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DEFINING EQUATIONS OF RATIONAL CURVES IN SMOOTH QUADRIC SURFACE

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Abstract. For a nondegenerate irreducible projective variety, it is a classical problem to study the defining equations of a variety with respect to the given embedding. In this paper we precisely determine the defining equations of certain types of rational curves in $\mathbb{P}^3$.

1. Introduction

We work over an algebraically closed field $\Bbbk$ of arbitrary characteristic. Let $\mathbb{P}^r$ and $R = \Bbbk[X_0, X_1, \ldots, X_r]$ denote respectively the projective r-space over $\Bbbk$ and the homogeneous coordinate ring of $\mathbb{P}^r$. Let $Z \subset \mathbb{P}^r$ be a nondegenerate irreducible variety and let I_Z be the homogeneous ideal of Z in R. To understand the variety Z, it is natural to study the defining equations of Z and the syzygies among them. Also the classical problem to find an upper bound of the maximal degree of a minimal generator of I_Z has been reformulated as describing the minimal free resolution of I_Z. About these problems, there were several results [7], [9], [11], [12], [13], [14], [15], [17], [23], [27] and so on. Nevertheless, it is still the most fundamental but difficult problem to determine the defining equations of Z precisely, namely the minimal generators of I_Z.

In this paper, we would like to focus our interest on the problem to describe the equations defining the rational curves. Let $T := \Bbbk[s, t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. For each $k \geq 1$, we denote by T_k the k-th graded component of T. Let $C \subset \mathbb{P}^r$ be a nondegenerate smooth rational curve of degree $d \geq r$. As is well known, there exists a subset $\{f_0, f_1, \ldots, f_r\} \subset T_d$ of $\Bbbk$-linearly independent forms of degree d such that the curve C is given by a parametrization

$$C = \{[f_0(P) : f_1(P) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1\}.$$

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As the most simplest case, the rational normal curve $C \subset \mathbb{P}^r$ of degree d is a smooth rational curve with the condition $r = d$. Then it can be defined to be a image of the map $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ parameterized by

$$C = \{[s^d(P) : s^{d-1}t(P) : \cdots : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}. \tag{1}$$

The almost all algebraic and geometric properties of C are very well understood. Also it is well-known that C is defined by the common zero locus of the polynomials $F_{i,j} = X_iX_j - X_{i-1}X_{j-1}$ for $1 \leq i \leq j \leq r-1$. For the next case where $d = r+1$, there are several results about algebraic and geometric properties of C ([1], [2], [3], [4], [5], [6], [16], [18], [19],[21], [22], [24], [25], [26],[26] and so on). However about the problem to describe the defining equations of C, almost nothing is known in general for the author's knowledge. In this short note, as a beginning of this problem we study the rational curve $C_d \subset \mathbb{P}^3$ parameterized as

$$C_d = \{[s^d(P) : s^{d-1}t(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}.$$

This parametrization of C_d is a kind of generalization of the rational normal curve $C \subset \mathbb{P}^3$ of degree 3 in (1). Then the curve C_d is a smooth rational curve of degree d contained in a smooth rational rational normal surface scroll $S(1,1)$(see Lemma 3.1 and Proposition 3.2). These investigations enable us to determine the precise shapes of the minimal generators of the homogenous ideal I_{C_d} of C_d. The following is the our main result.

Theorem 1.1. *Let $C_d \subset \mathbb{P}^3$ be a rational curve defined as the parametrization*

$$C_d = \{[s^d(P) : s^{d-1}t(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}$$

where $d \geq 3$. Then

(1) *The curve C_d is a smooth rational curve of degree d contained in the rational normal surface scroll $S(1,1)$ as a divisor linear equivalent to $H + (d-2)F$ where H and F are respectively the hyperplane section and a ruling line.*

(2) *C_d is of maximal regularity $d-1$ in the sense of Castelnuovo-Mumford.*

(3) *The defining ideal I_{C_d} of C_d is minimally generated as following:*

$$I_{C_d} = \langle X_0X_3 - X_1X_2, F_{d,1}, F_{d,2}, \ldots, F_{d,d-1}\rangle$$

where $F_{d,i} = X_0^{d-i-1}X_2^i - X_1^{d-i}X_3^{i-1}$ for $1 \leq i \leq d-1$.

2. Preliminaries

2.1. Rational normal surface scrolls and its divisors

We begin with recalling a standard description of rational normal surface scrolls (cf. [28]). For the vector bundle

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$$

on $\mathbb{P}^1$ where $0 \le a_1 \le a_2$ and $a_2 > 0$, the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of $\mathbb{P}(\mathcal{E})$ is globally generated and we write $S(a_1, a_2)$ for the image of the map defined by $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

(A) It is well-known that $S(a_1, a_2)$ is a normal variety and has only rational singularities. Also the homogeneous ideal of $S(a_1, a_2)$ is generated by quadratic equations.
(B) The divisor class group of $\mathbb{P}(\mathcal{E})$ is freely generated by $\widetilde{H} \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$ and a ruling subspace $\widetilde{F}$ of the bundle map $j : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. Moreover, if $a_1 > 0$ then the morphism $\varphi : \mathbb{P}(\mathcal{E}) \to S(a_1, a_2)$ induces an isomorphism between the divisor class groups and hence the divisor class group of $S(a_1, a_2)$ is freely generated by the hyperplane divisor H and a ruling line F of X.

Notation and Remark 2.1. (1) Let $S(1,1) \subset \mathbb{P}^3$ be the rational normal surface scroll of degree 2. Let $S = \Bbbk[X_0, X_1, X_2, X_3]$ be the homogeneous coordinate ring of $\mathbb{P}^3$. Then $S(1,1)$ is defined by the quadratic equation $X_0X_3 - X_1X_2$.
(2) Let $C \subset \mathbb{P}^3$ be a rational normal curve of degree 3. Then C can be defined by the parameterization

$$C = \{[s^3(P) : s^2t(P) : st^2(P) : t^3(P)] \mid P \in \mathbb{P}^1\}$$

and the ideal I_C of C is generated by the following three quadratic equations:

$$\{X_0X_2 - X_1^2, \quad X_1X_3 - X_2^2, \quad X_0X_3 - X_1X_2\}.$$

Thus C is contained the rational normal surface scroll $S(1,1)$. Furthermore, C is linear equivalent to a divisor $H + F$ where H and F are respectively a hyperplane section and a ruling line of $S(1,1)$ (For details, see [24, Theorem 5.10]).
(3) For a smooth curve $Z \subset \mathbb{P}^r$ and an integer $s \ge 2$, we defined the closure Z^s, say the s-th join of Z with itself, of the set of points lying in $(s-1)$-dimensional linear subspaces spanned by general collections of s points in Z. Then there is a strictly ascending filtration

$$Z \subsetneqq Z^2 \subsetneqq Z^3 \subsetneqq \cdots \subsetneqq Z^{\mathrm{ord(Z)}-1} \subsetneqq Z^{\mathrm{ord(Z)}} = \mathbb{P}^r$$

where the number $\mathrm{ord}(Z) = \min\{s \mid Z^s = \mathbb{P}^r\}$ is called the order of Z. Then it is well known that the linear projection map $\pi_q : Z \to \mathbb{P}^{r-1}$ of Z from a point $q \in \mathbb{P}^r \setminus Z^2$ is an isomorphism. For details, we refer to the reader to [29].
(4) Let $Z \subset \mathbb{P}^r$ be a nondegenerate irreducible projective curve of degree d. Z is said to be m-regular if its sheaf of ideal $\mathcal{I}_Z$ satisfies the vanishing

$$H^i(\mathbb{P}^r, \mathcal{I}_Z(m-i)) = 0 \quad \text{for all } i \ge 1.$$

The Castelnuovo-Mumford regularity (or simply the regularity) of Z, denoted by $\mathrm{reg}(Z)$, is defined as the least integer m such that Z is m-regular(cf. [23]). Another interest of this notion stems partly from the fact that Z is m-regular if and only if for every $j \ge 0$ the minimal generators of the j-th syzygy module of the homogeneous ideal $I(Z)$ of Z occur in degree $\leq m+j$ ([10]). In particular, $I(Z)$ is generated by forms of degree $\leq m$. Thus the existence of ℓ-secant line guarantees that $\mathrm{reg}(Z) \geq \ell$. By a well-known result of Gruson-Lazarsfeld-Peskine [14], the Castelnuovo-Mumford regularity $\mathrm{reg}(Z)$ of Z is bounded by $\mathrm{reg}(Z) \leq d-r+2$. They further classified the extremal curves which fail to be $(d-r+1)$-regular, showing in particular that if $d \geq r+2$ then Z is a smooth rational curve with a unique $(d-r+2)$-secant line.

2.2. Minimal set of generators of an ideal

Let $Z \subset \mathbb{P}^r$ be a nondegenerate projective irreducible curve and let I_Z be the homogeneous ideal of Z in R. Then we can choose the minimal set of homogeneous generators for I_Z as I_Z is finitely generated. For the convenience of the reader, we revisit the notion of minimal set of generators of an ideal I_Z. Let

$$M = \{G_{i,j} \in K[X_0, X_1, \ldots, X_r] \quad | \quad G_{i,j} \in I_Z \quad \text{for } 2 \leq i \leq m \text{ and } 1 \leq j \leq \ell_i\}$$

be the set of homogeneous polynomials of degree $\deg(G_{i,j}) = i$. Let $(I_Z)_{\leq t}$ be the ideal generated by the homogeneous polynomials in I_Z of degree at most t. Then M is the minimal set of generators of I_Z if and only if the following three conditions hold:

(*i*) I_Z is generated by the polynomials in M (i.e., $I_Z = \langle M \rangle$).

(*ii*) $G_{i,1}, G_{i,2}, \ldots, G_{i,\ell_i}$ are $\mathbb{K}$-linearly independent forms of degree i for each $2 \leq i \leq m$.

(*iii*) $G_{i,j} \notin (I_Z)_{\leq i-1}$ for each $2 \leq i \leq m$.

3. Proof of Main Theorem

This section is devoted to prove Theorem 1.1. We keep the notations in the previous section. Let $C_d \subset \mathbb{P}^3$ $(d \geq 3)$ be a rational curve defined as the parametrization

$$C_d = \{[s^d(P) : s^{d-1}t(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}. \tag{2}$$

Lemma 3.1. *Let C_d be a curve just stated as above. Then C_d is smooth and of degree d.*

Proof. The case where $d = 3$ follows from Notation and Remark 2.1.(2). Suppose that $d > 3$. Then we can see that the parametrization (2) comes from the embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ by

$$P \hookrightarrow [s^d(P) : s^{d-1}t(P) : \cdots : st^{d-1}(P) : t^d(P)] \quad \text{for} \quad P \in \mathbb{P}^1$$

of a projective line $\mathbb{P}^1$. More precisely, we denote $\widetilde{C}_d$ the image of $\mathbb{P}^1$ by the map ν_d and let $\mathbb{L}$ be a $(d-4)$-dimensional linear subspace of $\mathbb{P}^d$ spanned by

$(d-3)$ standard coordinate points

$$\{[0,0,1,0,\ldots,0,0],[0,0,0,1,0,\ldots,0,0],\ldots,[0,0,\cdots,0,1,0,0]\}.$$

Then C_d is obtained by the linear projection map $\pi_{\mathbb{L}} : \widetilde{C}_d \to \mathbb{P}^3$ of $\widetilde{C}_d$ from $\mathbb{L}$. Since $\mathbb{L} \subset \mathbb{P}^r \setminus C_d^2$, the map $\pi_{\mathbb{L}}$ is an isomorphism by Notation and Remark 2.1.(3). Thus C_d is a smooth rational curve of degree d. □

Proposition 3.2. *Let C_d be as in Lemma 3.1. Then we have*

(1) *The curve C_d is contained in the rational normal surface scroll $S(1,1)$ as a divisor linear equivalent to $H+(d-2)F$ where H and F are respectively the hyperplane section and a ruling line.*

(2) *The curve C_d is of maximal regularity $d-1$.*

Proof. (1) Consider the parametrization (2) of C_d. Then it is easy to see that the defining ideal I_{C_d} of C_d contains the quadratic equation $X_0X_3 - X_1X_2$ and hence C_d is a divisor of $S := S(1,1)$ by Notation and Remark 2.1.(1). Now assume that C_d is a divisor of S in the class $aH + bF$ for some $a \geq 1$. To show that $a = 1$, suppose that $a \geq 2$. First note that the curve C_d is not linearly normal as it is an image of an isomorphic projection of a rational normal curve of degree d by Lemma 3.1. Consider the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_{C_d} \to \mathcal{O}_S(-aH-bF) \to 0.$$

Then since S is arithmetically Cohen-Macaulay, we have the exact sequence

$$0 \to H^1(\mathbb{P}^r, \mathcal{I}_{C_d}(1)) \to H^1(S, \mathcal{O}_S((1-a)H-bF)) \to \cdots.$$

Then one can see that $H^1(S, \mathcal{O}_S((1-a)H-bF)) = 0$ for $a \geq 2$ and hence C_d is linearly normal. This is a contradiction. Now it can be shown that $b = d-2$ by degree counting of the divisor $H + bF$.

(2) It suffices to show that the line section $S(1)$ of S is a $(d-1)$-secant line to C_d because the regularity of C_d is bounded by $d-1$ (see Notation and Remark 2.1.(4)). Indeed since $C_d \cong H+(d-2)F$ and $S(1) \cong H - F$, it is easy to see that the intersection number $\sharp(C_d \cap S(1)) = d-1$ (cf. See [20, Lemma 2.1]). □

Example 3.3. For $d = 4,5,6,7,8,9,10$, let $C_d \subset \mathbb{P}^3$ be curves defined as the parametrization (2). For the simplicity, put

$$F_{d,i} = X_0^{d-i-1}X_2^i - X_1^{d-i}X_3^{i-1}$$

for $4 \leq d \leq 10$ and $1 \leq i \leq d-1$. Then by means of the Computer Algebra System Singular [8], the defining ideal I_{C_d} for $d = 4,5,6,7,8,9,10$ are respectively minimally generated as followings:

(*i*) $I_{C_4} = \langle X_0X_3 - X_1X_2, F_{4,1}, F_{4,2}, F_{4,3} \rangle$,

(*ii*) $I_{C_5} = \langle X_0X_3 - X_1X_2, F_{5,1}, F_{5,2}, F_{5,3}, F_{5,4} \rangle$

(*iii*) $I_{C_6} = \langle X_0X_3 - X_1X_2, F_{6,1}, F_{6,2}, F_{6,3}, F_{6,4}, F_{6,5}\rangle$

(*iv*) $I_{C_7} = \langle X_0X_3 - X_1X_2, F_{7,1}, F_{7,2}, F_{7,3}, F_{7,4}, F_{7,5}, F_{7,6}\rangle$

(*v*) $I_{C_8} = \langle X_0X_3 - X_1X_2, F_{8,1}, F_{8,2}, F_{8,3}, F_{8,4}, F_{8,5}, F_{8,6}, F_{8,7}\rangle$

(*vi*) $I_{C_9} = \langle X_0X_3 - X_1X_2, F_{9,1}, F_{9,2}, F_{9,3}, F_{9,4}, F_{9,5}, F_{9,6}, F_{9,7}, F_{9,8}\rangle$

(*vii*) $I_{C_{10}} = \langle X_0X_3 - X_1X_2, F_{10,1}, F_{10,2}, F_{10,3}, F_{10,4}, F_{10,5}, F_{10,6}, F_{10,7}, F_{10,8}, F_{10,9}\rangle$.

These examples and the observations about the pattern of the minimal generators of defining ideals I_{C_d} enable us to pose the following proposition.

Proposition 3.4. *Let C_d be as in Lemma 3.1. Then the defining ideal I_{C_d} of C_d is minimally generated as following:*

$$I_{C_d} = \langle X_0X_3 - X_1X_2, F_{d,1}, F_{d,2}, \ldots, F_{d,d-1}\rangle$$

where $F_{d,i} = X_0^{d-i-1}X_2^i - X_1^{d-i}X_3^{i-1}$ for $1 \le i \le d-1$.

Proof. If $d = 3$, then C_d is a rational normal curve (see Notation and Remark 2.1.(2)). So we may assume that $d \ge 4$. Put $M_d = \{X_0X_3 - X_1X_2, F_{d,1}, F_{d,2}, \ldots, F_{d,d-1}\}$. Then since $C_d \subset S(1,1)$ by Proposition 3.2.(1), we can see that $X_0X_3 - X_1X_2 \in I_{C_d}$. Also it can be shown that $F_{d,i}([s^d : s^{d-1}t : st^{d-1} : t^d]) = 0$ for all $1 \le i \le d-1$ as the parametrization (2). This shows that $M_d \subset I_{C_d}$. Now we will show that $I_{C_d} = \langle M_d\rangle$ by verifying the three conditions (ii), (iii) and (i) in subsection 2.2 hold for the set M_d in tern. For the condition (ii), it suffices to show that $\{F_{d,i}\}$ are $\mathbb{K}$-linearly independent polynomials of degree $d-1$. To do this, consider the degree of X_0 in each $F_{d,i}$ for $1 \le i \le d-1$. Then one can see that $F_{d,i}$ for each i can not be written by a linear combination of the other $F'_{d,j}s$. For the condition (iii), suppose that $F_{d,i} \in (I_{C_d})_{\le d-2}$ for some $F_{d,i} \in M$ and consider the following short exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_{C_d} \to \mathcal{O}_S(-H - (d-2)F) \to 0$$

comes from the inclusion $C_d \subset S$. First we have

$$\begin{aligned} H^0(\mathbb{P}^3, \mathcal{O}_S((j-1)H - (d-2)F)) &= H^0(\mathbb{P}^3, sym^{j-1}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-d+2)) \\ &= H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1}(j-d+1)) \\ &= 0 \quad \text{for } j \le d-2. \end{aligned}$$

This yields that

$$H^0(\mathbb{P}^3, \mathcal{I}_S(j)) = H^0(\mathbb{P}^3, \mathcal{I}_{C_d}(j)) = (I_{C_d})_{\le j} = \langle X_0X_3 - X_1X_2\rangle \quad \text{for } 2 \le j \le d-2. \tag{3}$$

So $F_{d,i}$ can be represented as following:

$$F_{d,i} = X_0^{d-i-1}X_2^i - X_1^{d-i}X_3^{i-1} = (X_0X_3 - X_1X_2)G \tag{4}$$

where $G \in \mathbb{K}[X_0, X_1, X_2, X_3]$ is a homogeneous polynomial of degree $d-3$. Then for any point $[X_0, X_1, X_2, 0] \in \mathbb{P}^3$ the equality (4) above should hold. However, it is impossible that

$$F_{d,i}([X_0, X_1, X_2, 0]) = X_0^{d-i-1} X_2^i = -X_1 X_2 \ G([X_0, X_1, X_2, 0]).$$

For the condition (i), we denote $(I_{C_d})_{d-1}$ the degree $(d-1)$-piece of the ideal I_{C_d}. Since $\operatorname{reg}(C_d) = d-1$ and $(I_{C_d})_{\leq d-2} = \langle X_0X_3 - X_1X_2 \rangle$ by (3), we get the equivalence condition that I_{C_d} is generated by the set M_d if and only if the homogeneous polynomials in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-3)) \times (X_0X_3 - X_1X_2)$ and $F'_{d,i}s$ consist the degree $(d-1)$-piece $(I_{C_d})_{d-1}$ of the ideal I_{C_d}. So we finish the proof by showing $\dim_{\mathbb{K}}(I_{C_d})_{d-1} = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-3)) + d - 1$. To see this, consider the short exact sequence

$$0 \to \mathcal{I}_{C_d} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{C_d} \to 0$$

Then since C_d is $(d-1)$-normal by the regularity of C_d, we have

$$\begin{aligned} h^0(\mathbb{P}^3, \mathcal{I}_{C_d}(d-1)) &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)) - h^0(\mathbb{P}^3, \mathcal{O}_{C_d}(d-1)) \\ &= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-1)) - h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d(d-1))) \\ &= \frac{d^3 - 3d^2 + 8d - 6}{6}. \end{aligned}$$

On the other hand, we see that $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-3)) + d - 1 = \frac{d^3-3d^2+8d-6}{6}$. □

Proof of Theorem 1.1. This follows from Lemma 3.1, Proposition 3.2 and Proposition 3.4.

References

[1] M. Brodmann and E. Park, *On varieties of almost minimal degree I: Secant loci of rational normal scrolls.* J. Pure Appl. Algebra 214 (2010), 2033-2043

[2] M. Brodmann and E. Park, *On varieties of almost minimal degree III: Tangent spaces and embedding scrolls.* J. Pure Appl. Algebra 215 (2011), no. 12, 2859-2872

[3] M. Brodmann, E. Park and P. Schenzel, *On varieties of almost minimal degree II: A rank-depth formula.* Proc. Amer. Math. Soc. 139 (2011), no. 6, 2025-2032

[4] M. Brodmann and P. Schenzel, *Curves of Degree r+2 in* $\mathbb{P}^r$ *:Cohomological, Geometric, and Homological Aspects.* J. Algebra 242 (2001), 577-623

[5] M. Brodmann and P. Schenzel, *On varieties of almost minimal degree in small codimension.* J. Algebra 305 (2006), no.2, 789-801.

[6] M. Brodmann and P. Schenzel, *Arithmetic properties of projective varieties of almost minimal degree.* J. Algebraic Geometry 16 (2007), 347 - 400.

[7] G. Castelnuovo, *Sui multipli di une serie lineare di gruppi di punti appartenente ad une curva algebraic.* Rend. Circ. Mat. Palermo (2) 7 (1893), 89-110.

[8] M. Decker, G.M. Greuel and H. Schönemann, Singular $3-1-2$ – A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2011).

[9] D. Eisenbud, M. Green, K. Hulek and S. Popescu, *Restriction linear syzygies: algebra and geometry*, Compositio Math. 141 (2005), 1460-1478.

[10] D. Eisenbud and S. Gôto, *Linear free resolutions and minimal multiplicity.* Journal of Algebra 88 (1984) 89-133.

[11] La. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*. Invention. Math. 111 (1993), 51-67.

[12] T. Fujita, *Defining equations for certain types of polarized varieties*. in: Complex Analysis and Algebraic Geometry, Cambridge University Press, Cambridge (1977), 165-173.

[13] M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*. Compos. Math. 67 (1988), 301-314.

[14] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnovo, and the equations defining space curves* . Inventiones mathematicae 72 (1983), 491-506.

[15] J. Harris (with the collaboration of D. Eisenbud), *Curves in Projective Space*. Les Presses de l'Universitê de Montrêal, Montrêal, 1982.

[16] L.T. Hoa, *On minimal free resolutions of projective varieties of degree = codimension+2*. J. Pure Appl. Algebra 87 (1993), 241-250.

[17] S. L'vovsky, *On inflection points, monomial curves, and hypersurfaces containing projective curves*. Math. Ann. 306 (1996), 719-735.

[18] W. Lee and E. Park, *On non-normal del Pezzo varieties*. J. Algebra 387 (2013), 11-28

[19] W. Lee and E. Park, *Projective curves of degree=codimension+2 II*. Internat. J. Algebra Comput. 26 (2016), no. 1, 95-104.

[20] W. Lee and E. Park, *On the minimal free resolution of curves of maximal regularity*. Bull. Korean Math. Soc. 53 (2016), no. 6, 1707-1714.

[21] W. Lee, E. Park and P. Schenzel, *On the classification of non-normal cubic hypersurfaces*. J. Pure Appl. Algebra 215 (2011), 2034-2042.

[22] W. Lee, E. Park and P. Schenzel, *On the classification of non-normal complete intersection of two quadrics*. J. Pure Appl. Algebra 216 (2012), no. 5, 1222-1234.

[23] D. Mumford, *Lectures on curves on an algebraic surface*. With a section by G. M. Bergman. Annals of Mathematics Studies, No. 59 Princeton University Press, Princeton, N.J. 1966 xi+200 pp.

[24] U. Nagel, *Arithmetically Buchsbaum divisors on varieties of minimal degree*. Transactions of the American Mathematical Society 351, 4381-4409 (1999)

[25] U. Nagel, *Minimal free resolutions of projective subschemes of small degree*. Syzygies and Hilbert functions, 209232, Lect. Notes Pure Appl. Math., 254, Chapman and Hall/CRC, Boca Raton, FL, 2007.

[26] E. Park, *Projective curves of degree = codimension+2*. Math. Z. 256 (2007), no. 3, 685-697

[27] E. Park, *On hypersurfaces containing projective varieties*. Forum Math. 27 (2015), no. 2, 843-875

[28] Schreyer,F-O, *Syzygies of canonical curves and special linear series*, Math.Ann.275,105-137 (1986).

[29] Fyodor Zak, *Determinants of projective varieties and their degrees*. Algebraic transformation groups and algebraic varieties, 207–238, Encyclopaedia Math. Sci., 132, Springer, Berlin, 2004.

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