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FUZZY NONLINEAR RANDOM VARIATIONAL INCLUSION PROBLEMS INVOLVING ORDERED RME-MULTIVALUED MAPPING IN BANACH SPACES

Jong Kyu Kim and Salahuddin

Abstract. In this paper, we consider a *fuzzy nonlinear random variational inclusion problems* involving ordered RME-multivalued mapping in ordered Banach spaces. By using the random relaxed resolvent operator and its properties, we suggest an random iterative algorithm. Finally both the existence of the random solution of the original problem and the convergence of the random iterative sequences generated by random algorithm are proved.

1. Introduction

In 1972, the number of solutions of nonlinear equation has been introduced and studied by Amann [7] and recent years, the nonlinear mappings, fixed point theories and their applications have been extensively studied in ordered Banach spaces, [16, 17]. Very recently, Li [19, 20, 21] has studied the approximation solution for general nonlinear ordered variational inequalities and ordered equations in ordered Banach spaces.

Uncertain or imprecise data are inherent and pervasive in many important applications in the areas such as business management, computer sciences, engineering, environment, social sciences and medical sciences. Uncertain data in those applications could be caused by data randomness, information incompleteness, limitations of measuring instrument, delayed data updates, and so forth. Due to the importance of those applications and the rapidly increasing amount of uncertain data collected and accumulated, research on effective and efficient techniques that are dedicated to modeling uncertain data and tackling uncertainties has attracted much interest in recent years and yet remained challenging at large. There have been a great amount of research and applications

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in the literature concerning some special tools like probability theory, (intuitionistic) fuzzy set theory, rough set theory, vague set theory, random game theory, random networks, economics theory and interval mathematics. However, all of these have theirs advantages as well as inherent limitations in dealing with uncertainties. One major problem shared by those theories is their incompatibility with the parameterizations tools, *see* [1, 2, 3, 11, 14].
Fuzzy sets were founded by Professor L. A. Zadeh in year 1965 [29]. The address of fuzzy set theory, since its introduction has been dramatic and breathtaking, several research papers have published in different journals devoted entirely to theoretical and application aspects of fuzzy sets. In 1989, Chang and Zhu [10] introduced the concept of variational inequalities in fuzzy mappings in abstract spaces and investigated existence theorem for some kinds of variational inequalities for fuzzy mappings. Afterwards, on several kinds of variational inequalities, variational inclusions and complementarity problems for fuzzy mappings were considered and studied by many authors see for instance, Ahmad and Salahuddin [4], Agarwal *et al.* [5, 6], Anastassiou *et al.* [8], Chang and Salahuddin [12], Ding and Park [15], Huang [18], Lee *et al.* [22, 23, 24], Salahuddin [25], Salahuddin and Verma [26], Salahuddin *et al.* [27] and Zhang and Bi [28], *etc.*
Inspired by the above recent research works, here we solve a *fuzzy nonlinear random variational inclusion problems* involving ordered RME-multivalued mapping in ordered Banach spaces. By using the relaxed randomize resolvent operator and its properties, we construct a new random iterative algorithm. Finally, both the existence of the random solution of *fuzzy nonlinear random variational inclusion problems* and the convergence of the random iterative sequences generated by random iterative algorithm are proved.

2. Preliminaries

Throughout this work, we assume that (Ω, Σ, μ) is a complete σ- finite measurable space and X is a separable real Banach space endowed with dual space X^*, the norm $\|\cdot\|$ and the dual pair $\langle\cdot,\cdot\rangle$ between X and X^*. We denote by $\mathfrak{B}(X)$ the class of Borel σ- field in X. Let 2^X and $CB(X)$ denote the family of all nonempty subset of X and the family of all nonempty bounded closed sets of X, *respectively.*

Definition 1. A mapping $x : \Omega \to X$ is said to be the measurable if for any $\mathrm{B} \in \mathfrak{B}(X), \{t \in \Omega, x(t) \in \mathrm{B}\} \in \Sigma$.

Definition 2. A mapping $f : \Omega \times X \to X$ is called a random operator if for any $x \in X, f(t, x) = x(t)$ is a measurable. A random operator f is said to be continuous if for any $t \in \Omega$, the mapping $f(t, \cdot) : X \to X$ is continuous.

Definition 3. A mapping $T : \Omega \times X \to 2^X$ is said to be measurable if for any $\mathrm{B} \in \mathfrak{B}(X), T^{-1}(\mathrm{B}) = \{t \in \Omega, T(t) \cap \mathrm{B} \neq \emptyset\} \in \Sigma$.

Definition 4. A mapping $u : \Omega \to X$ is called a measurable selection of a measurable mapping $T : \Omega \to 2^X$ if u is a measurable and for any $t \in \Omega, u(t) \in T(t)$.

Definition 5. A mapping $T : \Omega \times X \to 2^X$ is called a random multivalued mapping if for any $x \in X, T(\cdot, x)$ is a measurable. A random set valued mapping $T : \Omega \times X \to CB(X)$ is said to be $\mathcal{H}$-continuous if for any $t \in \Omega, T(t, \cdot)$ is continuous in Hausdorff metric.

Definition 6. A fuzzy mapping $F : \Omega \to \mathfrak{F}(X)$ is called measurable if for any $\alpha \in (0, 1), (F(\cdot))_\alpha : \Omega \to 2^X$ is a measurable mapping.

Definition 7. A fuzzy mapping $F : \Omega \times X \to \mathfrak{F}(X)$ is called random fuzzy mapping if for any $x \in X, F(\cdot, x) \to \mathfrak{F}(X)$ is a measurable fuzzy mapping.

Let $\mathfrak{F}(X)$ be a collection of all fuzzy sets over X. A mapping $F : X \to \mathfrak{F}(X)$ is called a fuzzy mapping on X. If F is a fuzzy mapping on X, the $F(x)$ (denote it by F_x, in the sequel) is a fuzzy set on X and $(F_x)(y)$ is the membership function of y in F_x. Let $N \in \mathfrak{F}(x), q \in [0, 1]$, then the set

$$(N)_q = \{x \in X : N(x) \geq q\}$$

is called a q-cut set of N.
A fuzzy mapping $F : X \to \mathfrak{F}(X)$ is said to be closed if for each $x \in X$, the function $y \to F_x(y)$ is upper semi continuous, that is, for any given net $\{y_\alpha\} \subset X$, satisfying $y_\alpha \to y_0 \in X$ we have

$$\limsup_\alpha F_x(y_\alpha) \leq F_x(y_0).$$

Let $T : \Omega \times X \to \mathfrak{F}(X)$ be the random fuzzy mapping satisfying the following condition (S):
(S): There exists a mapping $a : X \to [0, 1]$ such that for each $t \in \Omega, x \in X$ the set $(T_{t,x})_{a(x)} = \{t \in \Omega, y \in X : T_{t,x}(y) \geq a(x)\}$ is nonempty bounded subset of X. If T is a closed fuzzy mapping satisfying the condition (S), then for each $t \in \Omega, x \in X, (T_{t,x})_{a(x)} \in CB(X)$. In fact let $\{y_\alpha\} \subset (T_{t,x})_{a(x)}$ be a net and $y_\alpha \to y_0 \in X$, then $(T_{t,x})_{a(x)} \geq a(x)$, for each α. Since T is closed, we have

$$T_{t,x}(y_0) \geq \limsup_\alpha T_{t,x}(y_\alpha) \geq a(x).$$

which implies that

$$y_0 \in (T_{t,x})_{a(x)} \text{ and so } (T_{t,x})_{a(x)} \in CB(X).$$

Let X be a real ordered Banach space with a norm $\| \cdot \|$ and θ be a zero vector in X. Let C be a normal cone of X and $\leq$ be a partial ordered relation defined by the cone C and κ be the normal constant of C. Let $A, B, T : \Omega \times X \to \mathfrak{F}(X)$ be the closed fuzzy mappings satisfying the following condition (S). Then there exist mappings $a, b, c : X \to [0, 1]$, such that for each $t \in$

$\Omega, x \in X, (A_{t,x})_{a(x)} \in CB(X), (B_{t,x})_{b(x)}, (T_{t,x})_{c(x)}$. Thus we define the random multivalued fuzzy mappings by

$\tilde{A}(t,x) = (A_{t,x})_{a(x)}, \tilde{B}(t,x) = (B_{t,x})_{b(x)}$ and $\tilde{T}(t,x) = (T_{t,x})_{c(x)}, \forall (t,x) \in \Omega \times X$

where $\tilde{A}, \tilde{B}$ and $\tilde{T}$ are called random multivalued mappings induced by the fuzzy mappings A, B and T, *respectively.* Suppose that $M : \Omega \times X \to 2^X$ is an ordered RME-multivalued mapping and $N : \Omega \times X \times X \times X \to X$ is a random single valued mapping. We consider the following *fuzzy nonlinear random variational inclusion problems* involving ordered RME-multivalued mapping: Find measurable mappings $x, u, v, w : \Omega \to X$ such that for all $t \in \Omega, x(t) \in X, A_{(t,x(t))}(u(t)) \geq a(x(t)), B_{(t,x(t))}(v(t)) \geq b(x(t)), T_{(t,x(t))}(w(t)) \geq c(x(t))$ such that

$$0 \in N_t(u(t), v(t), w(t)) + M_t(x(t)). \tag{1}$$

Definition 8. [13] Let X be a real Banach space with a norm $\|\cdot\|$, θ be a zero element in the X. A nonempty closed convex subsets C of X is said to be a cone if

(a) for any $x \in C$, and any $\lambda > 0$, $\lambda x \in C$ holds;
(b) if $x \in C$ and $-x \in C$, then $x = \theta$.

Definition 9. [13] Let $\leq$ be a partial ordered relation defined in C. For arbitrary elements $x, y \in X$ if $x \leq y$ or $y \leq x$ then x and y is said to be the comparable and denoted by $x \propto y$.

Definition 10. [19] C is said to be a normal cone if and only if there exists a constant $\kappa > 0$ such that for $\theta \leq x \leq y$, implies $\|x\| \leq \kappa \|y\|$ where κ is called normal constant of C.

Lemma 2.1. [16] *If for any natural number n, $x \propto y_n$ and $y_n \to y^*$ $(n \to \infty)$ then $x \propto y^*$.*

Lemma 2.2. [13] *For arbitrary $x, y \in X$, $x \leq y$ if and only if $x - y \in C$, then the relation $\leq$ in X is a partial ordered relation in X where the Banach space X with a ordered relation $\leq$ defined by a normal cone C is called ordered Banach space.*

Definition 11. [13] Let X be an ordered Banach space and $\leq$ be a partial ordered relation defined by the cone C. For arbitrary elements $x, y \in X, lub\{x,y\}$ and $glb\{x,y\}$ express the least upper bound of the set $\{x,y\}$ and the greatest lower bound of the set $\{x,y\}$ on the partial ordered relation $\leq$, *respectively.* Let $\vee, \wedge$ and $\oplus$ be the OR, AND and XOR operators define by $x \vee y = lub\{x,y\}$, $x \wedge y = glb\{x,y\}$, $x \oplus y = lub\{x-y, y-x\}$ and $x \odot y = (x-y) \wedge (y-x)$. Then the following relations are hold:

(1) if $x \leq y$ then $x \vee y = y$, $x \wedge y = x$;
(2) $(x+u) \vee (y+u)$ exists and $(x+u) \vee (y+u) = (x \vee y) + u$;
(3) $(x+u) \wedge (y+u)$ exists and $(x+u) \wedge (y+u) = (x \wedge y) + u$;
(4) if $\lambda \geq 0$ then $\lambda(x \vee y) = \lambda x \vee \lambda y$;

(5) if $\lambda \leq 0$ then $\lambda(x \wedge y) = \lambda x \wedge \lambda y$;
(6) if $x \propto y$ then $x - y \propto y - x$ and $\theta \leq (x - y) \vee (y - x)$;
(7) if $x \propto y$ then $(x + y) \vee ((-x) + (-y)) \leq ((x \vee (-x)) + (y \vee (-y)))$;
(8) if $x \odot y = y \odot x, x \odot x = \theta, x \odot y = y \odot x = -(x \oplus y)$;
(9) $x \odot 0 \leq \theta$, if $x \propto 0$;
(10) $\theta \leq x \oplus y$ if $x \propto y$;
(11) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(12) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(13) $\alpha x \oplus \beta x =\mid \alpha - \beta \mid x$, if $x \propto 0$.

Lemma 2.3. [19] *Let X be an ordered Banach space, C be a normal cone with normal constant κ in X, then for each $x, y \in X$, then the following conditions are hold:*

(1) $\|\theta \oplus \theta\| = \|\theta\| = 0$,
(2) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$,
(3) $\|x \oplus y\| \leq \|x - y\| \leq \kappa \|x \oplus y\|$,
(4) *if $x \propto y$, then* $\|x \oplus y\| = \|x - y\|$,
(5) $\lim_{x \to x_0} \|A(x) - A(x_0)\| = 0$, *if and only if*

$$\lim_{x \to x_0} A(x) \oplus A(x_0) = \theta.$$

Definition 12. Let $M : \Omega \times X \to 2^X$ be a mapping such that $M_t(x(t))$ is a nonempty closed subset of X. Then

(i) M_t is said to be random comparison mapping, if for any $v(t)_{x(t)} \in M_t(x(t)), x(t) \propto v(t)_{x(t)}$ and if $x(t) \propto y(t)$, then for any $v(t)_{x(t)} \in M_t(x(t))$ and any $v(t)_{y(t)} \propto M_t(x(t)), v(t)_{x(t)} \propto v(t)_{y(t)}, \forall x(t), y(t) \in X$,
(ii) a random comparison mapping M_t is said to be random ordered rectangular if for each $x(t), y(t) \in X, u(t)_{x(t)} \in M_t(x(t))$, $u(t)_{y(t)} \in M_t(y(t))$ and
$$\langle u(t)_{x(t)} \odot u(t)_{y(t)}, -(x(t) \oplus y(t))\rangle = 0,$$
(iii) a random comparison mapping M_t is said to be randomly λ_t-ordered accretive, if there exists a measurable mapping $\lambda : \Omega \to (0, 1)$ such that
$$\lambda_t(v(t)_{x(t)} - v(t)_{y(t)}) \geq x(t) - y(t), \forall x(t), y(t) \in X, v(t)_{x(t)} \in M_t(x(t))$$
and $v(t)_{y(t)} \in M_t(y(t))$,
(iv) a random comparison mapping M_t is said to be randomly β_t-ordered extended, if there exists a measurable mapping $\beta : \Omega \to (0, 1)$ such that
$$\beta_t(x(t) \oplus y(t)) \leq v(t)_{x(t)} \oplus v(t)_{y(t)}, \forall x(t), y(t) \in X, v(t)_{x(t)} \in M_t(x(t))$$
and $v(t)_{y(t)} \in M_t(y(t))$,
(v) a random comparison mapping M_t is said to be ordered RME with respect to J_{M_t,λ_t} if M_t is random ordered rectangular, random λ_t-ordered accretive with respect to J_{M_t,λ_t}, random β_t-ordered extended

and $(I_t + \lambda_t M_t)(X) = X$, where $\lambda, \beta : \Omega \to (0,1)$ are measurable mappings and $I : \Omega \times X \to X$ an identity mapping.

Definition 13. Let X be a real ordered Banach space and $A, B : \Omega \times X \to X$ be the two random mappings.

(i) A_t is said to be randomly comparison if for any $t \in \Omega$ and each $x(t), y(t) \in X$, $x(t) \propto y(t)$ then $A_t(x(t)) \propto A_t(y(t))$, $x(t) \propto A_t(x(t))$ and $y(t) \propto A_t(y(t))$.

(ii) A_t and B_t are said to be randomly comparison if for each $t \in \Omega, x(t) \in X, A_t(x(t)) \propto B_t(x(t))$ (denoted by $A_t \propto B_t$).

Obviously, if A_t is a randomly comparison, then $A_t \propto I_t$ (where I_t is a random identity mapping on X).

Definition 14. A random mapping $A : \Omega \times X \to X$ is said to be randomly β_t-order compression with respect to a measurable mapping $\beta : \Omega \to (0,1)$ if A_t is a randomly comparative with respect to the measurable mapping $\beta : \Omega \to (0,1)$ such that for any $t \in \Omega$,

$$A_t(x(t)) \oplus A_t(y(t)) \leq \beta_t(x(t) \oplus y(t)).$$

Lemma 2.4. [9] *Let $T : \Omega \times X \to CB(X)$ be a $\mathcal{H}$-continuous random set valued mapping. Then for any measurable mapping $w : \Omega \to X$, the set valued mapping $T(\cdot, w(\cdot)) : \Omega \to CB(X)$ is a measurable.*

Lemma 2.5. *Let $T, S : \Omega \to CB(X)$ be the two measurable set valued mappings and $\upsilon : \Omega \to H$ be a measurable selection of S then there exists a measurable selection $w : \Omega \to H$ of T such that for all $t \in \Omega$*

$$\|\upsilon(t) - w(t)\| \leq \mathcal{H}(S(t), T(t)).$$

Definition 15. A mapping $N : \Omega \times X \times X \times X \to X$ is said to be randomly (μ_t, η_t, ξ_t)-ordered Lipschitz continuous, if $x(t) \propto y(t), u(t) \propto y(t)$ and $p(t) \propto q(t)$ then $N_t(x(t), u(t), p(t)) \propto N_t(y(t), v(t), q(t))$ and there exist measurable mappings $\mu_t, \eta_t, \xi_t : \Omega \to (0,1)$ such that

$$\begin{aligned} N_t(x(t), u(t), p(t)) \oplus N_t(y(t), v(t), q(t)) \leq \mu_t(x(t) \oplus y(t)) + \eta_t(u(t) \oplus v(t)) \\ + \xi_t(p(t) \oplus q(t)). \end{aligned}$$

Definition 16. Let $R : \Omega \times X \to X$ be a random mapping. Then a comparison mapping $M : \Omega \times X \to 2^X$ is a said to be random ordered RME with respect to $J^{(I_t - R_t)}_{M_t, \lambda_t}$, if M_t is random ordered rectangular and random λ_t-ordered accretive with respect to $J^{(I_t - R_t)}_{M_t, \lambda_t}$ and random β_t-ordered extended and $[(I_t - R_t) + \lambda_t M_t]X = X$ with measurable mappings $\lambda, \beta : \Omega \to (0,1)$, where $I : \Omega \times X \to X$ is random identity mapping.

Definition 17. Let C be a normal cone with normal constant κ and $M : \Omega \times X \to 2^X$ be a random multivalued ordered rectangular mapping. Let $I : \Omega \times X \to X$ be the random identity mapping and $R : \Omega \times X \to X$ be a random mapping. The randomize relaxed resolvent operator $J^{(I_t-R_t)}_{M_t,\lambda_t} : \Omega \times X \to X$ associated with random mappings I_t, R_t and M_t is defined by

$$J^{(I_t-R_t)}_{M_t,\lambda_t}(x(t)) = [(I_t - R_t) + \lambda_t M_t]^{-1}(x(t)), \forall t \in \Omega, x(t) \in X \tag{2}$$

where $\lambda : \Omega \to (0, 1)$ is a measurable mapping.

Lemma 2.6. *Let $R : \Omega \times X \to X$ be a random comparison and random γ_t-ordered compression mapping and let $M : \Omega \times X \to 2^X$ be the random multivalued ordered rectangular mapping. Then the random mapping $J^{(I_t-R_t)}_{M_t,\lambda_t} = [(I_t - R_t) + \lambda_t M_t]^{-1} : X \to X$ is a single valued.*

Lemma 2.7. *Let $M : \Omega\times X \to 2^X$ be the random multivalued ordered rectangular, random comparison and random λ_t-ordered accretive mapping with respect to $J^{(I_t-R_t)}_{M_t,\lambda_t}$. Let $R : \Omega \times X \to X$ be a random strongly comparison mapping and $(I_t - R_t)$ be a random strongly compression mapping with respect to $J^{(I_t-R_t)}_{M_t,\lambda_t}$. Then the randomize relaxed resolvent operator $J^{(I_t-R_t)}_{M_t,\lambda_t} : \Omega \times X \to X$ is a random comparison mapping.*

Lemma 2.8. *Let $M : \Omega \times X \to 2^X$ be a random ordered RME multivalued mapping with respect to $J^{(I_t-R_t)}_{M_t,\lambda_t}$. Let $R : \Omega \times X \to X$ be a random comparison and random γ_t-ordered compression mapping with respect to $J^{(I_t-R_t)}_{M_t,\lambda_t}$. Then the following condition is hold:*

$$J^{(I_t-R_t)}_{M_t,\lambda_t}(x(t)) \oplus J^{(I_t-R_t)}_{M_t,\lambda_t}(y(t)) \le \frac{1}{\beta_t\lambda_t - \gamma_t - 1}(x(t) \oplus y(t)).$$

3. Main Results

In this section, we define a random iterative algorithm to obtain the random solution of problem (1).

Algorithm 3.1. *Let $R : \Omega \times X \to X, N : \Omega \times X \times X \times X \to X$ be the random single valued mappings and $I : \Omega\times X \to X$ be the random identity mapping. Let $A, B, T : \Omega\times X \to \mathfrak{F}(X)$ be the closed fuzzy mapping satisfying the condition (S) and $\tilde{A}, \tilde{B}, \tilde{T}$ be the random multivalued mappings induced by the fuzzy mappings. Suppose that $M : \Omega \times X \to 2^X$ is a random ordered RME multivalued mapping. We define the following scheme:*
For any given $u_0(t) \in (A_t(x_0(t)))_{a(x_0(t))}$, $v_0(t) \in (B_t(x_0(t)))_{b(x_0(t))}$ and $w_0(t) \in (T_t(x_0(t)))_{c(x_0(t))}$, let

$$x_1(t) = J^{(I_t-R_t)}_{M_t,\lambda_t}[(I_t - R_t)x_0(t) - \lambda_t N_t(u_0(t), v_0(t), w_0(t))].$$

Since $u_0(t) \in (A_t(x_0(t)))_{a(x_0(t))} \in CB(X)$, $v_0(t) \in (B_t(x_0(t)))_{b(x_0(t))} \in CB(X)$ and $w_0(t) \in (T_t(x_0(t)))_{c(x_0(t))} \in CB(X)$, there exist $u_1(t) \in (A_t(x_1(t)))_{a(x_1(t))}$, $v_1(t) \in (B_t(x_1(t)))_{b(x_1(t))}$ and $w_1(t) \in (T_t(x_1(t)))_{c(x_1(t))}$ and suppose that $x_0(t) \propto$

$x_1(t)$, $u_0(t) \propto u_1(t)$, $v_0(t) \propto v_1(t), w_0(t) \propto w_1(t)$ *such that*

$$\|u_1(t) \oplus u_0(t)\| = \|u_1(t) - u_0(t)\| \leq \mathcal{H}((A_t(x_1(t)))_{a(x_1(t))}, (A_t(x_0(t)))_{a(x_0(t))}),$$
$$\|v_1(t) \oplus v_0(t)\| = \|v_1(t) - v_0(t)\| \leq \mathcal{H}((B_t(x_1(t)))_{b(x_1(t))}, (B_t(x_0(t)))_{b(x_0(t))}),$$
$$\|w_1(t) \oplus w_0(t)\| = \|w_1(t) - w_0(t)\| \leq \mathcal{H}((T_t(x_1(t)))_{c(x_1(t))}, (T_t(x_0(t)))_{c(x_0(t))}).$$

Continuing the above process inductively with the supposition that $x_n(t) \propto x_{n+1}(t)$, $u_n(t) \propto u_{n+1}(t)$, $v_n(t) \propto v_{n+1}(t)$, $w_n(t) \propto w_{n+1}(t)$, *for all* $n \in \mathbf{N}$, *we have the following:*

$$x_{n+1}(t) = J^{(I_t - R_t)}_{M_t,\lambda_t}[(I_t - R_t)x_n(t) - \lambda_t N_t(u_n(t), v_n(t), w_n(t))], \tag{3}$$

$$\begin{aligned} u_{n+1}(t) \in (A_t(x_{n+1}(t)))_{a(x_{n+1}(t))}, &\|u_{n+1}(t) \oplus u_n(t)\| = \|u_{n+1}(t) - u_n(t)\| \\ &\leq \mathcal{H}((A_t(x_{n+1}(t)))_{a(x_{n+1}(t))}, (A_t(x_n(t)))_{a(x_n(t))}); \\ v_{n+1}(t) \in (B_t(x_{n+1}(t)))_{b(x_{n+1}(t))}, &\|v_{n+1}(t) \oplus v_n(t)\| = \|v_{n+1}(t) - v_n(t)\| \\ &\leq \mathcal{H}((B_t(x_{n+1}(t)))_{b(x_{n+1}(t))}, (B_t(x_n(t)))_{b(x_n(t))}); \\ w_{n+1}(t) \in (T_t(x_{n+1}(t)))_{c(x_{n+1}(t))}, &\|w_{n+1}(t) \oplus w_n(t)\| = \|w_{n+1}(t) - w_n(t)\| \\ &\leq \mathcal{H}((T_t(x_{n+1}(t)))_{c(x_{n+1}(t))}, (T_t(x_n(t)))_{c(x_n(t))}) \end{aligned} \tag{4}$$

where $\lambda : \Omega \to (0,1)$ *is a measurable mapping.*

Now we establish the random fixed point problem for *fuzzy nonlinear random variational inclusion problems* (1).

Lemma 3.2. *Let* $x(t) \in X, u(t) \in (A_t x(t))_{a(x(t))}, v(t) \in (B_t x(t))_{b(x(t))}$ *and* $w(t) \in (T_t x(t))_{c(x(t))}$ *is a random solution of fuzzy nonlinear random variational inclusion problems* (1) *if and only if* $(x(t), u(t), v(t), w(t))$ *satisfies the following equation:*

$$x(t) = J^{(I_t - R_t)}_{M_t,\lambda_t}[(I_t - R_t)x(t) - \lambda_t N_t(u(t), v(t), w(t))]$$

where

$$J^{(I_t - R_t)}_{M_t,\lambda_t} = [(I_t - R_t) + \lambda_t M_t]^{-1}$$

where $\lambda : \Omega \to (0,1)$ *is a measurable mapping.*

Proof. The proof directly follows from the definition of the random relaxed resolvent operator $J^{(I_t - R_t)}_{M_t,\lambda_t}$. □

Now we prove the following existence and convergence result for *fuzzy nonlinear random variational inclusion problems* (1).

Theorem 3.3. *Let* $R : \Omega \times X \to X$ *be a random comparison and random* γ_t*-ordered compression mapping. Let* $N : \Omega \times X \times X \times X \to X$ *be a random* (μ_t, η_t, ξ_t)*-ordered Lipschitz continuous mapping with measures* $\mu, \eta, \xi : \Omega \to (0,1)$*. Let* $A, B, T : \Omega \times X \to \mathfrak{F}(X)$ *be the random fuzzy mappings satisfying condition* (S) *and* $\tilde{A}, \tilde{B}, \tilde{T} : \Omega \times X \to CB(X)$ *be the random continuous multivalued mappings induced by* A, B, T *respectively. Let the mappings* $\tilde{A}, \tilde{B}, \tilde{T}$

be the random $\mathcal{H}$-Lipschitz continuous ordered compression mappings with the measures $\delta_t^{A_t}, \delta_t^{B_t}, \delta_t^{T_t} : \Omega \to (0,1)$, respectively. Suppose that $M : \Omega \times X \to 2^X$ is a random ordered RME-multivalued mapping and if the following condition is satisfied

$$\kappa\lambda_t(\mu_t\delta_t^{A_t} + \eta_t\delta_t^{B_t} + \xi_t\delta_t^{T_t}) + (1+\kappa)(1+\gamma_t) < \beta_t\lambda_t, \tag{5}$$

then the random iterative sequences $\{x_n(t)\}, \{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ generated by Algorithm 3.1 converges strongly to $x(t), u(t), v(t)$ and $w(t)$, respectively and $(x(t), u(t), v(t), w(t))$ is a random solution of fuzzy nonlinear random variational inclusion problems (1).

Proof. Since R_t is random γ_t-ordered compression mapping and N_t is randomly (μ_t, η_t, ξ_t)-ordered Lipschitz continuous mapping. By Algorithm 3.1, Lemma 2.1 and Lemma 2.8, we have

$$\begin{aligned}
\theta &\le x_{n+1}(t) \oplus x_n(t) \\
&\le J_{M_t,\lambda_t}^{(I_t-R_t)}[(I_t - R_t)x_n(t) - \lambda_t N_t(u_n(t), v_n(t), w_n(t))] \\
&\qquad \oplus J_{M_t,\lambda_t}^{(I_t-R_t)}[(I_t - R_t)x_{n-1}(t) - \lambda_t N_t(u_{n-1}(t), v_{n-1}(t), w_{n-1}(t))] \\
&\le \tfrac{1}{\beta_t\lambda_t - \gamma_t - 1}[(I_t - R_t)x_n(t) - \lambda_t N_t(u_n(t), v_n(t), w_n(t))] \\
&\qquad \oplus [(I_t - R_t)x_{n-1}(t) - \lambda_t N_t(u_{n-1}(t), v_{n-1}(t), w_{n-1}(t))] \\
&\le \tfrac{1}{\beta_t\lambda_t - \gamma_t - 1}[(I_t - R_t)x_n(t) \oplus (I_t - R_t)x_{n-1}(t) \\
&\qquad + \lambda_t(N_t(u_n(t), v_n(t), w_n(t)) \oplus N_t(u_{n-1}(t), v_{n-1}(t), w_{n-1}(t)))] \\
&\le \tfrac{1}{\beta_t\lambda_t - \gamma_t - 1}[(x_n(t) \oplus x_{n-1}(t)) + R_t(x_n(t)) \oplus R_t(x_{n-1}(t)) \\
&\qquad + \lambda_t(N_t(u_n(t), v_n(t), w_n(t)) \oplus N_t(u_{n-1}(t), v_{n-1}(t), w_{n-1}(t)))] \\
&\le \tfrac{1}{\beta_t\lambda_t - \gamma_t - 1}[(x_n(t) \oplus x_{n-1}(t)) + \gamma_t(x_n(t) \oplus x_{n-1}(t)) \\
&\qquad + \lambda_t(N_t(u_n(t), v_n(t), w_n(t)) \oplus N_t(u_{n-1}(t), v_{n-1}(t), w_{n-1}(t)))] \\
&\le \tfrac{1}{\beta_t\lambda_t - \gamma_t - 1}[(x_n(t) \oplus x_{n-1}(t)) + \gamma_t(x_n(t) \oplus x_{n-1}(t)) \\
&\qquad + \lambda_t(\mu_t(u_n(t) \oplus u_{n-1}(t)) + \eta_t(v_n(t) \oplus v_{n-1}(t)) + \xi_t(w_n(t) \oplus w_{n-1}(t)))] \\
&\le \tfrac{1}{\beta_t\lambda_t - \gamma_t - 1}[(x_n(t) \oplus x_{n-1}(t)) + \gamma_t(x_n(t) \oplus x_{n-1}(t)) \\
&\qquad + \lambda_t\mu_t(u_n(t) \oplus u_{n-1}(t)) + \lambda_t\eta_t(v_n(t) \oplus v_{n-1}(t)) + \lambda_t\xi_t(w_n(t) \oplus w_{n-1}(t))].
\end{aligned}$$

Using the Definition 10, Definition 11, Lemma 2.3 and randomly $\mathcal{H}$-Lipschitz continuity of $\tilde{A}, \tilde{B}$ and $\tilde{T}$, we have

$$
\begin{aligned}
&\|x_{n+1}(t) - x_n(t)\| \\
&\le \kappa_t \| \frac{1}{\beta_t\lambda_t - \gamma_t - 1}[(x_n(t) \oplus x_{n-1}(t)) + \gamma_t(x_n(t) \oplus x_{n-1}(t)) \\
&\quad + \lambda_t\mu_t(u_n(t) \oplus u_{n-1}(t)) + \lambda_t\eta_t(v_n(t) \oplus v_{n-1}(t)) + \lambda_t\xi_t(w_n(t) \oplus w_{n-1}(t))]\| \\
&\le \frac{\kappa_t}{\beta_t\lambda_t - \gamma_t - 1}[(1+\lambda_t)\|x_n(t) - x_{n-1}(t)\| + \lambda_t\mu_t\|u_n(t) - u_{n-1}(t)\| \\
&\quad + \lambda_t\eta_t\|v_n(t) - v_{n-1}(t)\| + \lambda_t\xi_t\|w_n(t) - w_{n-1}(t)\|] \\
&\le \frac{\kappa_t}{\beta_t\lambda_t - \gamma_t - 1}[(1+\lambda_t)\|x_n(t) - x_{n-1}(t)\| + \lambda_t\mu_t\delta_t^{A_t}\|x_n(t) - x_{n-1}(t)\| \\
&\quad + \lambda_t\eta_t\delta_t^{B_t}\|x_n(t) - x_{n-1}(t)\| + \lambda_t\xi_t\delta_t^{T_t}\|x_n(t) - x_{n-1}(t)\|] \\
&\le \frac{\kappa_t}{\beta_t\lambda_t - \gamma_t - 1}[(1+\lambda_t) + \lambda_t(\mu_t\delta_t^{A_t} + \eta_t\delta_t^{B_t} + \xi_t\delta_t^{T_t})]\|x_n(t) - x_{n-1}(t)\|
\end{aligned}
$$

i.e.,

$$\|x_{n+1}(t) - x_n(t)\| \le \Theta_t\|x_n(t) - x_{n-1}(t)\| \tag{6}$$

where

$$\Theta_t = \frac{\kappa[(1+\lambda_t) + \lambda_t(\mu_t\delta_t^{A_t} + \eta_t\delta_t^{B_t} + \xi_t\delta_t^{T_t})]}{\beta_t\lambda_t - \gamma_t - 1}.$$

By condition (5), we have $0 < \Theta_t < 1$, thus $\{x_n(t)\}$ is a random Cauchy sequence in X and X is complete, there exists $x(t) \in X$ such that $x_n(t) \to x(t)$ as $n \to \infty$. From (4) of Algorithm 3.1 and random $\mathcal{H}$-Lipschitz continuity of $\tilde{A}, \tilde{B}$ and $\tilde{T}$, we have

$$
\begin{aligned}
\|u_{n+1}(t) - u_n(t)\| &\le \mathcal{H}((A_t(x_{n+1}(t)))_{a(x_{n+1}(t))}, (A_t(x_n(t)))_{a(x_n(t))}) \\
&\le \delta_t^{A_t}\|x_{n+1}(t) - x_n(t)\|; \\
\|v_{n+1}(t) - v_n(t)\| &\le \mathcal{H}((B_t(x_{n+1}(t)))_{b(x_{n+1}(t))}, (B_t(x_n(t)))_{b(x_n(t))}) \\
&\le \delta_t^{B_t}\|x_{n+1}(t) - x_n(t)\|; \\
\|w_{n+1}(t) - w_n(t)\| &\le \mathcal{H}((T_t(x_{n+1}(t)))_{c(x_{n+1}(t))}, (T_t(x_n(t)))_{c(x_n(t))}) \\
&\le \delta_t^{T_t}\|x_{n+1}(t) - x_n(t)\|. \qquad (7)
\end{aligned}
$$

It is clear from (7) that $\{u_n(t)\}, \{v_n(t)\}$ and $\{w_n(t)\}$ are also Cauchy sequences in X, there exist $u(t), v(t)$ and $w(t)$ in X such that $u_n(t) \to u(t), v_n(t) \to v(t)$ and $w_n(t) \to w(t)$ as $n \to \infty$. By using the continuity of the random operators $N_t, \tilde{A}, \tilde{B}, \tilde{T}, J_{M_t,\lambda_t}^{(I_t - R_t)}$ and Algorithm 3.1, we have

$$x(t) = J_{M_t,\lambda_t}^{(I_t - R_t)}[(I_t - R_t)x(t) - \lambda_t N_t(u(t), v(t), w(t))].$$

By Lemma 3.2, we conclude that $(x(t), u(t), v(t), w(t))$ is a random solution of problem (1). It remains to show that $u(t) \in (A_t(x(t)))_{a(x(t))}$, $v(t) \in (B_t(x(t)))_{b(x(t))}$

and $w(t) \in (T_t(x(t)))_{c(x(t))}$. In fact

$$\begin{aligned} d(u(t), (A_t(x(t)))_{a(x(t))}) &\leq \|u(t) - u_n(t)\| + d(u_n(t), (A_t(x(t)))_{a(x(t))}) \\ &\leq \|u(t) - u_n(t)\| + \mathcal{H}((A_t(x_n(t)))_{a(x_n(t))}, (A_t(x(t)))_{a(x(t))}) \\ &\leq \|u(t) - u_n(t)\| + \delta_t^{A_t} \|x_n(t) - x(t)\| \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence $u(t) \in (A_t(x(t)))_{a(x(t))}$. Similarly, we can show that $v(t) \in (B_t(x(t)))_{b(x(t))}$ and $w(t) \in (T_t(x(t)))_{c(x(t))}$. This completes the proof. □

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Jong Kyu Kim
Department of Mathematics Education
Kyungnam University
Changwon, Gyeongnam 51767, Korea
E-mail address: jongkyuk@kyungnam.ac.kr

Salahuddin
Department Of Mathematics
Jazan University, Jazan
Kingdom of Saudi Arabia
E-mail address: salahuddin12@mailcity.com