac{-R_{ijkl}y^i v^j y^k v^l}{(g_{ik}g_{jl} - g_{il}g_{jk})y^i v^j y^k v^l}, \tag{8}$$

where $R_{ijkl} := R^{\ s}_{i\ kl} g_{sj}$ and $\Pi_y = span\{y, v\}$ is a section of dimension 2, with $y = y^i \partial_i$ and $v = v^i \partial_i$.

If we define $R^{i}_{kl} = y^s R_{s\,kl}^{\,i}$, then (5) combining with the relation $N^i_j = y^k \Gamma^i_{jk}$ shows that

$$R^i_{jk} = \frac{\partial N^i_k}{\partial x^j} - \frac{\partial N^i_j}{\partial x^k} + N^s_k \frac{\partial N^i_j}{\partial y^s} - N^s_j \frac{\partial N^i_k}{\partial y^s}. \tag{9}$$

If we define

$$R^i_k := y^j R_{j\,kl}^{\,i} y^l, \quad R_{jk} := g_{ij} R^i_k = -R_{ijlk} y^i y^l,$$

the flag curvature can be concisely expressed by

$$K(\Pi_y)(v) = F^{-2} R_{jk} v^j v^k. \tag{10}$$

So R^i_k or R_{jk} is also called the *flag curvature tensor*. It is also denoted by R_y for short since the flag curvature is obtained by contracting the Chern-Riemannian curvature tensor with y.

The *Chern-non-Riemannian curvature tensor* is locally defined by

$$P_{j\,kl}^{\,i} := \frac{\partial \Gamma^i_{jk}}{\partial y^l}. \tag{11}$$

The symmetry of jk in Γ^i_{jk} implies that

$$P_{j\,kl}^{\,i} = P_{k\,jl}^{\,i}. \tag{12}$$

A *complete lifting* of vector field $V = V^i \partial_i$ on a Finsler manifold is denoted by $\hat{V}$ and locally defined by

$$\hat{V} = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}. \tag{13}$$

For any Finsler metric, the *mean Ricci curvature* $\widetilde{Ricci}$ is first introduced in [10], which reduces to the Ricci curvature when the metric is Riemannian. The *related Riemannian metric* can be found in [6], which is

$$a_{ij}(x) = \int_{S_x M} g_{ij}(x,y)\omega_x = \int_{S_x M} \frac{\partial^2 F^2}{\partial y^i \partial y^j}\omega_x, \tag{14}$$

where ω_x is a volume form on $S_1 := \{\xi \in \mathbf{R}^n \,|\, F(\xi) = 1\}$. There are some different volume forms on a Finsler manifold. In this article, we adopt the *Holmes-Thompson volume form*. That is,

$$dV_F := \sigma_H(x) dx, \tag{15}$$

$$\sigma_H(x) := \frac{1}{c_{n-1}} \int_{S_x M} \sqrt{\det(g_{ij})}\, d\nu, \tag{16}$$

$$d\nu := \sqrt{\det(g_{ij})} \sum_i (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n, \tag{17}$$

where "$\widehat{\ }$" means the term is suppressed and c_{n-1} denotes the volume of the Euclidean sphere $\mathbf{S}^{n-1}$ of dimension $(n-1)$. We know that $d\nu$ is the volume form on the tangent sphere $S_x M$.

At last, the following mean Ricci curvature can be used to solve a lot of problems.

Definition 2.1 ([10])**.** The mean Ricci curvature $\widetilde{Ricci}$ is the integral of the flag curvature tensor on the indicatrix S_xM of each point, i.e.,

$$\widetilde{Ricci}(v) = \frac{1}{c_{n-1}}\int_{S_xM} K(\Pi_y)(v)\frac{\sqrt{\det g_{ij}}}{\sqrt{\det a_{ij}}}d\nu = \frac{1}{c_{n-1}}\int_{S_xM} F^{-2}R_{jk}v^j v^k \frac{\sqrt{\det g_{ij}}}{\sqrt{\det a_{ij}}}d\nu, \tag{18}$$

where a_{ij} is the related Riemannian metric defined in (14) and $d\nu$ is defined in (17).

3. Projective vector fields on Riemannian manifolds

In this section, we recall some results about the Riemannian projective vector fields. For the convenience of the reader, we deduce the relations between different definitions. At last, we give the vanishing theorem of the projective fields in the Riemannian version without proof.

Let (M, g) be a Riemannain manifold. Denote the Lie derivative by $\mathcal{L}$ as usual. It defines on Riemannian manifolds in [3] that

$$\mathcal{L}_V\nabla(Y, Z) = [V, \nabla_Y Z] - \nabla_{[V,Y]}Z - \nabla_Y[V, Z], \tag{19}$$

where $V = V^i\partial_i$, $Y = Y^j\partial_j$ and $Z = Z^k\partial_k$ are arbitrary vector fields. Customarily, a vector field V is called a projective field if there is a 1-form ψ such that V satisfies

$$\mathcal{L}_V\nabla(Y, Z) = \psi(Y)Z + \psi(Z)Y. \tag{20}$$

It is equivalent to the following equations

$$V_{i|j|k} + V_{j|i|k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_k g_{ij}. \tag{21}$$

From the point of view of the Spray geometry, (19) actually curves the Lie derivative of Spray [12]. V is a projective vector field if and only if the 1-parameter transformation generated by V is a locally projective transformation, which locally preserves the geodesics. So (19) is not simply considered as the definition. Indeed, it is a formula of projective fields which can be acquired from computation and the local definition of Lie derivative of Levi-Civita connection that $\mathcal{L}_V\nabla(Y, Z) = \mathcal{L}_V\Gamma^i_{jk}Y^jZ^k\partial_i$, which can be calculated by infinitesimal analysis [14]. We prove the following property.

Proposition 3.1. *On any Riemannian manifold,* (19) *holds for any vector fields V, Y, Z. Furthermore, V is a projective field,* (20) *and* (21) *are equivalent mutually.*

Proof. From the definition of the Lie derivative, we know that

$$\mathcal{L}_V \Gamma^i_{jk} = \frac{\partial^2 V^i}{\partial x^j \partial x^k} + \Gamma^i_{jl}\frac{\partial V^l}{\partial x^k} + \Gamma^i_{lk}\frac{\partial V^l}{\partial x^j} - \Gamma^l_{jk}\frac{\partial V^i}{\partial x^l} + V^l \frac{\partial \Gamma^i_{jk}}{\partial x^l}. \tag{22}$$

On the other hand,

$$\begin{aligned}[X, \nabla_Y Z] &= [X^i\partial_i, (Y^j\partial_j Z^k + Y^j Z^l \Gamma^k_{jl})\partial_k] \\ &= \big(X^i(\partial_i Y^j \partial_j Z^k + Y^j \partial_i\partial_j Z^k + \partial_i Y^j Z^l \Gamma^k_{jl} + Y^j Z^l \partial_i \Gamma^k_{jl}) \\ &\quad - (Y^j\partial_j Z^i \partial_i X^k + Y^j Z^l \Gamma^i_{jl}\partial_i X^k)\big)\partial_k,\end{aligned} \tag{23}$$

$$\begin{aligned}\nabla_{[X,Y]}Z &= \nabla_{(X^i\partial_i Y^j - Y^i\partial_i X^j)\partial_j}(Z^k\partial_k) \\ &= (X^i\partial_i Y^j \partial_j Z^k + X^i \partial_i Y^j Z^l \Gamma^k_{jl} - Y^i\partial_i X^j\partial_j Z^k - Y^i\partial_i X^j Z^l \Gamma^k_{jl})\partial_k,\end{aligned} \tag{24}$$

and

$$\begin{aligned}\nabla_Y[X,Z] &= \nabla_{Y^i\partial_i}(X^j\partial_j Z^k - Z^j\partial_j X^k)\partial_k \\ &= Y^i(\partial_i X^j\partial_j Z^k + X^j\partial_i\partial_j Z^k - \partial_i Z^j \partial_j X^k - Z^j\partial_i\partial_j X^k \\ &\quad + X^j\partial_j Z^l \Gamma^k_{il} - Z^j\partial_j X^l \Gamma^k_{il})\partial_k.\end{aligned} \tag{25}$$

So (19) follows from (22)-(25) directly.

Furthermore, suppose V is a projective field. From [11], we know it preserves the geodesics and is equal to

$$\mathcal{L}_V G^i = P y^i, \tag{26}$$

where P is a y-homogeneous function of degree one. Taking the second derivative of y, noticing that V is a Riemannian projective field and $\mathcal{L}_V y^i = 0$, we get

$$\mathcal{L}_V \Gamma^i_{jk} = P_j \delta^i_k + P_k \delta^i_j. \tag{27}$$

Hence (20) holds with $\psi_i = P_i$. Since the manifold is Riemannian, P_i is only depend on x, because $P_{jk} = 0$ for any j, k. Therefore, $\psi = \psi_i dx^i$ is a 1-form. Contracting (27) with $y^i y^j$ yields (26).

It follows from (20) and (19) that

$$\frac{1}{2}(\partial_i V^l \Gamma^k_{jl} + \partial_j V^l \Gamma^k_{il} + \partial_i\partial_j V^k + V^l \partial_l \Gamma^k_{ij} - \partial_l V^k \Gamma^l_{ij}) = \frac{1}{2}(\psi_j \delta^k_i + \psi_i \delta^k_j), \tag{28}$$

which implies that

$$V^i{}_{|j|k} - V^p R_j{}^i{}_{kp} = \psi_j \delta^i_k + \psi_k \delta^i_j, \tag{29}$$

and

$$V_{i|j|k} - V^p R_{jikp} = \psi_j g_{ik} + \psi_k g_{ij}. \tag{30}$$

Adding up (30) and the equation exchanged i, j in (30) yields (21).

Conversely, suppose (21) holds. We can obtain from the Ricci identity and the first Bianchi identity that

$$\begin{aligned}
&2\psi_k g_{ij} + \psi_j g_{ik} + \psi_i g_{jk} \\
&= V_{i|j|k} + V_{j|i|k} \\
&= V_{i|j|k} + V_{j|k|i} + V_m R_j{}^m{}_{ki} \\
&= V_{i|j|k} - V_{k|j|i} + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij} + V_m R_j{}^m{}_{ki} \\
&= V_{i|j|k} - V_{k|i|j} - V_m R_k{}^m{}_{ji} + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij} + V_m R_j{}^m{}_{ki} \\
&= V_{i|j|k} + V_{i|k|j} - 2\psi_j g_{ik} - \psi_i g_{jk} - \psi_k g_{ij} \\
&\quad - V_m R_k{}^m{}_{ji} + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij} + V_m R_j{}^m{}_{ki} \\
&= V_{i|j|k} + V_{i|j|k} + V_m R_i{}^m{}_{kj} - 2\psi_j g_{ik} - \psi_i g_{jk} - \psi_k g_{ij} \\
&\quad - V_m R_k{}^m{}_{ji} + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij} + V_m R_j{}^m{}_{ki} \\
&= 2V_{i|j|k} + V_m(R_j{}^m{}_{ki} - R_k{}^m{}_{ji} + R_i{}^m{}_{kj}) + \psi_i g_{jk} - \psi_j g_{ik} \\
&= 2V_{i|j|k} - 2V_m R_i{}^m{}_{jk} + \psi_i g_{jk} - \psi_j g_{ik}
\end{aligned}$$

which implies (30). □

It follows from Proposition 3.1 and the definition of Ricci curvature that:

Theorem 3.2. *Suppose (M, g) is a compact Riemannian manifold with non-positive Ricci curvature $Ric \leq 0$. Then every projective field V is parallel and $Ric(V, V) = 0$. Furthermore, if the Ricci curvature is negative $Ric < 0$, then there is no nontrivial (nonzero) projective field.*

The proof of the theorem can be reduced from the proof of the vanishing theorem of the Finsler version, which will be presented in the next section. So we omit it.

4. Projective vector fields on Finsler manifolds

As emphasised in Section 1, through this paper, we use the Chern connection. We denote the horizontal covariant derivative about the Chern connection by "|" and the vertical covariant derivative about the Chern connection by ";". Noticing the infinitesimal coordinate transformation on TM

$$\overline{x}^i = x^i + V^i dt, \qquad \overline{y}^i = y^i + y^j \frac{\partial V^i}{\partial x^j} dt, \tag{31}$$

and (11), the Lie derivative of Chern-Riemannian connection coefficients with respect to the complete lifting $\hat{V}$ of V is

$$\mathcal{L}_{\hat{V}} \Gamma^i_{jk} = \frac{\partial^2 V^i}{\partial x^j \partial x^k} + \Gamma^i_{jl} \frac{\partial V^l}{\partial x^k} + \Gamma^i_{lk} \frac{\partial V^l}{\partial x^j} - \Gamma^l_{jk} \frac{\partial V^i}{\partial x^l} + V^l \frac{\partial \Gamma^i_{jk}}{\partial x^l} + y^s \frac{\partial V^l}{\partial x^s} P_j{}^i{}_{kl}. \tag{32}$$

The reason why we use the Chern connection is based on the Bochner technique in Finsler geometry [10]. Unfortunately, we don't get a similar expression of the Lie derivative of the Chern connection $\mathcal{L}_{\hat{V}} \nabla$ as brief as (19). Although we can get a formula by lifting Y, Z and V into the TTM (or HTM) and then projecting back onto TM several times, which seems too artificial. We think it is a nice work for anyone to find a natural and integral way to express the Lie derivative of the Chern connection $\mathcal{L}_{\hat{V}}\nabla$.

If V is a projective field on a Finsler manifold (M, F), then

$$\mathcal{L}_{\hat{V}}G = Py^i\frac{\partial}{\partial y^i}, \quad \text{i.e.,} \quad \mathcal{L}_{\hat{V}}G^i = Py^i, \tag{33}$$

where $P = P(x, y)$ is a y-homogeneous function of degree one [5].

For applications, we focus on the following projective vector fields on Finlser manifolds.

Definition 4.1. A vector field V on a Finsler manifold (M, F) is called a strongly projective vector field if the complete lifting of V satisfies that

$$2\mathcal{L}_{\hat{V}}\Gamma^i_{jk} = P_j\delta^i_k + P_k\delta^i_j + P_{jk}y^i, \tag{34}$$

where $P_i = \frac{\partial P}{\partial y^i}$ and $P_{ij} = \frac{\partial^2 P}{\partial y^i \partial y^j}$.

Hence a strongly projective vector field satisfies that

$$\mathcal{L}_{\hat{V}}\nabla(Y, Z) = \psi(Y)Z + \psi(Z)Y + \tau(Y, Z)\hat{l}, \tag{35}$$

where $\psi = \psi_i dx^i$ is a 1-form with $\psi_i = P_i$, $\tau = \tau_{ij}dx^i \otimes dx^j$ is a 2-form with $\tau_{ij} = \frac{1}{2}P_{ij}$ and $\hat{l} = y^i\frac{\partial}{\partial y^i}$.

Remark 4.2. It follows from $\mathcal{L}_{\hat{V}}y^i = 0$ and $G^i = \Gamma^i_{jk}y^j y^k$ that (34) implies (33). It means a strongly projective vector field must be a projective vector field.

Now we give the following lemma.

Lemma 4.3. *V is a strongly projective field on Finsler manifolds if and only if there are a 0-homogeneous 1-form ψ and a (-1)-homogeneous 2-form τ such that*

$$V^i_{|j|k} - V^p R_j{}^i{}_{kp} + y^s V^r_{|s} P_j{}^i{}_{kr} = \psi_j\delta^i_k + \psi_k\delta^i_j + \tau_{jk}y^i, \tag{36}$$

where $R_j{}^i{}_{kp}$ and $P_j{}^i{}_{kr}$ are given in (5) *and* (11) *respectively. Therefore,*

$$\begin{aligned} V_{i|j|k} + V_{j|i|k} = {} & 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik} + \tau_{ik}y_j + \tau_{jk}y_i \\ & - 2V^p C_{ijs}R^s{}_{kp} - 2y^s V^r_{|s}(C_{ijr|k} - 2y^p P_k{}^l{}_{pr}C_{ijl}), \end{aligned} \tag{37}$$

where $R^s{}_{kp}$ are given in (9) *and C_{ijk} are components of the Cartan tensor.*

Proof. It follows directly that $V^i_{|j} = \frac{\partial V^i}{\partial x^j} + V^l\Gamma^i_{lj}$, and

$$\begin{aligned} V^i_{|j|k} = {} & \frac{\partial^2 V^i}{\partial x^j \partial x^k} + \frac{\partial V^l}{\partial x^k}\Gamma^i_{lj} + V^l(\frac{\partial \Gamma^i_{lj}}{\partial x^k} - N^r_k\frac{\partial \Gamma^i_{lj}}{\partial y^r}) \\ & + \frac{\partial V^l}{\partial x^j}\Gamma^i_{lk} + V^r\Gamma^l_{rj}\Gamma^i_{lk} - \frac{\partial V^i}{\partial x^l}\Gamma^l_{jk} - V^r\Gamma^i_{rl}\Gamma^l_{jk}, \end{aligned} \tag{38}$$

which can deduce the following equation by comparing with (32),

$$V^i_{|j|k} = \mathcal{L}_{\hat{V}}\Gamma^i_{jk} - \frac{\partial V^l}{\partial x^s} y^s P_j{}^i{}_{kl} - V^l N^r_k \frac{\partial \Gamma^i_{lj}}{\partial y^r} + V^r \Gamma^l_{rj}\Gamma^i_{lk} - V^r \Gamma^i_{rl}\Gamma^l_{jk}. \tag{39}$$

On the other hand

$$\begin{aligned} V^p R_j{}^i{}_{kp} &= V^p(\frac{\delta \Gamma^i_{pj}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^p} + \Gamma^l_{jp}\Gamma^i_{kl} - \Gamma^i_{lp}\Gamma^l_{jk}) \\ &= V^p(\frac{\partial \Gamma^i_{pj}}{\partial x^k} - N^r_k \frac{\partial \Gamma^i_{pj}}{\partial y^r} - \frac{\partial \Gamma^i_{jk}}{\partial x^p} + N^r_p \frac{\partial \Gamma^i_{jk}}{\partial y^r} + \Gamma^l_{jp}\Gamma^i_{kl} - \Gamma^i_{lp}\Gamma^l_{jk}). \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} V^i_{|j|k} - V^p R_j{}^i{}_{kp} &= \mathcal{L}_V \Gamma^i_{jk} - \frac{\partial V^l}{\partial x^s} y^s P_j{}^i{}_{kl} - V^p N^r_p \frac{\partial \Gamma^i_{jk}}{\partial y^r} \\ &= \mathcal{L}_V \Gamma^i_{jk} - (\frac{\partial V^r}{\partial x^s} y^s + V^p \Gamma^r_{ps} y^s) P_j{}^i{}_{kr} \\ &= \mathcal{L}_V \Gamma^i_{jk} - y^s V^r_{|s} P_j{}^i{}_{kr}, \end{aligned} \tag{41}$$

which implies (36) provided that V is a projective field.

Applying the analogous method in the Riemannian case, one can easily obtain that

$$\begin{aligned} V_{i|j|k} + V_{j|i|k} &= 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik} + V^p(R_{jikp} + R_{ijkp}) \\ &\quad + y^s V^r_{|s}(P_{jikr} + P_{ijkr}) + \tau_{jk} y_i + \tau_{ik} y_j \\ &= 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik} - 2V^p C_{ijs} R^s{}_{kp} \\ &\quad - y^s V^r_{|s}(P_{jikr} + P_{ijkr}) + \tau_{jk} y_i + \tau_{ik} y_j. \end{aligned} \tag{42}$$

It follows from (11) that

$$P_{jikr} = g_{si} P_j{}^s{}_{kr} = g_{si} \frac{\partial \Gamma^s_{jk}}{\partial y^r} = \frac{\partial}{\partial y^r}(g_{si}\Gamma^s_{jk}) - 2\Gamma^s_{jk} C_{sir},$$

and

$$P_{ijkr} = \frac{\partial}{\partial y^r}(g_{sj}\Gamma^s_{ik}) - 2\Gamma^s_{ik} C_{sjr}.$$

Then

$$\begin{aligned} \frac{\partial}{\partial y^s}(g_{si}\Gamma^s_{jk} + g_{sj}\Gamma^s_{jk}) &= \frac{\partial}{\partial y^r}\frac{\delta}{\delta x^k} G_{ij} \\ &= \frac{\delta}{\delta x^k}(\frac{\partial}{\partial y^r} g_{ij}) - (\frac{\partial}{\partial y^r} N^l_k)\frac{\partial}{\partial y^l} g_{ij} \\ &= 2\frac{\delta}{\delta x^k} C_{ijr} - 2\Gamma^l_{kr} C_{ijl} - 2y^p P_k{}^l{}_{pr} C_{ijl}, \end{aligned} \tag{43}$$

which yields

$$\begin{aligned} P_{jikr} + P_{ijkr} &= 2(\frac{\delta}{\delta x^k} C_{ijr} - \Gamma^l_{kr} C_{ijl} - \Gamma^s_{jk} C_{sir} - \Gamma^s_{ik} C_{sjr} - y^p P_k{}^l{}_{pr} C_{ijl}) \\ &= 2C_{ijr|k} - 2y^p P_k{}^l{}_{pr} C_{ijl}. \end{aligned} \tag{44}$$

Hence (37) follows from (42) and (44). □

Before presenting the analogous characterizations of Finsler strongly projective vector fields as the ones in the Riemmannian geometry, we introduce the following Ricci identity in Finsler geometry.

Lemma 4.4 (Ricci type formula [10]). *For any vector field $v = v^i(x, y)\partial_i$ on a Finsler manifold, the exchange of horizontal covariant derivatives about the Chern connection satisfies*

$$v_{j|k|l} - v_{j|l|k} = R_j{}^m{}_{kl} v_m + R^m{}_{kl} v_{j;m}, \tag{45}$$

where $R_j{}^m{}_{lk}$ is the Chern-Riemannian curvature tensor.

One can deduce from Lemma 4.4 that for any vector $V = V^i(x)\partial_i$,

$$V_{i|j|k} - V_{i|k|j} = R_i{}^m{}_{jk} V_m + 2R^m{}_{kj} V_p C^p_{im}. \tag{46}$$

Now we prove Theorem 1.1, which is equal to:

Theorem 4.5. *Suppose V is a strongly projective field on Finsler manifolds.* (36) *and* (37) *are equivalent mutually.*

Proof. We only need to prove that (37) implies (36) by Lemma 4.3. It follows directly from (37) and Lemma 4.4 that

$$\begin{aligned}
&V_{i|j|k} + V_{j|i|k}\\
&= V_{i|j|k} + V_{j|k|i} + V_m R_j{}^m{}_{ik} + 2R^m{}_{ik} V_p C^p{}_{mj}\\
&= V_{i|j|k} + V_{k|j|i} + V_m R_j{}^m{}_{ik} + 2R^m{}_{ik} V_p C^p{}_{mj} + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij}\\
&\quad + \tau_{ij} y_j + \tau_{ij} y_k - 2V^p C_{kjs} R^s{}_{ip} - y^s V^r{}_{|s}(P_{jkir} + P_{kjir})\\
&= V_{i|j|k} - V_{k|i|j} - V_m R_k{}^m{}_{ji} - 2R^m{}_{ji} V_p C^p{}_{mk} + V_m R_j{}^m{}_{ik} + 2R^m{}_{ik} V_p C^p{}_{mj}\\
&\quad + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij} + \tau_{ij} y_j + \tau_{ij} y_k\\
&\quad - 2V^p C_{kjs} R^s{}_{ip} - y^s V^r{}_{|s}(P_{jkir} + P_{kjir})\\
&= V_{i|j|k} - V_{i|k|j} - V_m R_k{}^m{}_{ji} - 2R^m{}_{ji} V_p C^p{}_{mk} + V_m R_j{}^m{}_{ik} + 2R^m{}_{ik} V_p C^p{}_{mj}\\
&\quad - 2\psi_j g_{ik} - \psi_i g_{jk} - \psi_k g_{ij} - \tau_{ji} y_k - \tau_{jk} y_i + 2V^p C_{iks} R^s{}_{jp}\\
&\quad - y^s V^r{}_{|s}(P_{ikjr} + P_{kijr}) + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij} + \tau_{ij} y_j + \tau_{ij} y_k\\
&\quad - 2V^p C_{kjs} R^s{}_{ip} - y^s V^r{}_{|s}(P_{jkir} + P_{kjir})\\
&= V_{i|j|k} + V_{i|j|k} + V_m R_i{}^m{}_{kj} + 2R^m{}_{kj} V_p C^p{}_{mi} - V_m R_k{}^m{}_{ji} - 2R^m{}_{ji} V_p C^p{}_{mk}\\
&\quad + V_m R_j{}^m{}_{ik} + 2R^m{}_{ik} V_p C^p{}_{mj} - 2\psi_j g_{ik} - \psi_i g_{jk} - \psi_k g_{ij} - \tau_{ji} y_k - \tau_{jk} y_i\\
&\quad + 2V^p C_{iks} R^s{}_{jp} - y^s V^r{}_{|s}(P_{ikjr} + P_{kijr}) + 2\psi_i g_{jk} + \psi_j g_{ik} + \psi_k g_{ij}\\
&\quad + \tau_{ij} y_j + \tau_{ij} y_k - 2V^p C_{kjs} R^s{}_{ip} - y^s V^r{}_{|s}(P_{jkir} + P_{kjir})\\
&= 2V_{i|j|k} - \psi_j g_{ik} + \psi_i g_{jk} + \tau_{ik} y_j - \tau_{jk} y_i\\
&\quad + V_m(R_i{}^m{}_{kj} - R_k{}^m{}_{ji} + R_j{}^m{}_{ik}) + 2V^l(C_{lmi} R^m{}_{kj} - C_{lmk} R^m{}_{ji} + C_{lmj} R^m{}_{ik})\\
&\quad + 2V^p(C_{iks} R^s{}_{jp} - C_{kjs} R^s_{ip}) + y^s V^r{}_{|s}(P_{ikjr} + P_{kijr} - P_{jkir} - P_{kjir}).
\end{aligned}$$

Plugging it back into (37) yields that

$$
\begin{aligned}
(47)\qquad & 2(\psi_k g_{ij} + \psi_j g_{ik} + \tau_{jk} y_i) \\
&= 2V_{i|j|k} + V_m(R_i{}^m{}_{kj} - R_k{}^m{}_{ji} + R_j{}^m{}_{ik}) \\
&\quad + 2V^l(C_{lmi}R^m{}_{kj} - C_{lmk}R^m{}_{ji} + C_{lmj}R^m{}_{ik}) \\
&\quad + 2V^p(C_{iks}R^s{}_{jp} - C_{kjs}R^s_{ip} + C_{ijs}R^s{}_{kp}) \\
&\quad + y^s V^r_{|s}(P_{ikjr} + P_{kijr} - P_{jkir} - P_{kjir} + P_{jikr} + P_{ijkr}) \\
&= 2V_{i|j|k} + \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}.
\end{aligned}
$$

By (7), we can compute that

$$
\begin{aligned}
(48)\quad \mathcal{A} &= V_m(R_i{}^m{}_{kj} - R_k{}^m{}_{ji} + R_j{}^m{}_{ik}) \\
&= -2V^m R_{kmji} \\
&= -2V^m R_{jikm} + 2V^m(C_{kms}R^s{}_{ji} - C_{jis}R^s{}_{km} + C_{kis}R^s{}_{mj} + C_{mjs}R^s{}_{ki} \\
&\qquad\qquad + C_{ims}R^s{}_{jk} + C_{jks}R^s{}_{im}).
\end{aligned}
$$

From (12), it reduces that

$$
\begin{aligned}
(49)\qquad \mathcal{D} &= y^s V^r_{|s}(P_{ikjr} + P_{kijr} - P_{jkir} - P_{kjir}) \\
&= 2y^s V^r_{|s} P_{jikr}.
\end{aligned}
$$

Plugging (48) and (49) into (47) yields (36). □

5. Vanishing theorem of projective vector fields

In this section, we present the vanishing theorem of projective fields on Finsler manifolds. Firstly, we need the definition of the degenerate elliptic operator $\triangle^{SD}$ in Finsler geometry [10].

Definition 5.1 ([10])**.** A degenerate elliptic operator $\triangle^{SD}$ is defined as the second order derivative about the Chern connection contracting with a symmetric semi-positive definite matrix $a^{ij} = \frac{y^i y^j}{F^2}$, i.e.,

$$
(50)\qquad \triangle^{SD} := \frac{y^k y^l}{F^2}\left(\nabla_{\frac{\delta}{\delta x^k}} \nabla_{\frac{\delta}{\delta x^j}} - \nabla_{\nabla_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j}}\right),
$$

where ∇ means the horizontal covariant derivative with respect to the Chern connection.

The descriptions of strongly projective vector field are enough to prove vanishing theorems of projective vector field.

Lemma 5.2. *A Finsler projective vector field V satisfies that*

$$
(51)\qquad V_{i|0|0} + V^m R_{im} = 2\psi(y) y_i.
$$

Proof. Let V be a projective vector field on (M,F). Then V satisfies that (33) which is equal to $y^j y^k \mathcal{L}_{\hat{V}}\Gamma^i_{jk} = Py^i$. Since $(G^i)_{y^j y^k} = \Gamma^i_{jk} + y^p P_p{}^i{}_{jk}$ and $\mathcal{L}_{\hat{V}} y^p = 0$, a projective vector field satisfies that

$$y^k \mathcal{L}_{\hat{V}}\Gamma^i_{jk} = \psi_j y^i + P\delta^i_j, \tag{52}$$

where $\psi_j = \frac{\partial P}{\partial y^j}$. According to (41), (52) is equal to

$$y^j(V_{i|j|k} - V^p R_{jikp} + y^s V^r{}_{|s} P_{jikr} - y^j \psi_j g_{ik} - \psi_k y_i) = 0, \tag{53}$$

or

$$y^k(V_{i|j|k} - V^p R_{jikp} + y^s V^r{}_{|s} P_{jikr} - y^j \psi_j g_{ik} - \psi_k y_i) = 0. \tag{54}$$

Interchanging i,j in (54) and adding the two equations, one can obtain that

$$\begin{aligned} y^k[&V_{i|j|k} + V_{j|i|k} + 2V^p C_{ijs} R^s{}_{kp} + 2y^s V^r{}_{|s}(C_{ijr|k} - 2y^p P_k{}^l{}_{pr} C_{ijl}) \\ &- 2\psi_k g_{ij} - \psi_i g_{jk} - \psi_j g_{ik}] = 0. \end{aligned} \tag{55}$$

Applying Lemma 4.4 to (56) yields that

$$\begin{aligned} y^k[&V_{i|j|k} + V_{j|k|i} + R_j{}^m{}_{ik} V_m + 2R^m{}_{ki} V_p C^p_{jm} + 2V^p C_{ijs} R^s{}_{kp} \\ &+ 2y^s V^r{}_{|s}(C_{ijr|k} - 2y^p P_k{}^l{}_{pr} C_{ijl}) - 2\psi_k g_{ij} - \psi_i g_{jk} - \psi_j g_{ik}] = 0. \end{aligned} \tag{56}$$

Contracting (56) with y^j yields

$$V_{i|0|0} + V_{0|0|i} + V^m R_{im} = 3\psi(y) y_i + F^2 \psi_i. \tag{57}$$

On the other hand, contracting (53) with y^i, we find that

$$V_{0|0|k} - y^j \psi_j y_k - \psi_k F^2 = 0. \tag{58}$$

Plugging (58) back into (57) yields (51). □

We now prove the following theorem by using the flag curvature, which can be considered as the generalization of Tian's result in [14]. This work has also been contained in [1]. However, the method we utilized is different and more general.

Theorem 5.3 ([14]). *Let (M,F) be a compact Finsler manifold with non-positive flag curvature. Suppose V is a projective vector field on M. Then*

$$R_y(V,V) = 0, \quad \text{and} \quad \nabla V(y) = \lambda(x,y) y,$$

where $\lambda(x,y)$ is a scalar function on TM. Moreover, there is no nontrivial (nonzero) projective vector field on compact Finsler manifold M with negative flag curvature.

Proof. Taking derivatives with respect to the Chern connection of $|V|^2$ shows

$$D_k|V|^2 = 2V_{i|k}V^i, \quad \text{and} \quad D_l D_k |V|^2 = 2V_{i|k}V^i_{|l} + 2V^i V_{i|k|l}.$$

One can obtain from Lemma 5.2 that

$$\triangle^{SD}|V|^2 = 2|\nabla V(\frac{y}{F})|^2 + \frac{2V^i}{F^2} V_{i|0|0} \tag{59}$$

$$= 2|\nabla V(\frac{y}{F})|^2 - \frac{2V^i}{F^2}\big(V^m R_{im} - 2\psi(y)y_i\big)$$
$$= 2|\nabla V(\frac{y}{F})|^2 + 4\psi(\frac{y}{F})\frac{V_0}{F} - 2R_y(V,V).$$

On the other hand, taking derivatives with respect to the Chern connection of V_0^2 shows

$$D_k V_0^2 = 2V_0 V_{i|k} y^i, \quad \text{and} \quad D_l D_k(V_0^2) = 2V_{j|l}V_{i|k}y^i y^j + 2V_0 y^i V_{i|k|l}.$$

According to Lemma 5.2, it provides that

$$\triangle^{SD}(\frac{V_0}{F})^2 = 2(V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2 + 4\frac{V_0}{F}\psi(\frac{y}{F}). \tag{60}$$

Combing (59) and (60), we get

$$\triangle^{SD}\left(|V|^2 - (\frac{V_0}{F})^2\right) = 2\left(|\nabla V(\frac{y}{F})|^2 - (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2\right) - 2R_y(V,V). \tag{61}$$

By the Cauchy-Schwartz inequality, one can find

$$|V|^2 - (\frac{V_0}{F})^2 \geq 0, \quad \text{and} \quad |\nabla V(\frac{y}{F})|^2 - (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2 \geq 0. \tag{62}$$

When the flag curvature R_y is negative, the right hand side of (61) is strictly positive, unless the flag curvature R_y vanishes along V, that is $R_y(V,V) = 0$. Since the manifold is compact, at the maximum point of $|V|^2 - (\frac{V_0}{F})^2$, the left hand is non-positive, which implies $R_y(V,V) = 0$, i.e., $V = 0$ is a trivial vector field.

When the flag curvature R_y is non-positive, then $\triangle^{SD}\big(|V|^2 - (\frac{V_0}{F})^2\big) \geq 0$. By the strong maximum principle of the degenerate elliptic operator [7, 10], $|V|^2 - (\frac{V_0}{F})^2$ is a constant on the manifold. Plugging it back into (61) shows that

$$|\nabla V(\frac{y}{F})|^2 - (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2 = 0, \quad \text{and} \quad R_y(V,V) = 0.$$

The former equation holds if and only if the two vectors are linearly dependent, that is $\nabla V(\frac{y}{F}) = \lambda(x,y)\frac{y}{F}$ for some scaler $\lambda(x,y)$. Both $\nabla V(y) = \lambda(x,y)y$ and $R_y(V,V) = 0$ are the conclusions of Theorem 1.1 in [14]. □

One of the conclusion in Theorem 5.3 is $\nabla V(y) = \lambda(x,y)y$ for some scalar function $\lambda(x,y)$ on TM. We recall that a vector on Riemannian manifold (M,g,∇) is said to be parallel if $\nabla V = 0$ with respect to the Levi-Civita connection. So we give the following definition.

Definition 5.4. A vector field V on a Finsler manifold (M,F) is called almost parallel if $\nabla V(y) = \lambda(x,y)y$ for some scalar function $\lambda(x,y)$ on TM. In our theorem, ∇ denotes the Chern connection.

Using the terminology in Definition 2.1, we can further get Theorem 1.2 which reduces to Theorem 3.2 when the manifolds are Riemannian.

Proof of Theorem 1.2. Taking integral of both sides in (61) on the sphere bundle SM yields that

$$\begin{aligned}0 &= \int_{SM} \triangle^{SD}\left(|V|^2 - (\frac{V_0}{F})^2\right) d\omega \\ &= 2\int_{SM}\left(|\nabla V(\frac{y}{F})|^2 - (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2\right) d\omega - 2\int_{SM} R_y(V,V)d\omega,\end{aligned} \tag{63}$$

where $d\omega$ is the volume form on SM, which can induce the Holmes-Thompson volume form on M. The left hand side of (63) is equal to zero by the self-adjoint property of the $\triangle^{SD}$ [10]. (63) is equal to

$$\begin{aligned}0 &= \int_{SM} \triangle^{SD}\left(|V|^2 - (\frac{V_0}{F})^2\right) d\omega \\ &= 2\int_{SM}\left(|\nabla V(\frac{y}{F})|^2 - (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2\right) d\omega \\ &\quad - 2\int_M \left[\int_{S_xM} R_y(V,V)\frac{\sqrt{\det g_{ij}}}{\sqrt{\det a_{ij}}}d\nu\right]\sqrt{\det a_{ij}}dx \\ &= 2\vartheta - 2\int_M \widetilde{Ricci}(V,V)\sqrt{\det a_{ij}}dx,\end{aligned} \tag{64}$$

where $\vartheta = \int_{SM}(|\nabla V(\frac{y}{F})|^2 - (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2)d\omega$ is non-negative.

If the mean Ricci curvature is negative, i.e., for any $V \neq 0$, $\widetilde{Ricci}(V,V) < 0$, then (64) becomes

$$\begin{aligned}0 &= \int_{SM} \triangle^{SD}\left(|V|^2 - (\frac{V_0}{F})^2\right) d\omega \\ &= 2\vartheta - 2\int_M \widetilde{Ricci}(V,V)\sqrt{\det a_{ij}}dx > 0.\end{aligned} \tag{65}$$

It implies

$$\int_M \widetilde{Ricci}(V,V)\sqrt{\det a_{ij}}dx = 0, \quad \text{i.e.,} \quad \widetilde{Ricci}(V,V) = 0, \tag{66}$$

hence $V = 0$ is a trivial vector field. When the mean Ricci curvature is non-positive, then (64) becomes

$$\begin{aligned}0 &= \int_{SM} \triangle^{SD}\left(|V|^2 - (\frac{V_0}{F})^2\right) d\omega \\ &= 2\vartheta - 2\int_M \widetilde{Ricci}(V,V)\sqrt{\det a_{ij}}dx \geq 0.\end{aligned} \tag{67}$$

It implies that

$$\widetilde{Ricci}(V,V) = 0, \quad \text{and} \quad \vartheta = 0, \tag{68}$$

which means

$$|\nabla V(\frac{y}{F})|^2 = (V_{i|j}\frac{y^i}{F}\frac{y^j}{F})^2. \tag{69}$$

By the equality condition of the Cauchy-Schwartz inequality, we obtain again that

$$V_{i|j}\frac{y^i}{F} = \lambda(x,y)\frac{y^i}{F},$$

equivalently, it is

$$\nabla V(y) = \lambda(x,y)y,$$

where $\lambda(x,y)$ is a scalar function on TM. □

Remark 5.5. Theorem 1.2 implies Theorem 3.2, and Theorem 3.2 contains the Killing vanishing result in [15]. Therefore, the mean Ricci curvature $\widetilde{Ricci}$, which reduces to the Ricci curvature in Riemann geometry, is a suitable condition in the research of vanishing properties in Finsler geometry,

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References

[1] H. Akbar-Zadeh, *Initiation to Global Finslerian Geometry*, North-Holland Mathematical Library, **68**, Elsevier Science B.V., Amsterdam, 2006.

[2] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics, **200**, Springer-Verlag, New York, 2000.

[3] G. S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific Lecture Notes in Physics, **46**, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

[4] B. Li, *On the classification of projectively flat Finsler metrics with constant flag curvature*, Adv. Math. **257** (2014), 266–284.

[5] R. L. Lovas, *Affine and projective vector fields on spray manifolds*, Period. Math. Hungar. **48** (2004), no. 1-2, 165–179.

[6] V. S. Matveev, H. Rademacher, M. Troyanov, and A. Zeghib, *Finsler conformal Lichnerowicz-Obata conjecture*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 3, 937–949.

[7] D. D. Monticelli, *Maximum principles and the method of moving planes for a class of degenerate elliptic linear operators*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 3, 611–654.

[8] P. Petersen, *Riemannian Geometry*, Second edition, Graduate Texts in Mathematics, **171**, Springer, New York, 2006.

[9] M. Schottenloher, *A mathematical Introduction to Conformal Field Theory*, translated from the German, Lecture Notes in Physics. New Series m: Monographs, **43**, Springer-Verlag, Berlin, 1997.

[10] B. Shen, *Vanishing of Killing vector fields on Finsler manifolds*, Kodai Math. J., to be appeared.

[11] Z. Shen, *Lectures on Finsler Geometry*, World Scientific Publishing Co., Singapore, 2001.

[12] ———, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[13] ———, *Projectively flat Finsler metrics of constant flag curvature*, Trans. Amer. Math. Soc. **355** (2003), no. 4, 1713–1728.

[14] H. Tian, *Projective vector fields on Finsler manifolds*, Appl. Math. J. Chinese Univ. Ser. B **29** (2014), no. 2, 217–229.

[15] H. H. Wu, *The Bochner technique in differential geometry*, Math. Rep. **3** (1988), no. 2, i–xii and 289–538.

[16] K. Yano, *Integral Formulas in Riemannian Geometry*, Pure and Applied Mathematics, No. 1, Marcel Dekker, Inc., New York, 1970.

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