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# RECURRENCE RELATIONS FOR HIGHER ORDER MOMENTS OF A COMPOUND BINOMIAL RANDOM VARIABLE

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Abstract. We present new recurrence formulas for the raw and central moments of a compound binomial random variable. Our approach involves relating two compound binomial random variables that have parameters with a difference of 1 for the number of trials, but which have the same parameters for the success probability for each trial. As a consequence of our recursions, the raw and central moments of a binomial random variable are obtained in a recursive manner without the use of Stirling numbers.

## 1. Introduction

Let $\{X_i, i = 1, 2, \dots\}$ be a sequence of independent and identically distributed random variables. A compound random variable $S_N$ is defined as

$$S_N = X_1 + X_2 + \cdots + X_N,$$

where $N$ is a non-negative integer-valued random variable and is independent of $\{X_i, i = 1, 2, \dots\}$. If $N = 0$, then $S_N = 0$. This compound random variable has received special attention in collective risk theory, in which the random variables $X_i, N$, and $S_N$ represent the amount of a claim, the number of claims in a certain period, and the aggregate claims of a portfolio, respectively (see e.g., [8]).

In this paper, we consider $S_N$ when $N$ follows a binomial distribution. In this case, $S_N$ is called a compound binomial random variable. The compound binomial distribution reduces to a binomial distribution if $X_i$ is degenerate with $\mathbb{P}(X_i = 1) = 1$. In this paper, we address the moments of a compound binomial

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random variable. We also address the moments of a binomial random variable as a special case of a compound binomial random variable.

For some class of compound random variables, recurrence formulas for higher-order moments have been presented by De Pril [2], Hesselager [6], Murat and Szynal [9], [10], and Sundt [13]. In particular, the results in [2], [9], [10], [13] apply to compound binomial random variables (we present the details in Section 5). Grubbström and Tang [5] presented closed-form formulas for the moments of a compound random variable having $X_i \geq 0$. In computing the moments of a binomial random variable, Stirling numbers have often been utilized. Bényi [1] and Griffiths [4] derived recurrence formulas for the moments of a binomial random variable using Stirling numbers of the first kind. Knoblauch [7] derived closed-form formulas for the moments of a binomial random variable using Stirling numbers of the second kind. González and Santana [3] presented a new recurrence formula for the moments of a binomial random variable using a combinatorial identity.

The main purpose of this paper is two-fold. (i) We derive a new recurrence formula for the raw and central moments of a compound binomial random variable. (ii) From this recursion, one can obtain the raw and central moments of a binomial random variable in a recursive manner without the use of Stirling numbers. To this end, in Section 2, we first give a relation between two compound binomial random variables that have parameters with a difference of 1 for the number of trials, but which have the same parameters for the success probability for each trial. Based on this relation, we next derive our main results on the raw and central moments in Sections 3 and 4, respectively. Finally, in Section 5, we deduce recursions from results in the existing literature on the moments of a compound random variable.

## 2. Preliminary lemma

Let $N_1$ be a binomial random variable with parameters $n$ and $p$, and $N_2$ be the one with parameters $n-1$ and $p$, where $n \in \{2, 3, \dots\}$ and $p \in [0, 1]$. Lemma 2.1 presents a relation between $S_{N_1}$ and $S_{N_2}$.

**Lemma 2.1.** *For any function $f : \mathbb{R} \to \mathbb{R}$ and constant $c \in \mathbb{R}$,*

$$\mathbb{E}\big[(S_{N_1} - c)f(S_{N_1} - c)\big] = \mathbb{E}[N_1]\mathbb{E}\big[Xf(S_{N_2} + X - c)\big] - c\mathbb{E}\big[f(S_{N_1} - c)\big],$$

*where $X \stackrel{d}{=} X_i \, (i \in \mathbb{N})$ and is independent of $S_{N_2}$.*

*Proof.* By the linearity of expectation, we immediately have

$$\mathbb{E}\big[(S_{N_1} - c)f(S_{N_1} - c)\big] = \mathbb{E}\big[S_{N_1}f(S_{N_1} - c)\big] - c\mathbb{E}\big[f(S_{N_1} - c)\big]. \tag{2.1}$$

In the following, we will show that the first term on the right-hand side of (2.1) satisfies

$$\mathbb{E}\big[S_{N_1}f(S_{N_1} - c)\big] = \mathbb{E}[N_1]\mathbb{E}\big[Xf(S_{N_2} + X - c)\big].$$

By conditioning on $N_1$,

$$\mathbb{E}\big[S_{N_1} f(S_{N_1} - c)\big] = \sum_{k=0}^{n} \mathbb{P}(N_1 = k)\, \mathbb{E}\big[S_{N_1} f(S_{N_1} - c) \,|\, N_1 = k\big]$$
$$= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}\, \mathbb{E}\big[S_k f(S_k - c)\big], \tag{2.2}$$

where the second equality follows from the independence of $\{X_i, i = 1, 2, \dots\}$ and $N_1$. Since $X_1, X_2, \dots$ are independent and identically distributed, the expectation $\mathbb{E}\big[S_k f(S_k - c)\big]$ is simplified as

$$\mathbb{E}\big[S_k f(S_k - c)\big] = \sum_{i=1}^{k} \mathbb{E}\left[X_i f\left(\sum_{j=1}^{k} X_j - c\right)\right] = k\, \mathbb{E}\big[X_k f(S_k - c)\big]. \tag{2.3}$$

Substituting (2.3) into (2.2) and then using the change of variables $l = k - 1$,

$$\mathbb{E}\big[S_{N_1} f(S_{N_1} - c)\big] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}\, k\, \mathbb{E}\big[X_k f(S_k - c)\big]$$
$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k}\, \mathbb{E}\big[X_k f(S_k - c)\big]$$
$$= \mathbb{E}[N_1] \sum_{k=1}^{n} \mathbb{P}(N_2 = k-1) \mathbb{E}\big[X_k f(S_k - c)\big]$$
$$= \mathbb{E}[N_1] \sum_{l=0}^{n-1} \mathbb{P}(N_2 = l) \mathbb{E}\big[X_{l+1} f(S_{l+1} - c)\big]. \tag{2.4}$$

By the independence of $N_2$ and $\{X_i, i = 1, 2, \dots\}$, the expectation on the right-hand side of (2.4) can be expressed as

$$\mathbb{E}\big[X_{l+1} f(S_{l+1} - c)\big] = \mathbb{E}\big[X_{l+1} f(S_l + X_{l+1} - c)\big]$$
$$= \mathbb{E}\big[X_{l+1} f(S_l + X_{l+1} - c) \,|\, N_2 = l\big]$$
$$= \mathbb{E}\big[X_{N_2+1} f(S_{N_2} + X_{N_2+1} - c) \,|\, N_2 = l\big]$$
$$= \mathbb{E}\big[X f(S_{N_2} + X - c) \,|\, N_2 = l\big]. \tag{2.5}$$

Therefore, substituting (2.5) into (2.4), we have

$$\mathbb{E}\big[S_{N_1} f(S_{N_1} - c)\big] = \mathbb{E}[N_1] \sum_{l=0}^{n-1} \mathbb{P}(N_2 = l) \mathbb{E}\big[X f(S_{N_2} + X - c) \,|\, N_2 = l\big]$$
$$= \mathbb{E}[N_1] \mathbb{E}\Big[\mathbb{E}\big[X f(S_{N_2} + X - c) \,|\, N_2\big]\Big]$$
$$= \mathbb{E}[N_1] \mathbb{E}\big[X f(S_{N_2} + X - c)\big].$$

This completes the proof. □

Note that Peköz and Ross [12, Theorem 2.1] presented an identity similar to Lemma 2.1 in the case $c = 0$ and utilized the identity to obtain recurrence formulas for the probability mass function of compound random variables having positive integer-valued $X_i$.

## 3. Raw moments

Using Lemma 2.1 with $f(x) = x^k$ and $c = 0$, we can derive a recurrence formula for the raw moments as follows.

**Theorem 3.1.** *For a binomial random variable $N$ with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, let $\alpha_k(n,p) = \mathbb{E}[(S_N)^k]$ and $\mu_k(n,p) = \mathbb{E}[N^k]$ denote the $k$th raw moments of $S_N$ and $N$, respectively. Then,*

$$\alpha_{k+1}(n,p) = np \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}[X^{k+1-i}] \alpha_i(n-1,p),$$

$$\mu_{k+1}(n,p) = np \sum_{i=0}^{k} \binom{k}{i} \mu_i(n-1,p),$$

*where $X \overset{d}{=} X_i$ $(i \in \mathbb{N})$.*

*Proof.* We apply $f(x) = x^k$ and $c = 0$ to Lemma 2.1. Then,

$$\mathbb{E}\big[S_{N_1}(S_{N_1})^k\big] = \mathbb{E}[N_1]\mathbb{E}\big[X(S_{N_2} + X)^k\big]. \tag{3.1}$$

Since $N_1 \overset{d}{=} N$, the left-hand side of (3.1) can be expressed as

$$\mathbb{E}\big[S_{N_1}(S_{N_1})^k\big] = \mathbb{E}\big[(S_N)^{k+1}\big] = \alpha_{k+1}(n,p). \tag{3.2}$$

In addition, the right-hand side of (3.1) satisfies

$$\begin{aligned} \mathbb{E}[N_1]\mathbb{E}\big[X(S_{N_2}+X)^k\big] &= np\,\mathbb{E}\left[X \sum_{i=0}^{k} \binom{k}{i} (S_{N_2})^i X^{k-i}\right] \\ &= np \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}\left[(S_{N_2})^i\right] \mathbb{E}[X^{k+1-i}] \\ &= np \sum_{i=0}^{k} \binom{k}{i} \alpha_i(n-1,p)\, \mathbb{E}[X^{k+1-i}], \end{aligned} \tag{3.3}$$

where the second equality follows from the independence of $S_{N_2}$ and $X$. Substituting (3.2) and (3.3) into (3.1) gives the first recursion in Theorem 3.1.

If $X_i$ is degenerate with $\mathbb{P}(X_i = 1) = 1$, we have $\alpha_k(n,p) = \mu_k(n,p)$ and $\mathbb{E}[X^j] = 1$ for all $j \in \mathbb{N}$. Hence, the second recursion in Theorem 3.1 follows immediately from the first recursion for $X_i$ such that $\mathbb{P}(X_i = 1) = 1$. □

Starting from the initial value $\alpha_0(n,p)=1$, we can find $\alpha_k(n,p)$ in a recursive manner using Theorem 3.1. For example, we have

$$\begin{aligned}
\alpha_0(n,p) &= 1,\\
\alpha_1(n,p) &= np\binom{0}{0}\mathbb{E}[X^1]\underbrace{\alpha_0(n-1,p)}_{=1} = np\mathbb{E}[X],\\
\alpha_2(n,p) &= np\Bigg[\binom{1}{0}\mathbb{E}[X^2]\underbrace{\alpha_0(n-1,p)}_{=1}+\binom{1}{1}\mathbb{E}[X^1]\underbrace{\alpha_1(n-1,p)}_{=(n-1)p\mathbb{E}[X]}\Bigg]\\
&= np\mathbb{E}[X^2]+n(n-1)p^2(\mathbb{E}[X])^2,\\
\alpha_3(n,p) &= np\Bigg[\binom{2}{0}\mathbb{E}[X^3]\underbrace{\alpha_0(n-1,p)}_{=1}+\binom{2}{1}\mathbb{E}[X^2]\underbrace{\alpha_1(n-1,p)}_{=(n-1)p\mathbb{E}[X]}\\
&\qquad+\binom{2}{2}\mathbb{E}[X^1]\underbrace{\alpha_2(n-1,p)}_{=(n-1)p\mathbb{E}[X^2]+(n-1)(n-2)p^2(\mathbb{E}[X])^2}\Bigg]\\
&= np\mathbb{E}[X^3]+3n(n-1)p^2\mathbb{E}[X^2]\mathbb{E}[X]+n(n-1)(n-2)p^3(\mathbb{E}[X])^3.
\end{aligned}$$

Similarly as above, we can find $\mu_k(n,p)$ in a recursive manner using Theorem 3.1. For example, we have

$$\begin{aligned}
\mu_0(n,p) &= 1,\\
\mu_1(n,p) &= np\binom{0}{0}\underbrace{\mu_0(n-1,p)}_{=1} = np,\\
\mu_2(n,p) &= np\Bigg[\binom{1}{0}\underbrace{\mu_0(n-1,p)}_{=1}+\binom{1}{1}\underbrace{\mu_1(n-1,p)}_{=(n-1)p}\Bigg]\\
&= np+n(n-1)p^2,\\
\mu_3(n,p) &= np\Bigg[\binom{2}{0}\underbrace{\mu_0(n-1,p)}_{=1}+\binom{2}{1}\underbrace{\mu_1(n-1,p)}_{=(n-1)p}+\binom{2}{2}\underbrace{\mu_2(n-1,p)}_{=(n-1)p+(n-1)(n-2)p^2}\Bigg]\\
&= np+3n(n-1)p^2+n(n-1)(n-2)p^3.
\end{aligned}$$

## 4. Central moments

Using Lemma 2.1 with $f(x)=x^k$ and $c=\mathbb{E}[S_N]$, we can derive a recurrence formula for the central moments as follows.

**Theorem 4.1.** *For a binomial random variable $N$ with parameters $n\in\mathbb{N}$ and $p\in[0,1]$, let $\beta_k(n,p)=\mathbb{E}\big[(S_N-\mathbb{E}[S_N])^k\big]$ and $\sigma_k(n,p)=\mathbb{E}\big[(N-\mathbb{E}[N])^k\big]$*

*denote the kth central moments of $S_N$ and $N$, respectively. Then,*

$$\beta_{k+1}(n,p) = np\left\{\sum_{i=0}^{k}\binom{k}{i}\mathbb{E}\big[X(X-p\mathbb{E}[X])^{k-i}\big]\beta_i(n-1,p) - \mathbb{E}[X]\beta_k(n,p)\right\},$$

$$\sigma_{k+1}(n,p) = np\left\{\sum_{i=0}^{k}\binom{k}{i}(1-p)^{k-i}\sigma_i(n-1,p) - \sigma_k(n,p)\right\},$$

*where $X \overset{d}{=} X_i$ $(i \in \mathbb{N})$.*

*Proof.* The proof is similar to the one used for Theorem 3.1. We first apply $f(x) = x^k$ and $c = \mathbb{E}[S_{N_1}]$ to Lemma 2.1. Then,

$$\begin{aligned}&\mathbb{E}\big[(S_{N_1} - \mathbb{E}[S_{N_1}])(S_{N_1} - \mathbb{E}[S_{N_1}])^k\big]\\ &\quad= \mathbb{E}[N_1]\mathbb{E}\big[X(S_{N_2} + X - \mathbb{E}[S_{N_1}])^k\big] - \mathbb{E}[S_{N_1}]\mathbb{E}\big[(S_{N_1} - \mathbb{E}[S_{N_1}])^k\big].\end{aligned} \tag{4.1}$$

Since $N_1 \overset{\mathrm{d}}{=} N$, the left-hand side of (4.1) can be expressed as

$$\mathbb{E}\big[(S_{N_1} - \mathbb{E}[S_{N_1}])(S_{N_1} - \mathbb{E}[S_{N_1}])^k\big] = \mathbb{E}\big[(S_N - \mathbb{E}[S_N])^{k+1}\big] = \beta_{k+1}(n,p). \tag{4.2}$$

The first term on the right-hand side of (4.1) satisfies

$$\begin{aligned}&\mathbb{E}[N_1]\mathbb{E}\big[X(S_{N_2} + X - \mathbb{E}[S_{N_1}])^k\big]\\ &\quad= np\,\mathbb{E}\big[X(S_{N_2} - \mathbb{E}[S_{N_2}] + X + \mathbb{E}[S_{N_2}] - \mathbb{E}[S_{N_1}])^k\big]\\ &\quad= np\,\mathbb{E}\left[X\sum_{i=0}^{k}\binom{k}{i}\left(S_{N_2} - \mathbb{E}[S_{N_2}]\right)^i\left(X + \mathbb{E}[S_{N_2}] - \mathbb{E}[S_{N_1}]\right)^{k-i}\right]\\ &\quad= np\sum_{i=0}^{k}\binom{k}{i}\mathbb{E}\big[(S_{N_2} - \mathbb{E}[S_{N_2}])^i\big]\mathbb{E}\big[X(X + \mathbb{E}[S_{N_2}] - \mathbb{E}[S_{N_1}])^{k-i}\big].\end{aligned}$$

Note that $\mathbb{E}\big[(S_{N_2} - \mathbb{E}[S_{N_2}])^i\big] = \beta_i(n-1,p)$. In addition, by Wald's equality,

$$\mathbb{E}[S_{N_2}] - \mathbb{E}[S_{N_1}] = (n-1)p\mathbb{E}[X] - np\mathbb{E}[X] = -p\mathbb{E}[X].$$

Hence, the first term on the right-hand side of (4.1) reduces to

$$\begin{aligned}&\mathbb{E}[N_1]\mathbb{E}\big[X(S_{N_2} + X - \mathbb{E}[S_{N_1}])^k\big]\\ &\quad= np\sum_{i=0}^{k}\binom{k}{i}\beta_i(n-1,p)\mathbb{E}\big[X(X - p\mathbb{E}[X])^{k-i}\big],\end{aligned} \tag{4.3}$$

whereas the second term on the right-hand side of (4.1) reduces to

$$\mathbb{E}[S_{N_1}]\mathbb{E}\big[(S_{N_1} - \mathbb{E}[S_{N_1}])^k\big] = np\,\mathbb{E}[X]\beta_k(n,p). \tag{4.4}$$

Substituting (4.2), (4.3), and (4.4) into (4.1) gives the first recursion in Theorem 4.1. The second recursion in Theorem 4.1 follows immediately from the first recursion for $X_i$ such that $\mathbb{P}(X_i = 1) = 1$. □

Starting from the initial value $\beta_0(n,p) = 1$, we can find $\beta_k(n,p)$ in a recursive manner using Theorem 4.1. For example, we have

$$
\begin{aligned}
\beta_0(n,p) &= 1,\\
\beta_1(n,p) &= np\left[\binom{0}{0}\mathbb{E}[X]\underbrace{\beta_0(n-1,p)}_{=1} - \mathbb{E}[X]\underbrace{\beta_0(n,p)}_{=1}\right] = 0,\\
\beta_2(n,p) &= np\left[\binom{1}{0}\mathbb{E}\big[X(X-p\mathbb{E}[X])\big]\underbrace{\beta_0(n-1,p)}_{=1} + \binom{1}{1}\mathbb{E}[X]\underbrace{\beta_1(n-1,p)}_{=0}\right.\\
&\qquad\left. - \mathbb{E}[X]\underbrace{\beta_1(n,p)}_{=0}\right]\\
&= np\,\mathbb{E}[X^2] - np^2(\mathbb{E}[X])^2,\\
\beta_3(n,p) &= np\left[\binom{2}{0}\mathbb{E}\big[X(X-p\mathbb{E}[X])^2\big]\underbrace{\beta_0(n-1,p)}_{=1}\right.\\
&\qquad + \binom{2}{1}\mathbb{E}\big[X(X-p\mathbb{E}[X])\big]\underbrace{\beta_1(n-1,p)}_{=0}\\
&\qquad\left. + \binom{2}{2}\mathbb{E}[X]\underbrace{\beta_2(n-1,p)}_{=(n-1)p\mathbb{E}[X^2]-(n-1)p^2(\mathbb{E}[X])^2} - \mathbb{E}[X]\underbrace{\beta_2(n,p)}_{=np\mathbb{E}[X^2]-np^2(\mathbb{E}[X])^2}\right]\\
&= np\mathbb{E}[X^3] - 3np^2\mathbb{E}[X^2]\mathbb{E}[X] + 2np^3(\mathbb{E}[X])^3.
\end{aligned}
$$

Similarly as above, we can find $\sigma_k(n,p)$ in a recursive manner using Theorem 4.1. For example, we have

$$
\begin{aligned}
\sigma_0(n,p) &= 1,\\
\sigma_1(n,p) &= np\left[\binom{0}{0}\underbrace{\sigma_0(n-1,p)}_{=1} - \underbrace{\sigma_0(n,p)}_{=1}\right] = 0,\\
\sigma_2(n,p) &= np\left[\binom{1}{0}(1-p)\underbrace{\sigma_0(n-1,p)}_{=1} + \binom{1}{1}\underbrace{\sigma_1(n-1,p)}_{=0} - \underbrace{\sigma_1(n,p)}_{=0}\right]\\
&= np - np^2,\\
\sigma_3(n,p) &= np\left[\binom{2}{0}(1-p)^2\underbrace{\sigma_0(n-1,p)}_{=1} + \binom{2}{1}(1-p)\underbrace{\sigma_1(n-1,p)}_{=0}\right.\\
&\qquad\left. + \binom{2}{2}\underbrace{\sigma_2(n-1,p)}_{=(n-1)p-(n-1)p^2} - \underbrace{\sigma_2(n,p)}_{=np-np^2}\right]\\
&= np - 3np^2 + 2np^3.
\end{aligned}
$$

## 5. Discussion

In this section, we deduce recursions from results in the existing literature on the moments of a compound random variable.

Consider a class of random variables satisfying

$$\mathbb{P}(N=n)=\left(a+\frac{b}{n}\right)\mathbb{P}(N=n-1), \qquad n=1,2,\dots, \tag{5.1}$$

for some constants $a<1$ and $a+b\geq 0$. This family of distributions was considered by Panjer [11] and is often referred to as Panjer's class. Note that a binomial random variable with parameters $n$ and $p$ belongs to the Panjer's class with $a=-p/(1-p)$ and $b=(n+1)p/(1-p)$. For a compound random variable $S_N$ with $N$ satisfying (5.1), the following result was proved in various ways by De Pril [2, Theorem], Murat and Szynal [9, Theorem 3.1], [10, Theorem 4.1], and Sundt [13, Equations (6) and (8)]:

$$\begin{aligned}(1-a)&\mathbb{E}[(S_N-c)^{k+1}]\\ &=\sum_{i=0}^{k}\binom{k}{i}\left\{\left(\frac{k+1}{i+1}a+b\right)\mathbb{E}[X^{i+1}]+ac\mathbb{E}[X^i]\right\}\mathbb{E}[(S_N-c)^{k-i}]\\ &\quad-c\mathbb{E}[(S_N-c)^k].\end{aligned} \tag{5.2}$$

Applying (5.2) with $c=0$, $a=-p/(1-p)$ and $b=(n+1)p/(1-p)$ gives a recurrence formula for the raw moments $\alpha_k(n,p)$ and $\mu_k(n,p)$ as follows:

$$\alpha_{k+1}(n,p)=p\sum_{i=0}^{k}\binom{k}{i}\left(n-\frac{k-i}{i+1}\right)\mathbb{E}[X^{i+1}]\alpha_{k-i}(n,p),$$

$$\mu_{k+1}(n,p)=p\sum_{i=0}^{k}\binom{k}{i}\left(n-\frac{k-i}{i+1}\right)\mu_{k-i}(n,p).$$

Similarly, the central moments $\beta_k(n,p)$ and $\sigma_k(n,p)$ are obtained as follows:

$$\begin{aligned}\beta_{k+1}(n,p)=p&\sum_{i=0}^{k}\binom{k}{i}\left\{\left(n-\frac{k-i}{i+1}\right)\mathbb{E}[X^{i+1}]-np\mathbb{E}[X]\mathbb{E}[X^i]\right\}\beta_{k-i}(n,p)\\ &-np(1-p)\mathbb{E}[X]\beta_k(n,p),\end{aligned}$$

$$\sigma_{k+1}(n,p)=p\sum_{i=0}^{k}\binom{k}{i}\left\{n(1-p)-\frac{k-i}{i+1}\right\}\sigma_{k-i}(n,p)-np(1-p)\sigma_k(n,p).$$

Clearly, the recurrence relations presented in Theorems 3.1 and 4.1 have different expressions than those deduced from (5.2). This difference arises due to the following reason. In order to derive (5.2), moment-generating functions or some identities were used for fixed $a$ and $b$ (i.e., fixed $n$ and $p$). In this paper, we take a different approach by relating two compound binomial random variables with parameter pairs $(n,p)$ and $(n-1,p)$ for the number of trials and the success probability (i.e., varying $n$ and fixed $p$). This approach leads to a new recurrence formula for the moments of a compound binomial random variable. Our new formula can be useful when dealing with compound binomial random variables with varying number of trials.

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