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ASYMPTOTIC ANALYSIS FOR PORTFOLIO OPTIMIZATION PROBLEM UNDER TWO-FACTOR HESTON'S STOCHASTIC VOLATILITY MODEL

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Abstract. We study an optimization problem for hyperbolic absolute risk aversion (HARA) utility function under two-factor Heston's stochastic volatility model. It is not possible to obtain an explicit solution because our financial market model is complicated. However, by using asymptotic analysis technique, we find the explicit forms of the approximations of the optimal value function and the optimal strategy for HARA utility function.

1. Introduction

Based on the pioneering work by Merton [7], the portfolio optimization problem has been studied by many authors. The purpose of the the portfolio optimization is to find the optimal value function and the optimal strategy which gives the maximal expected utility of the final wealth at the maturity time.

In this study, we suppose that an investor manages his or her initial wealth X_0 by investing in a financial market consisting of a risky asset and a risk-free asset whose price processes are given as follows. The price B_t of the risk-free asset at time t follows the ordinary differential equation (ODE)

$$dB_t = rB_t dt, \tag{1.1}$$

where $r > 0$ is a constant interest rate. The price S_t of risky one is given by the following stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu(Y_t, Z_t)dt + f(Y_t)\sqrt{Z_t}dW_t^s, \tag{1.2}$$

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where

$$dY_t = \frac{Z_t}{\epsilon}\beta(Y_t)dt + \sqrt{\frac{Z_t}{\epsilon}}\alpha(Y_t)dW_t^y, \tag{1.3}$$

$$dZ_t = \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}dW_t^z. \tag{1.4}$$

Here W^s, W^y and W^z are correlated Brownian motions in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with correlation structure given by

$$d\langle W^s, W^y\rangle_t = \rho_{sy}dt,\ d\langle W^s, W^z\rangle_t = \rho_{sz}dt,\ d\langle W^y, W^z\rangle_t = \rho_{yz}dt.$$

The correlation coefficients ρ_{sy}, ρ_{sz} and ρ_{yz} are constants in $(-1,1)$ satisfying $\rho_{sy}^2 + \rho_{sz}^2 + \rho_{yz}^2 - 2\rho_{sy}\rho_{sz}\rho_{yz} < 1$, so that the covariance matrix of the Brownian motions is guaranteed to be positive definite. The process Z_t is a Cox-Ingersoll-Ross (CIR) process, where θ is the long run mean, κ is the rate of mean reversion, and σ is the the volatility of volatility. The parameters κ, θ and σ are positive constants and required to satisfy the Feller condition $2\kappa\theta \geq \sigma^2$ to ensure that the process Z_t is always positive starting with $Z_0 > 0$. We assume that the coefficients $\mu(y,z), f(y), \alpha(y)$ and $\beta(y)$ satisfy some conditions to guarantee the existence and uniqueness of strong solutions for (1.2) and (1.3).

We assume that given $Z_t = z$, the process Y_t in (1.3) is a mean-reverting process and $Y_t = Y_{t/\epsilon}^{(1)}$ in distribution, where $Y^{(1)}$ is an ergodic diffusion process with unique invariant distribution denoted by Φ (independent of ϵ) and has the infinitesimal generator $\mathcal{L}_0$ defined by

$$\mathcal{L}_0 = \frac{1}{2}\alpha^2(y)\frac{\partial^2}{\partial y^2} + \beta(y)\frac{\partial}{\partial y}. \tag{1.5}$$

We use the notation $\langle\cdot\rangle$ for averaging with respect to Φ, i.e.,

$$\langle g\rangle = \int g(y)\Phi(dy). \tag{1.6}$$

In this case we call (B_t, S_t) a financial market with two-factor Heston's stochastic volatility model.

We observe that in the case that $f(y) = 1$ and $\mu(y,z) = \mu(z)$ is independent of y, our model reduces the Heston stochastic volatility model considered by [6]. Note also that when $\beta(y) = m - y$ and $\alpha(y) = \sqrt{2}\nu$, with m and ν constant, the model becomes a generalized Heston model considered in [1] for an option pricing problem. So, we may call our system (1.2)-(1.4) an extended Heston stochastic volatility model.

We assume that the investor dynamically manages his or her portfolio by allocating a fraction π_t of the wealth at time $t \in [0,T]$ in the risky asset, while the remaining amount is held in the risk-free asset earning the risk-free interest of r. Assuming the investment strategy π is self-financing, the associated wealth process X_t^π satisfies

$$dX_t = X_t\left\{r + \pi_t\left(\mu(Y_t, Z_t) - r\right)\right\}dt + \pi_t f(Y_t)\sqrt{Z_t}X_t dW_t^s. \tag{1.7}$$

We assume that all coefficients of the above SDEs are $\mathcal{F}_t$–progressively measurable and that each of SDEs (1.2) - (1.4) and (1.7) has unique strong solution. Given for a fixed parameter ϵ and a strategy π_t, we denote the solution of (1.7) by $(X^{\epsilon,\pi}(t))_{t\in[0,T]}$. The control function π_t is said to be admissible if it is $\mathcal{F}_t$–progressively measurable and satisfies

$$E\Big[\int_0^T \pi_t^2 f^2(Y_t) Z_t X_t^2 dt\Big] < \infty.$$

We denote the set of all admissible strategies by $\mathcal{A}$. We assume that $\mu(Y_t, Z_t) - r = \mu(Y_t)Z_t$, so that the market price of risk ζ_t is given by

$$\zeta_t = \frac{\mu(Y_t, Z_t) - r}{f(Y_t)\sqrt{Z_t}} = \frac{\mu(Y_t)}{f(Y_t)}\sqrt{Z_t} := \lambda(Y_t)\sqrt{Z_t}. \tag{1.8}$$

We define the value function corresponding to an investment strategy π by

$$V^{\epsilon,\pi}(t,x,y,z) = \mathbb{E}\left[U(X_T^{\epsilon,\pi})\big|X_t^{\epsilon,\pi} = x, Y_t = y, Z_t = z\right].$$

for all $(t,x,y,z) \in [0,T] \times \mathbf{R^1} \times \mathbf{R^1} \times \mathbf{R^1}$, where U is a HARA utility function defined by

$$U(x;p,q,\eta) = \frac{1-p}{pq}\left(\frac{qx}{1-p} + \eta\right)^p, \quad q > 0, p < 1, p \neq 1 \tag{1.9}$$

and $\mathbb{E}[X|A]$ is the conditional expectation of a random variable X given an event A. The object of the investor is to find the optimal investment strategy π^* such that

$$V^{\epsilon,\pi^*}(t,x,y,z) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U(X_T^{\epsilon,\pi})\big|X_t^{\epsilon,\pi} = x, Y_t = y, Z_t = z\right].$$

and the optimal value function

$$V^{\epsilon}(t,x,y,z) = V^{\epsilon,\pi^*}(t,x,y,z).$$

In fact, the optimal value function is the value function corresponding to the optimal investment strategy π^*.

Since our financial market model (B_t, S_t) is very complicated, it is impossible to get the explicit forms of the optimal value function and the optimal investment strategy. So we use a power series representation of $V^\epsilon(t,x,y,z)$ in powers of $\sqrt{\epsilon}$ given by

$$V^\epsilon(t,x,y,z) = V^{(0)}(t,x,y,z) + \sqrt{\epsilon}V^{(1)}(t,x,y,z) + \epsilon V^{(2)}(t,x,y,z) + \cdots, \tag{1.10}$$

for any small positive parameter $\epsilon < 1$. The aim of this paper is to give the explicit form of the approximation $V^{(0)}(t,x,y,z) + \sqrt{\epsilon}V^{(1)}(t,x,y,z)$ up to the first order of $\sqrt{\epsilon}$

The structure of this paper is as follows. In Section 2, we give an explicit form of the leading order term $V^{(0)}$. In Section 3, we drive an explicit expression for $V^{(1)}$ in terms of $V^{(0)}$ for the first time, which give us an explicit form of the approximation $V^{(0)} + \sqrt{\epsilon}V^{(1)}$ up to the first order . In Section 4, by using the results in Section 3, we find an explicit form of the first order correction in a power series representation of the optimal strategy in powers of $\sqrt{\epsilon}$.

2. The leading order term $V^{(0)}$

In this section we give an explicit form of the leading order term $V^{(0)}$. In our model, the Hamilton-Jacobi-Bellman (HJB) equation (cf. Øksendal [8]) for V^ϵ is given by, for $t \in [0,T]$, $x \in \mathbb{R}^+$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^+$,

$$\begin{aligned} V_t^\epsilon + \frac{z}{\epsilon}\mathcal{L}_0 V^\epsilon + rxV_x^\epsilon + \kappa(\theta - z)V_z^\epsilon + \frac{1}{2}\sigma^2 z V_{zz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{yz}\sigma\alpha(y)zV_{yz}^\epsilon \\ + \sup_\pi \Bigg[\frac{1}{2}\pi^2 f^2(y) z x^2 V_{xx}^\epsilon + \pi z x\Bigg(\mu(y)V_x^\epsilon + \rho_{sz}\sigma f(y)V_{xz}^\epsilon \\ + \frac{1}{\sqrt{\epsilon}}\rho_{sy}\alpha(y)f(y)V_{xy}^\epsilon\Bigg)\Bigg] = 0 \end{aligned} \tag{2.1}$$

where the terminal condition is given by

$$V^\epsilon(T,x,y,z) = U(x). \tag{2.2}$$

Maximizing the quadratic expression in π, the optimal investment strategy is given in feedback form by

$$\pi^*(t,x,y,z) = -\frac{\lambda(y)V_x^\epsilon + \rho_{sz}\sigma V_{xz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{sy}\alpha(y)V_{xy}^\epsilon}{f(y)xV_{xx}^\epsilon}, \tag{2.3}$$

where λ is the function defined in (1.8). Substituting this optimal strategy into (2.1) yields

$$\begin{aligned} V_t^\epsilon + \frac{z}{\epsilon}\mathcal{L}_0 V^\epsilon + rxV_x^\epsilon + \kappa(\theta - z)V_z^\epsilon + \frac{1}{2}\sigma^2 z V_{zz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{yz}\sigma\alpha(y)zV_{yz}^\epsilon \\ - \frac{z\Big(\lambda(y)V_x^\epsilon + \rho_{sz}\sigma V_{xz}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho_{sy}\alpha(y)V_{xy}^\epsilon\Big)^2}{2V_{xx}^\epsilon} = 0. \end{aligned} \tag{2.4}$$

As in Fouque et al [2], we assume that the value function $V^\epsilon(t,x,y,z)$ is strictly increasing, strictly concave in x for each $t \in [0,T), y \in \mathbb{R}$ and $z \in \mathbb{R}^+$, and is smooth enough on the domain $[0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$. We also assume that it is the unique solution for the HJB equation (2.1) with terminal condition (2.2).

We now use asymptotic analysis developed in [2] to obtain approximations to the value function and optimal investment strategy for the HARA utility function defined in (1.9).

Now we substitute the power series representation (1.10) into (2.4) and successively group the terms by the powers of ϵ. Collecting the terms in ϵ^{-1}, we have

$$z\mathcal{L}_0 V^{(0)} - \frac{1}{2}\rho_{sy}^2\alpha^2(y)z\frac{\Big(V_{xy}^{(0)}\Big)^2}{V_{xx}^{(0)}} = 0. \tag{2.5}$$

Since all terms of the operator $\mathcal{L}_0$ in (1.5) take derivatives in y, we choose $V^{(0)}$ to be independent of y so that the equation (2.5) is satisfied. It follows from this choice that $V_y^{(0)} = 0$, and then the expansion of (2.4) is given by

$$\begin{aligned}
&V_t^{(0)} + \sqrt{\epsilon} V_t^{(1)} + z\mathcal{L}_0\Big(\frac{1}{\sqrt{\epsilon}} V^{(1)} + V^{(2)} + \sqrt{\epsilon} V^{(3)}\Big) + rx\big(V_x^{(0)} + \sqrt{\epsilon} V_x^{(1)}\big) \\
&+\kappa(\theta - z)\big(V_z^{(0)} + \sqrt{\epsilon} V_z^{(1)}\big) + \frac{1}{2}\sigma^2 z\big(V_{zz}^{(0)} + \sqrt{\epsilon} V_{zz}^{(1)}\big) \\
&+\rho_{yz}\sigma\alpha(y) z\big(V_{yz}^{(1)} + \sqrt{\epsilon} V_{yz}^{(2)}\big) \\
&-z\Big[\lambda(y)\Big(V_x^{(0)} + \sqrt{\epsilon} V_x^{(1)}\Big) + \rho_{sz}\sigma\big(V_{xz}^{(0)} + \sqrt{\epsilon} V_{xz}^{(1)}\big) \\
&\quad +\rho_{sy}\alpha(y)\Big(V_{xy}^{(1)} + \sqrt{\epsilon} V_{xy}^{(2)}\Big)\Big]^2 \frac{1}{2V_{xx}^{(0)}}\Bigg(1 - \sqrt{\epsilon}\frac{V_{xx}^{(1)}}{V_{xx}^{(0)}}\Bigg) + o(\sqrt{\epsilon}) = 0, \quad (2.6)
\end{aligned}$$

where $o(x)/x$ goes to 0 as $x \to 0$. Hence, we see from (2.6) that there is only one term in $(\sqrt{\epsilon})^{-1}$, which leads to

$$\mathcal{L}_0 V^{(1)} = 0. \quad (2.7)$$

By the definition of $\mathcal{L}_0$, $V^{(1)}$ must be independent of y (otherwise, $V^{(1)}$ would grow as much as $e^{y^2/2}$ as $y \to \infty$, which is not of interest). Using the fact that $V^{(0)}$ and $V^{(1)}$ are independent of y, the constant terms in the equation (2.6) lead to

$$z\mathcal{L}_0 V^{(2)} + U = 0, \quad (2.8)$$

where

$$U = V_t^{(0)} + rxV_x^{(0)} + \kappa(\theta - z)V_z^{(0)} + \frac{1}{2}\sigma^2 z V_{zz}^{(0)} - \frac{z\Big(\lambda(y)V_x^{(0)} + \rho_{sz}\sigma V_{xz}^{(0)}\Big)^2}{2V_{xx}^{(0)}}.$$

Viewing (2.8) as a Poisson equation for $V^{(2)}$ in y, the centering condition on the source term is given by

$$\langle U \rangle = 0, \quad (2.9)$$

where $\langle \cdot \rangle$ is the averaging operator defined in (1.6). Then it follows that

$$\begin{aligned}
V_t^{(0)} + rxV_x^{(0)} + \kappa(\theta - z)V_z^{(0)} + \frac{1}{2}\sigma^2 z V_{zz}^{(0)} - \frac{1}{2}\widetilde{\lambda}^2 z \frac{\Big(V_x^{(0)}\Big)^2}{V_{xx}^{(0)}} & \\
-\rho_{sz}\sigma\overline{\lambda} z \frac{V_x^{(0)} V_{zx}^{0}}{V_{xx}^{(0)}} - \frac{1}{2}\rho_{sz}^2\sigma^2 z \frac{\Big(V_{zx}^{(0)}\Big)^2}{V_{xx}^{(0)}} &= 0, \quad (2.10)
\end{aligned}$$

where we have used the fact that $V^{(0)}$ is independent of y and denoted $\overline{\lambda} = \langle \lambda \rangle$ and $\widetilde{\lambda} = \sqrt{\langle \lambda^2 \rangle}$. From (1.10), we have the terminal condition

$$V^{(0)}(T, x, z) = U(x). \quad (2.11)$$

Observe that when $\lambda(y) = \lambda$, the nonlinear PDE (2.10) is the HJB equation corresponding to the Heston model studied by [6]. We recall that [6] considered the power utility functions.

The following theorem contains an explicit formula of $V^{(0)}$ that satisfies the PDE (2.10) with terminal condition (2.11) for the HARA utility function. Since the proof is similar to [2, 3, 4, 6, 11], here we omit it.

Theorem 2.1. *For the HARA utility function $U(x)$ given in* (1.9)*, the PDE* (2.10) *with the terminal condition* (2.11) *has an explicit solution*

$$V^{(0)}(t,x,z) = \frac{1-p}{pq}\Big(\frac{q}{1-p}xe^{r(T-t)} + \eta\Big)^p e^{A(t)+B(t)z}, \tag{2.12}$$

where the functions $A(t)$ and $B(t)$ are given as in the following cases:

Case 1: $\Delta > 0$.

$$A(t) = \frac{\kappa\theta}{\sigma^2\left(1+\Gamma\rho_{sz}^2\right)}\left[\left(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda} + \sqrt{\Delta}\right)(T-t) - 2\ln\left(\frac{1-ge^{(T-t)\sqrt{\Delta}}}{1-g}\right)\right], \tag{2.13}$$

$$B(t) = \frac{\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda} + \sqrt{\Delta}}{\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}\left(\frac{1-e^{\sqrt{\Delta}(T-t)}}{1-ge^{\sqrt{\Delta}(T-t)}}\right). \tag{2.14}$$

Case 2: $\Delta = 0$ *and* $TK + 1 > 0$.

$$A(t) = \frac{\kappa\theta}{\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}\Big[\big(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda}\big)(T-t) - 2\ln\Big(1+\frac{1}{2}\big(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda}\big)(T-t)\Big)\Big], \tag{2.15}$$

$$B(t) = \frac{\big(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda}\big)^2}{2\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}\left(\frac{T-t}{1+\frac{1}{2}\big(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda}\big)(T-t)}\right). \tag{2.16}$$

Here, we define Δ, g, K and Γ by

$$\begin{aligned}
\Delta &= \kappa^2 - \Gamma(2\kappa\rho_{sz}\sigma\overline{\lambda} + \sigma^2\widetilde{\lambda}^2) + \Gamma^2\rho_{sz}^2\sigma^2\Big(\overline{\lambda}^2 - \widetilde{\lambda}^2\Big),\\
g &= \frac{\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda} + \sqrt{\Delta}}{\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda} - \sqrt{\Delta}},\\
K &= \frac{1}{2}\big(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda}\big),\\
\Gamma &= \frac{p}{1-p}.
\end{aligned} \tag{2.17}$$

3. The first order term $V^{(1)}$

In this section, we drive an explicit expression for $V^{(1)}$ in terms of $V^{(0)}$, which give us an explicit form of the approximation $V^{(0)}+\sqrt{\epsilon}V^{(1)}$ up to the first order . From Theorem 2.1 we can get some useful properties about $V^{(0)}$ which are given by the following remark.

Remark 1. (i) We can easily see from (2.12) that $V^{(0)}$ satisfies

$$V_z^{(0)}(t,x,z) = B(t)V^{(0)}(t,x,z) \tag{3.1}$$

for all $(t,x,z) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}^+$. This particular property leads to the explicit derivation of the first order correction term $V^{(1)}$ in terms of $V^{(0)}$.
(ii) From (2.12), we observe that the ratio $V_x^{(0)}/V_{xx}^{(0)}$ does not depend on z, and so we denote

$$R(t,x) = -\frac{V_x^{(0)}(t,x,z)}{V_{xx}^{(0)}(t,x,z)}. \tag{3.2}$$

Before continuing our asymptotic analysis, we define the differential operators D_j by

$$D_j = R^j(t,x)\frac{\partial^j}{\partial x^j}, \quad j=1,2,\ldots, \tag{3.3}$$

and the linear operator $\mathcal{L}_{t,x,z}(\lambda_1,\lambda_2)$ by

$$\begin{aligned}\mathcal{L}_{t,x,z}(\lambda_1,\lambda_2) &= \frac{\partial}{\partial t} + rx\frac{\partial}{\partial x} + \kappa(\theta - z)\frac{\partial}{\partial z} + \frac{1}{2}\sigma^2 z\frac{\partial^2}{\partial z^2} \\ &\quad + \left(\lambda_1^2 + \rho_{sz}\sigma\lambda_2 B(t)\right)zD_1 + \rho_{sz}\sigma\left(\lambda_2 + \rho_{sz}\sigma B(t)\right)zD_1\frac{\partial}{\partial z} \\ &\quad + \frac{1}{2}\left(\lambda_1^2 + 2\rho_{sz}\sigma\lambda_2 B(t) + \rho_{sz}^2\sigma^2 B^2(t)\right)zD_2.\end{aligned} \tag{3.4}$$

Then, by using of (3.1) direct computation shows that the equation (2.10) can be written as

$$\mathcal{L}_{t,x,z}\big(\widetilde{\lambda},\overline{\lambda}\big)V^{(0)} = 0. \tag{3.5}$$

Similarly, we can rewrite (2.8) as

$$z\mathcal{L}_0 V^{(2)} + \mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big)V^{(0)} = 0. \tag{3.6}$$

Then it follows from (3.5) and (3.6) that

$$\mathcal{L}_0 V^{(2)} = -\frac{1}{z}\Big(\mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big) - \mathcal{L}_{t,x,z}\big(\widetilde{\lambda},\overline{\lambda}\big)\Big)V^{(0)}. \tag{3.7}$$

Hence, up to a constant in y, we choose

$$V^{(2)} = -\frac{1}{z}\mathcal{L}_0^{-1}\Big(\mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big) - \mathcal{L}_{t,x,z}\big(\widetilde{\lambda},\overline{\lambda}\big)\Big)V^{(0)}, \tag{3.8}$$

where $\mathcal{L}_0^{-1}$ is the inverse operator of $\mathcal{L}_0$.

Now, we proceed asymptotic analysis to derive the first order term $V^{(1)}$. By using (3.1), the terms in $\sqrt{\epsilon}$ of the expanded PDE (2.6) lead to

$$
\begin{aligned}
&z\mathcal{L}_0V^{(3)}+V_t^{(1)}+rxV_x^{(0)}+\kappa(\theta-z)V_z^{(1)}+\frac{1}{2}\sigma^2zV_{zz}^{(1)}+\rho_{yz}\sigma\alpha(y)zV_{yz}^{(2)}\\
&+\frac{z}{2V_{xx}^{(0)}}\Bigg[\Big(\lambda(y)+\rho_{sz}\sigma B(t)\Big)^2\Big(V_x^{(0)}\Big)^2\frac{V_{xx}^{(1)}}{V_{xx}^{(0)}}\\
&-2\Big(\lambda(y)+\rho_{sz}\sigma B(t)\Big)V_x^{(0)}\Big(\lambda(y)V_x^{(1)}+\rho_{sz}\sigma V_{xz}^{(1)}+\rho_{sy}\alpha(y)V_{xy}^{(2)}\Big)\Bigg]=0. \qquad (3.9)
\end{aligned}
$$

Then using (3.2) and (3.3), we can write (3.9) as

$$
z\mathcal{L}_0V^{(3)}+\mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big)V^{(1)}+z\mathcal{L}_{x,y,z}V^{(2)}=0, \qquad (3.10)
$$

where the operator $\mathcal{L}_{x,y,z}$ is defined by

$$
\mathcal{L}_{x,y,z}=\rho_{yz}\sigma\alpha(y)\frac{\partial^2}{\partial y\partial z}+\rho_{sy}\alpha(y)\Big(\lambda(y)+\rho_{sz}\sigma B(t)\Big)R\frac{\partial^2}{\partial x\partial y}. \qquad (3.11)
$$

Viewing (3.10) as a Poisson equation for $V^{(3)}$ in y, the centering condition requires that

$$
\Big\langle\mathcal{L}_{t,x,z}(\lambda(y),\lambda(y))V^{(1)}+z\mathcal{L}_{x,y,z}V^{(2)}\Big\rangle=0. \qquad (3.12)
$$

Since $V^{(1)}$ does not depend on y and $\langle\mathcal{L}_{t,x,z}\big(\lambda(y),\lambda(y)\big)\rangle=\mathcal{L}_{t,x,z}\big(\tilde{\lambda},\bar{\lambda}\big)$, we deduce from (3.12) that

$$
\mathcal{L}_{t,x,z}\big(\widetilde{\lambda},\overline{\lambda}\big)V^{(1)}=-z\Big\langle\mathcal{L}_{x,y,z}V^{(2)}\Big\rangle. \qquad (3.13)
$$

Substituting $V^{(2)}$, given by (3.8) , into this equation yields

$$
\mathcal{L}_{t,x,z}(\widetilde{\lambda},\overline{\lambda})V^{(1)}=\mathcal{A}V^{(0)}, \qquad (3.14)
$$

where

$$
\mathcal{A}:=z\Bigg\langle\mathcal{L}_{x,y,z}\frac{1}{z}\mathcal{L}_0^{-1}\Big(\mathcal{L}_{t,x,z}(\lambda(y),\lambda(y))-\mathcal{L}_{t,x,z}(\widetilde{\lambda},\overline{\lambda})\Big)\Bigg\rangle.
$$

We explicitly compute the source term of (3.14). To do this, we introduce two functions ϕ and ψ that satisfy the following Poisson equations

$$
\begin{aligned}
\mathcal{L}_0\phi(y) &= \frac{1}{2}\Big(\lambda^2(y)-\widetilde{\lambda}^2\Big), \qquad (3.15)\\
\mathcal{L}_0\psi(y) &= \lambda(y)-\overline{\lambda}. \qquad (3.16)
\end{aligned}
$$

We observe from (2.12) that $D_1V^{(0)} = \Gamma V^{(0)}$, where Γ is defined in (2.17). Then, we have

$$\begin{aligned}
\mathcal{A}V^{(0)} &= z\Big\langle \mathcal{L}_{x,y,z}\frac{1}{z}\mathcal{L}_0^{-1}\Big(\frac{1}{2}\big(\lambda^2(y)-\widetilde{\lambda}^2\big)+\rho_{sz}\sigma B(t)\big(\lambda(y)-\overline{\lambda}\big)\Big)zD_1V^{(0)}\Big\rangle \\
&= z\Gamma\Big\langle \mathcal{L}_{x,y,z}\phi(y)V^{(0)}\Big\rangle + \rho_{sz}\sigma B(t)z\Gamma\Big\langle \mathcal{L}_{x,y,z}\psi(y)V^{(0)}\Big\rangle \\
&= z\Big(\rho_{sy}\Gamma^2F_3+\sigma\Gamma\Big(\rho_{yz}F_1+\rho_{sy}\rho_{sz}\Gamma(F_1+F_4)\Big)B(t) \\
&\quad +\rho_{sz}\sigma^2\Gamma\big(\rho_{yz}+\rho_{sy}\rho_{sz}\Gamma\big)F_2B^2(t)\Big)V^{(0)},
\end{aligned} \tag{3.17}$$

where the group parameters F_i are defined by

$$F_1 = \langle\alpha\phi'\rangle,\ F_2 = \langle\alpha\psi'\rangle,\ F_3 = \langle\alpha\lambda\phi'\rangle,\ F_4 = \langle\alpha\lambda\psi'\rangle. \tag{3.18}$$

From the expansion (1.10), the PDE (3.14) has the terminal condition

$$V^{(1)}(T,x,z) = 0. \tag{3.19}$$

Up to now, we have shown that the first order term $V^{(1)}$ satisfies the linear PDE (3.14) with the terminal condition (3.19). Many authors just gave a PDE satisfied by $V^{(1)}$, but they were not able to provide the explicit formula for $V^{(1)}$ in several types of market models (cf. [9], for example). In the following theorem, we derive an explicit expression for $V^{(1)}$ in terms of $V^{(0)}$ for the first time, which is the main result of this study.

Theorem 3.1. *The linear PDE* (3.14) *with terminal condition* (3.19) *has a solution of the form*

$$V^{(1)}(t,x,z) = \Big(\kappa\theta g_1(t)+g_2(t)z\Big)V^{(0)}(t,x,z), \tag{3.20}$$

where $V^{(0)}$ is given in Theorem 2.1, and $g_1(t)$ and $g_2(t)$ are defined in the following cases:
Case 1: $\Delta > 0$.

$$\begin{aligned}
g_1(t) &= \frac{1}{\Delta g\Big(1-ge^{\sqrt{\Delta}(T-t)}\Big)}\Bigg[c_2+(c_0+2c_1+c_2)g+c_0g^2 \\
&\quad +(c_0+c_1+c_2)\sqrt{\Delta}g(T-t)-\Big(c_2+(c_0+2c_1+c_2)g+c_0g^2 \\
&\quad -\big(c_1+2c_2-(c_0-c_2)g\big)\sqrt{\Delta}g(T-t)\Big)e^{\sqrt{\Delta}(T-t)}\Bigg] \\
&\quad -\frac{c_2(1-g)^2}{\Delta g^2}\ln\left(\frac{1-ge^{\sqrt{\Delta}(T-t)}}{1-g}\right),
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
g_2(t) = & \frac{1}{\sqrt{\Delta}\left(1-ge^{\sqrt{\Delta}(T-t)}\right)^2}\Big\{(c_0+c_1+c_2) \\
& -(c_0+c_1-c_1g-c_0g^2)e^{\sqrt{\Delta}(T-t)} + \big(c_1+2c_2+(2c_0+c_1)g)\big) \\
& \times\sqrt{\Delta}(T-t)e^{\sqrt{\Delta}(T-t)} - (c_2+c_1g+c_0g^2)e^{2\sqrt{\Delta}(T-t)}\Big\}, \quad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= \rho_{sy}\Gamma^2F_3, & (3.23)\\
c_1 &= \sigma\Gamma\Big(\rho_{yz}F_1+\rho_{sy}\rho_{sz}\Gamma(F_1+F_4)\Big)\left(\frac{\kappa-\Gamma\rho_{sz}\sigma\overline{\lambda}+\sqrt{\Delta}}{\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}\right), & (3.24)\\
c_2 &= \rho_{sz}\sigma^2\Gamma\big(\rho_{yz}+\rho_{sy}\rho_{sz}\Gamma\big)F_2\left(\frac{\kappa-\Gamma\rho_{sz}\sigma\overline{\lambda}+\sqrt{\Delta}}{\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}\right)^2. & (3.25)
\end{aligned}
$$

Case 2: $\Delta = 0$ *and* $TK + 1 > 0$.

$$
\begin{aligned}
g_1(t) = & \frac{1}{6K^3(1+K(T-t))}\{6\bar{c}_2(T-t)+3\big(\bar{c}_2-\bar{c}_0K^2\big)K(T-t)^2 \\
& -\big(\bar{c}_2+\bar{c}_1K+\bar{c}_0K^2\big)K^2(T-t)^3\} - \frac{\bar{c}_2}{K^4}\ln\big(1+K(T-t)\big), \quad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
g_2(t) = & \frac{-1}{6\big(1+K(T-t)\big)^2}\{6\bar{c}_0(T-t)+3(\bar{c}_1+2\bar{c}_0K)(T-t)^2 \\
& +2(\bar{c}_2+\bar{c}_1K+\bar{c}_0K^2)(T-t)^3\}, \quad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
\bar{c}_0 &= \rho_{sy}\Gamma^2F_3, & (3.28)\\
\bar{c}_1 &= \sigma\Gamma\Big(\rho_{yz}F_1+\rho_{sy}\rho_{sz}\Gamma(F_1+F_4)\Big)\frac{\big(\kappa-\Gamma\rho_{sz}\sigma\overline{\lambda}\big)^2}{2\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}, & (3.29)\\
\bar{c}_2 &= \rho_{sz}\sigma^2\Gamma\big(\rho_{yz}+\rho_{sy}\rho_{sz}\Gamma\big)F_2\left(\frac{\big(\kappa-\Gamma\rho_{sz}\sigma\overline{\lambda}\big)^2}{2\sigma^2\Big(1+\Gamma\rho_{sz}^2\Big)}\right)^2. & (3.30)
\end{aligned}
$$

Here, F_1, F_2, F_3 *and* F_4 *are given by* (3.18)*, respectively.* Δ, g, K *and* Γ *are defined as in Theorem 2.1.*

Proof. Now, we try to find the solution $V^{(1)}$ of the PDE (3.14) with the terminal condition (3.19) in the form

$$
V^{(1)}(t,x,z) = \Big(\kappa\theta g_1(t)+g_2(t)z\Big)V^{(0)}(t,x,z) \quad (3.31)
$$

with $g_1(T) = 0$ and $g_2(T) = 0$. Substituting (3.31) into (3.14) and using (3.1) and (3.5) yields

$$\kappa\theta g_1' + z g_2' + \kappa(\theta - z) g_2(t) + \sigma^2 z B(t) g_2(t) + \rho_{sz}\sigma\Gamma\Big(\overline{\lambda} + \rho_{sz}\sigma B(t)\Big) z g_2(t)$$
$$= z\Big[\rho_{sy}\Gamma^2 F_3 + \sigma\Gamma\Big(\rho_{yz}F_1 + \rho_{sy}\rho_{sz}\Gamma(F_1 + F_4)\Big)B(t)$$
$$+\rho_{sz}\sigma^2\Gamma\big(\rho_{yz} + \rho_{sy}\rho_{sz}\Gamma\big)F_2 B^2(t)\Big], \qquad (3.32)$$

where $B(t)$ is defined as in Theorem 2.1.

Therefore, (3.32) is separable in z and we can split it into two ODEs

$$g_1'(t) = -g_2(t), \qquad (3.33)$$
$$g_2'(t) + a(t) g_2(t) = b(t), \qquad (3.34)$$

where the functions $a(t)$ and $b(t)$ are defined by

$$a(t) = \sigma^2(1 + \rho_{sz}^2\Gamma)B(t) + (\rho_{sz}\sigma\overline{\lambda}\Gamma - \kappa), \qquad (3.35)$$
$$b(t) = \rho_{sy}\Gamma^2 F_3 + \sigma\Gamma(\rho_{yz}F_1 + \rho_{sy}\rho_{sz}\Gamma(F_1 + F_4))B(t)$$
$$+\rho_{sz}\sigma^2\Gamma\big(\rho_{yz} + \rho_{sy}\rho_{sz}\Gamma\big)F_2 B^2(t). \qquad (3.36)$$

The equation (3.34) can be solved by using the integral factor

$$w(t) = e^{-\int_t^T a(s)ds},$$

which leads to

$$g_2(t) = -\int_t^T \frac{w(s)}{w(t)} b(s) ds. \qquad (3.37)$$

Therefore, we study in two cases as follows:

- Case 1: $\Delta > 0$.

By making use of (3.35) and (2.14), direct computation gives

$$\frac{w(s)}{w(t)} = \exp\left(\sqrt{\Delta}(s - t) + 2\ln\frac{1 - ge^{\sqrt{\Delta}(T-s)}}{1 - ge^{\sqrt{\Delta}(T-t)}}\right),$$

where g and Δ are defined as in Theorem 2.1. Then it follows from (3.37) that

$$g_2(t) = -\int_t^T e^{\sqrt{\Delta}(s-t)}\left(\frac{1 - ge^{\sqrt{\Delta}(T-s)}}{1 - ge^{\sqrt{\Delta}(T-t)}}\right)^2 b(s) ds, \qquad (3.38)$$

We can write $b(t)$ in (3.36) as

$$b(t) = c_0 + c_1\left(\frac{1 - e^{\sqrt{\Delta}(T-t)}}{1 - ge^{\sqrt{\Delta}(T-t)}}\right) + c_2\left(\frac{1 - e^{\sqrt{\Delta}(T-t)}}{1 - ge^{\sqrt{\Delta}(T-t)}}\right)^2,$$

where the constants c_0, c_1 and c_2 are given by (3.23), (3.24) and (3.25), respectively. Then, it follows from (3.38) and (3.33) that

$$
\begin{aligned}
g_2(t) &= -\Big(c_0 I_0(t) + c_1 I_1(t) + c_2 I_2(t)\Big), && (3.39)\\
g_1(t) &= -\Big(c_0 J_0(t) + c_1 J_1(t) + c_2 J_2(t)\Big), && (3.40)
\end{aligned}
$$

where the functions $I_0(t), I_1(t), I_2(t), J_0(t), J_1(t)$ and $J_2(t)$ are defined by

$$
\begin{aligned}
I_0(t) &= \int_t^T e^{\sqrt{\Delta}(s-t)} \left(\frac{1-ge^{\sqrt{\Delta}(T-s)}}{1-ge^{\sqrt{\Delta}(T-t)}} \right)^2 ds,\\
I_1(t) &= \int_t^T e^{\sqrt{\Delta}(s-t)} \left(\frac{1-ge^{\sqrt{\Delta}(T-s)}}{1-ge^{\sqrt{\Delta}(T-t)}} \right)^2 \left(\frac{1-e^{\sqrt{\Delta}(T-s)}}{1-ge^{\sqrt{\Delta}(T-s)}} \right) ds,\\
I_2(t) &= \int_t^T e^{\sqrt{\Delta}(s-t)} \left(\frac{1-ge^{\sqrt{\Delta}(T-s)}}{1-ge^{\sqrt{\Delta}(T-t)}} \right)^2 \left(\frac{1-e^{\sqrt{\Delta}(T-s)}}{1-ge^{\sqrt{\Delta}(T-s)}} \right)^2 ds,\\
J_i(t) &= \int_t^T I_i(s) ds, \quad i = 0, 1, 2.
\end{aligned}
$$

Direct computation leads to

$$
I_0(t) = \frac{e^{\sqrt{\Delta}(T-t)} \left(\left(1-e^{-\sqrt{\Delta}(T-t)}\right) - 2g\sqrt{\Delta}(T-t) - g^2\left(1-e^{\sqrt{\Delta}(T-t)}\right) \right)}{\sqrt{\Delta}\left(1-ge^{\sqrt{\Delta}(T-t)}\right)^2}, \tag{3.41}
$$

$$
I_1(t) = \frac{e^{\sqrt{\Delta}(T-t)} \left(\left(1-e^{-\sqrt{\Delta}(T-t)}\right) - (1+g)\sqrt{\Delta}(T-t) - g\left(1-e^{\sqrt{\Delta}(T-t)}\right) \right)}{\sqrt{\Delta}\left(1-ge^{\sqrt{\Delta}(T-t)}\right)^2}, \tag{3.42}
$$

$$
I_2(t) = \frac{e^{\sqrt{\Delta}(T-t)} \left(\left(1-e^{-\sqrt{\Delta}(T-t)}\right) - 2\sqrt{\Delta}(T-t) - \left(1-e^{\sqrt{\Delta}(T-t)}\right) \right)}{\sqrt{\Delta}\left(1-ge^{\sqrt{\Delta}(T-t)}\right)^2}, \tag{3.43}
$$

$$J_0(t) = -\frac{1+g+\sqrt{\Delta}(T-t)-\big(1+g-\sqrt{\Delta}g(T-t)\big)e^{\sqrt{\Delta}(T-t)}}{\Delta\big(1-ge^{\sqrt{\Delta}(T-t)}\big)}, \tag{3.44}$$

$$J_1(t) = -\frac{2+\sqrt{\Delta}(T-t)-\big(2-\sqrt{\Delta}(T-t)\big)e^{\sqrt{\Delta}(T-t)}}{\Delta\big(1-ge^{\sqrt{\Delta}(T-t)}\big)}, \tag{3.45}$$

$$J_2(t) = -\frac{1+g+\sqrt{\Delta}g(T-t)-\big(1+g-(2-g)\sqrt{\Delta}g(T-t)\big)e^{\sqrt{\Delta}(T-t)}}{\Delta g\big(1-ge^{\sqrt{\Delta}(T-t)}\big)} + \frac{(1-g)^2}{\Delta g^2}\ln\left(\frac{1-ge^{\sqrt{\Delta}(T-t)}}{1-g}\right). \tag{3.46}$$

Then, by substituting (3.41), (3.42), (3.43) in (3.39), and (3.44), (3.45), (3.46) in (3.40) we easily see that $g_2(t)$ and $g_1(t)$ are given by (3.22) and (3.21), respectively.

- Case 2: $\Delta = 0$ and $TK + 1 > 0$.

By similar calculation to Case 1, we get

$$\frac{w(s)}{w(t)} = \exp\left\{2\ln\left(\frac{(1+\frac{1}{2}(\kappa-\rho_{sz}\sigma\overline{\lambda}\Gamma)(T-s))}{(1+\frac{1}{2}(\kappa-\rho_{sz}\sigma\overline{\lambda}\Gamma)(T-t))}\right)\right\}.$$

Then (3.37) is equivalent to

$$g_2(t) = -\int_t^T \left(\frac{1+K(T-s)}{1+K(T-t)}\right)^2 b(s)ds, \tag{3.47}$$

where $K = \frac{1}{2}\big(\kappa - \Gamma\rho_{sz}\sigma\overline{\lambda}\big)$. We can write $b(t)$ in (3.36) as

$$b(t) = \bar{c}_0 + \bar{c}_1\frac{T-t}{1+K(T-t)} + \bar{c}_2\left(\frac{T-t}{1+K(T-t)}\right)^2,$$

where the constants $\bar{c}_0, \bar{c}_1$ and $\bar{c}_2$ are defined by (3.28),(3.29) and (3.30), respectively. Then, it follows from (3.47) and (3.33) that

$$g_2(t) = -\Big(\bar{c}_0\bar{I}_0(t) + \bar{c}_1\bar{I}_1(t) + \bar{c}_2\bar{I}_2(t)\Big), \tag{3.48}$$

$$g_1(t) = -\big(\bar{c}_0\bar{J}_0(t) + \bar{c}_1\bar{J}_1(t) + \bar{c}_2\bar{J}_2(t)\big), \tag{3.49}$$

where the functions $\bar{I}_0, \bar{I}_1, \bar{I}_2(t), \bar{J}_0(t), \bar{J}_1(t)$ and $\bar{J}_2$ are defined by

$$\begin{aligned}
\bar{I}_0(t) &= \int_t^T \left(\frac{1+K(T-s)}{1+K(T-t)}\right)^2 ds, \\
\bar{I}_1(t) &= \int_t^T \left(\frac{1+K(T-s)}{1+K(T-t)}\right)^2 \frac{T-s}{1+K(T-s)} ds, \\
\bar{I}_2(t) &= \int_t^T \left(\frac{1+K(T-s)}{1+K(T-t)}\right)^2 \left(\frac{T-s}{1+K(T-s)}\right)^2 ds, \\
\bar{J}_i(t) &= \int_t^T \bar{I}_i(s) ds, \quad i = 0, 1, 2.
\end{aligned}$$

Direct computation shows that

$$\bar{I}_0(t) = \frac{1}{\big(1+K(T-t)\big)^2}\big((T-t) + K(T-t)^2 + \frac{K^2}{3}(T-t)^3\big), \tag{3.50}$$

$$\bar{I}_1(t) = \frac{1}{\big(1+K(T-t)\big)^2}\big(\frac{1}{2}(T-t)^2 + \frac{K}{3}(T-t)^3\big), \tag{3.51}$$

$$\bar{I}_2(t) = \frac{(T-t)^3}{3\big(1+K(T-t)\big)^2}, \tag{3.52}$$

$$\bar{J}_0(t) = \frac{(T-t)^2(3+K(T-t))}{6\big(1+K(T-t)\big)}, \tag{3.53}$$

$$\bar{J}_1(t) = \frac{(T-t)^3}{6\big(1+K(T-t)\big)} \tag{3.54}$$

and

$$\bar{J}_2(t) = \frac{-6(T-t) - 3K(T-t)^2 + K^2(T-t)^3}{6K^3\big(1+K(T-t)\big)} + \frac{1}{K^4}\ln\big(1+K(T-t)\big). \tag{3.55}$$

Then, by plugging (3.50), (3.51), (3.52) in (3.48), and (3.53), (3.54), (3.55) in (3.49) we find that $g_2(t)$ and $g_1(t)$ are given by (3.27) and (3.26), respectively. The proof is complete.

□

4. An explicit approximation of the optimal strategy

Since we have derived the first two terms $V^{(0)}$ and $V^{(1)}$ for the optimal value function in the previous section, we can proceed to derive an asymptotic approximation to the optimal strategy π^* given by (2.3). Like the case of the optimal value function, we look for the optimal strategy π^* of the form

$$\pi^*(t,x,y,z) = \pi^{*(0)} + \sqrt{\epsilon}\pi^{*(1)} + \epsilon\pi^{*(2)} + \cdots, \tag{4.1}$$

and we are interested to derive expressions for the first two terms, $\pi^{*(0)}$ and $\pi^{*(1)}$.

Substituting the expansion (1.10) for V^ϵ into (2.3) gives

$$\begin{aligned}\pi^* =&\Big(\lambda(y)+\rho_{sz}\sigma B(t)\Big)\frac{R(t,x)}{f(y)x}+\frac{\sqrt{\epsilon}}{f(y)xV_x^{(0)}}\Bigg[\Big(\lambda(y)+\rho_{sz}\sigma B(t)\Big)D_2V^{(1)}\\ &+\lambda(y)D_1V^{(1)}+\rho_{sz}\sigma D_1V_z^{(1)}-\rho_{sy}\alpha(y)\Big(\phi'(y)+\rho_{sz}\sigma B(t)\psi'(y)\Big)D_1^2V^{(0)}\Bigg]\\ &+\cdots .\end{aligned}$$

Here, we have used the fact that $V_z^{(0)} = B(t)V^{(0)}$. Using the explicit expressions of $V^{(0)}$ and $V^{(1)}$ given respectively in Theorem 2.1 and Theorem 3.1 and the fact that $D_2V^{(0)} = -D_1V^{(0)}$, we have the following asymptotic result for the optimal strategy.

Theorem 4.1. *The first order correction of the optimal strategy* $\pi^*(t,x,y,z)$ *is*

$$\tilde{\pi}^* = \pi^{*(0)} + \sqrt{\epsilon}\pi^{*(1)},$$

where $\pi^{*(0)}$ *and* $\pi^{*(1)}$ *are given by*

$$\begin{aligned}\pi^{*(0)} &= \frac{1}{qf(y)}\Big(\lambda(y)+\rho_{sz}\sigma B(t)\Big)\Big(\frac{q}{1-p}+\frac{\eta e^{r(t-T)}}{x}\Big),\\ \pi^{*(1)} &= \frac{\rho_{sz}\sigma g_2(t)-\Gamma\rho_{sy}\alpha(y)\Big(\phi'(y)+\rho_{sz}\sigma B(t)\psi'(y)\Big)}{qf(y)}\left(\frac{q}{1-p}+\frac{\eta e^{r(t-T)}}{x}\right).\end{aligned}$$

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