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A VARIANT OF THE QUADRATIC FUNCTIONAL EQUATION ON GROUPS AND AN APPLICATION

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Abstract. Let G be a group and $\mathbb{C}$ the field of complex numbers. Suppose $\sigma : G \to G$ is an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all x in G. In this paper, we first determine the central solution, $f : G$ or $G \times G \to \mathbb{C}$, of the functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x) + 2f(y) \quad \text{for all } x, y \in G,$$

which is a variant of the quadratic functional equation. Using the central solution of this functional equation, we determine the general solution of the functional equation $f(pr, qs) + f(sp, rq) = 2f(p, q) + 2f(r, s)$ for all $p, q, r, s \in G$, which is a variant of the equation $f(pr, qs) + f(ps, qr) = 2f(p, q) + 2f(r, s)$ studied by Chung, Kannappan, Ng and Sahoo in [3] (see also [16]). Finally, we determine the solutions of this equation on the free groups generated by one element, the cyclic groups of order m, the symmetric groups of order m, and the dihedral groups of order $2m$ for $m \geq 2$.

1. Introduction

The functional equation

$$f(xy) + f(xy^{-1}) = 2f(x) + 2f(y)$$

for all $x, y \in G$, where G is a group written multiplicatively and y^{-1} is the inverse of y, is known as the quadratic functional equation. It serves in certain abstract spaces for the definition of the norm. It was studied by many authors including Jensen [5, 6], Jordan and von Neumann [7], Kurepa [11], Aczél and Vincze [2], Aczél [1], Kannappan [8–10], and Yang [19]. Sinopoulos [17] considered the following generalization of the quadratic functional equation

$$f(xy) + f(x\sigma(y)) = 2f(x) + 2f(y) \tag{1.1}$$

for all $x, y \in S$, where S is a commutative semigroup and $\sigma : S \to S$ is an endomorphism of S such that $\sigma(\sigma(x)) = x$ for all $x \in S$. In this paper, we

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consider a variant of this functional equation, namely

$$f(xy) + f(\sigma(y)x) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in G$, where G is a group (not necessarily abelian). If G is an abelian group or f is a central function on a group G, then the equations (1.1) and (1.2) are equivalent.

This paper is organized as follows: In Section 2, we introduce the definition of relevant terminologies and notations that will be used in the subsequent sections of the paper. In Section 3, we prove some preliminary results that will be used to determine the solution of the equation (1.2) on groups. Section 4 contains the solution of (1.2) on groups when f is a central function. In Section 5, we solve the functional equation $f(pr, qs) + f(sp, rq) = 2f(p, q) + 2f(r, s)$ on groups as an application of a result related to equation (1.2). Section 6 contains the solution of the functional equation (1.2) on certain groups, such as, the free groups generated by one element, the cyclic groups of order m, the symmetric groups of order m, and the dihedral groups of order $2m$ for $m \geq 2$.

2. Notation and terminology

Let G be a group and let S be a semigroup written multiplicatively with the identity element e. Let $\mathbb{C}$ be the field of complex numbers. A function $f : S \to \mathbb{C}$ is said to be central if and only if $f(xy) = f(yx)$ for all $x, y \in S$. Similarly, a function $f : S \times S \to \mathbb{C}$ is said to be central if and only if $f(xy, uv) = f(yx, vu)$ for all $x, y, u, v \in S$. A function $A : S \to \mathbb{C}$ is said to be a homomorphism if $A(xy) = A(x) + A(y)$ for all $x, y \in S$. It is known that $A(e) = 0$. Similarly, $A : S \times S \to \mathbb{C}$ is said to be a homomorphism if $A(xy, uv) = A(x, u) + A(y, v)$ for all $x, y, u, v \in S$. It is known that $A(e, e) = 0$. A function $B : S \times S \to \mathbb{C}$ is said to be a bi-homomorphism if and only if B is a homomorphism in each variable, that is $B(xy, z) = B(x, z) + B(y, z)$ and $B(x, yz) = B(x, y) + B(x, z)$ for all $x, y, z \in S$ or equivalently B satisfies $B(xy, uv) = B(x, u) + B(x, v) + B(y, u) + B(y, v)$ for $x, y, u, v \in S$. The map $B : S \times S \to \mathbb{C}$ is said to be symmetric if and only if $B(x, u) = B(u, x)$ for all $x, u \in S$. It can be easily seen that $B(e, x) = B(x, e) = B(e, e) = 0$ for all $x \in S$ and $B(x^{-1}, u) = -B(x, u)$ for $x, u \in S$. The map $\sigma : S \to S$ is an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in S$.

Let $f : S \to \mathbb{C}$ be a function. The Cauchy difference $C_f : S \times S \to \mathbb{C}$ of a function f is defined by $C_f(x, y) := f(xy) - f(x) - f(y)$ for all $x, y \in S$. The Cauchy difference $C_f(x, y)$ measures how much f deviates from being a homomorphism of the semigroup S into the additive group $(\mathbb{C}, +)$. The second Cauchy difference of f, $C_f^{(2)} : S \times S \times S \to \mathbb{C}$, is defined by $C_f^{(2)}(x, y, z) := C_f(xy, z) - C_f(x, z) - C_f(y, z)$ for all $x, y, z \in S$.

3. Some preliminary results

Lemma 3.1. *Let $\mathbb{C}$ be the field of complex numbers. Let S be a semigroup and $\sigma : S \to S$ be an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in S$. Let $f : S \to \mathbb{C}$ be a function satisfying the functional equation* (1.2)*, that is*

$$f(xy) + f(\sigma(y)\,x) = 2\,f(x) + 2\,f(y)$$

for all $x, y \in S$. Then $f : S \to \mathbb{C}$ also satisfies the functional equation

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(xz) - f(y) \tag{3.1}$$

for all $x, y, z \in S$.

Proof. Let f be a solution of (1.2). Then, letting $x = xy$ and $y = z$ in (1.2) we get that

$$f(xyz) + f(\sigma(z)\,xy) = 2\,f(xy) + 2\,f(z) \tag{3.2}$$

for all $x, y, z \in S$. Similarly, letting $x = \sigma(z)x$ in (1.2) and using the fact that σ is an endomorphism, the resulting equation yields

$$f(\sigma(z)xy) + f(\sigma(yz)\,x) = 2\,f(\sigma(z)x) + 2\,f(y) \tag{3.3}$$

for all $x, y, z \in S$. Letting $y = yz$ in (1.2) and rewriting we get that

$$f(\sigma(yz)x) = 2\,f(x) + 2\,f(yz) - f(xyz) \tag{3.4}$$

for all $x, y, z \in S$. Letting $y = z$ in (1.2), rewriting, and multiplying by 2 yields

$$2\,f(\sigma(z)x) = 4\,f(x) + 4\,f(z) - 2f(xz) \tag{3.5}$$

for all $x, y, z \in S$. Substituting (3.4) and (3.5) into (3.3) and then rewriting the resulting expression we get

$$f(\sigma(z)xy) - f(xyz) = 2\,f(x) + 2\,f(y) + 4\,f(z) - 2\,f(xz) - 2\,f(yz) \tag{3.6}$$

for all $x, y, z \in S$. Subtracting the previous equation (3.6) from (3.2) and dividing by 2 we get the asserted equation:

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(xz) - f(y)$$

for all $x, y, z \in S$. This completes the proof. □

Remark 3.2. The functional equation (3.1) is in fact the kernel of the second Cauchy difference of f, that is $C_f^{(2)}(x, y, z) = 0$. This equation first appeared in a paper by J. H. C. Whitehead in 1950 [20]. He solved the functional equation (3.1) on abelian groups assuming that f is an even function. The equation has been referred to by Faiziev and Sahoo in [4] as Whitehead's functional equation. If f is a central function, then Whitehead's equation takes the form

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(zx) - f(y).$$

This equation is often referred to as Fréchet's functional equation. Kannappan, in [10], and Stetkær, in [18], deal with these equations in various settings.

The next lemma follows from Kannappan's work (see [8], [9], and [10]).

Lemma 3.3. *Let G be a group, $\mathbb{C}$ be the field of complex numbers and $f : G \to \mathbb{C}$ satisfy the functional equation*

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(zx) - f(y)$$

for all $x, y, z \in G$. Then f satisfies

$$8\, f(x) = 4\, a(x) + b(x, x)$$

for all $x \in G$, where $a : G \to \mathbb{C}$ is a homomorphism and $b : G \times G \to \mathbb{C}$ is a symmetric bi-homomorphism.

Remark 3.4. An examination of Kannappan's proof reveals that

(i) $a(x) := f(x) - f(x^{-1})$ and
(ii) $b(x, y) := q(xy) - q(xy^{-1})$, where $q(x) := f(x) + f(x^{-1})$

for all $x \in G$. That is, $a(x)$ and $q(x)$ are the odd and even parts of the solution f of Fréchet's functional equation respectively.

4. The central solution of the equation (1.2)

In this section, we determine the central solution of the functional equation (1.2).

Theorem 4.1. *Let G be a group and $\sigma : G \to G$ be an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Let $f : G \to \mathbb{C}$ be a central function satisfying the functional equation* (1.2) *for all $x, y \in G$. Then*

$$f(x) = A(x) + B(x, x), \tag{4.1}$$

where $A : G \to \mathbb{C}$ is a homomorphism satisfying $A(\sigma(x)) = A(x)$ for all $x \in G$ and $B : G \times G \to \mathbb{C}$ is a symmetric bi-homomorphism satisfying $B(\sigma(x), y) = -\,B(x, y)$ for all $x, y \in G$. The converse is also true.

Proof. Let f be a solution of (1.2). From Lemma 3.1 we have that f satisfies the functional equation

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(xz) - f(y)$$

for all $x, y, z \in G$. Using the fact that f is central the previous equation yields

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(zx) - f(y)$$

for all $x, y, z \in G$. From Lemma 3.3 we have that

$$8\, f(x) = 4\, a(x) + b(x, x)$$

for all $x \in G$, and hence

$$f(x) = A(x) + B(x, x), \tag{4.2}$$

where $A : G \to \mathbb{C}$ is a homomorphism and $B : G \times G \to \mathbb{C}$ is a symmetric bi-homomorphism.

One can see that if $x = y = e$ in (1.2), where e is the identity element in G, we get that $f(e) = 0$. Now, letting $x = e$ in (1.2) and using the fact that $f(e) = 0$ we have that

$$f(\sigma(y)) = f(y) \tag{4.3}$$

for all y in G. Therefore, by item (i) of Remark 3.4 we have that

$$A(\sigma(x)) = f(\sigma(x)) - f(\sigma(x)^{-1}) = f(x) - f(x^{-1}) = A(x) \tag{4.4}$$

for all $x \in G$, thus A is a σ-even function. By (4.2) and (4.3) we see that

$$A(\sigma(x)) + B(\sigma(x), \sigma(x)) = A(x) + B(x, x)$$

for all $x \in G$, thus (4.4) yields

$$B(\sigma(x), \sigma(x)) = B(x, x) \tag{4.5}$$

for all $x \in G$. Therefore, using (4.5), the fact that σ is an endomorphism, and the fact that B is a symmetric bi-homomorphism we have the following:

$$\begin{aligned}
&B(xy, xy)\\
&= B(\sigma(xy), \sigma(xy))\\
&= B(\sigma(x)\sigma(y), \sigma(x)\sigma(y))\\
&= B(\sigma(x), \sigma(x)) + B(\sigma(x), \sigma(y)) + B(\sigma(y), \sigma(x)) + B(\sigma(y), \sigma(y))\\
&= B(\sigma(x), \sigma(x)) + 2\,B(\sigma(x), \sigma(y)) + B(\sigma(y), \sigma(y))\\
&= B(x, x) + 2\,B(\sigma(x), \sigma(y)) + B(y, y)
\end{aligned}$$

for all $x, y \in G$. Since B is a symmetric bi-homomorphism

$$B(xy, xy) = B(x, x) + 2\,B(x, y) + B(y, y)$$

for all $x, y \in G$, thus

$$B(\sigma(x), \sigma(y)) = B(x, y)$$

for all $x, y \in G$.

Now, using (4.2) in (1.2) we have that

$$\begin{aligned}
&A(xy) + B(xy, xy) + A(\sigma(y)x) + B(\sigma(y)x, \sigma(y)x)\\
&= 2\,A(x) + 2\,B(x, x) + 2\,A(y) + 2\,B(y, y)
\end{aligned}$$

for all $x, y \in G$. Using the fact that A is a homomorphism and σ-even, that is A satisfies (4.4), we obtain

$$\begin{aligned}
&A(x) + A(y) + B(xy, xy) + A(\sigma(y)) + A(x) + B(\sigma(y)x, \sigma(y)x)\\
&= 2\,A(x) + 2\,B(x, x) + 2\,A(y) + 2\,B(y, y),
\end{aligned}$$

which simplifies to

$$B(xy, xy) + B(\sigma(y)x, \sigma(y)x) = 2\,B(x, x) + 2\,B(y, y)$$

for all $x, y \in G$. Now, using the fact that B is a symmetric bi-homomorphism the last equality gives us

$$B(x, x) + 2\,B(x, y) + B(y, y) + B(\sigma(y), \sigma(y)) + 2\,B(\sigma(y), x) + B(x, x)$$

$$= 2\,B(x,x) + 2\,B(y,y)$$

for all $x, y \in G$. Simplifying and using (4.5) and the fact that B is symmetric we get

$$B(\sigma(x), y) = -\,B(x,y)$$

for all $x, y \in G$.

It is easy to check that any function of the form (4.2) having the properties that $A(\sigma(x)) = A(x)$ and $B(\sigma(x), y) = -\,B(x,y)$ for all $x, y \in G$ is a solution of (1.2). This completes the proof. □

5. An application of the central solution of the equation (1.2)

The goal of this section is to determine the general solution $f : G \times G \to \mathbb{C}$ of the following functional equation

$$f(pr, qs) + f(sp, rq) = 2\,f(p,q) + 2\,f(r,s) \tag{5.1}$$

for all $p, q, r, s \in G$. This equation is a variant of the functional equation $f(pr, qs) + f(ps, qr) = 2f(p,q) + 2f(r,s)$ studied in [3] for all p, q, r, s in the open unit interval $(0,1)$ and equivalent to the above equation when G is an abelian group.

For the sake of convenience, throughout this section we will denote $G \times G$ by $\overline{G}$ and $S \times S$ by $\overline{S}$; then $\overline{G}$ (or $\overline{S}$) is a group (or semigroup) under componentwise multiplication. That is,

$$(p,q)(r,s) = (pr, qs)$$

for all $p, q, r, s \in G$ (or S). Define a function $\sigma : \overline{G} \to \overline{G}$ (or $\sigma : \overline{S} \to \overline{S}$) such that $\sigma(p,q) = (q,p)$ for all $p, q \in G$ (or S). Then it is easy to see that σ is an endomorphism. To see this, consider $\sigma(pr, qs)$. Using the definition of σ, we have

$$\sigma(pr, qs) = (qs, pr) = (q,p)\,(s,r) = \sigma(p,q)\,\sigma(r,s).$$

Further, σ satisfies the property

$$\sigma(\sigma(p,q)) = \sigma(q,p) = (p,q)$$

for all $x = (p,q) \in \overline{G}$ (or $\overline{S}$), that is $\sigma(\sigma(x)) = x$ for all $x \in \overline{G}$ (or $\overline{S}$).

One can see that if we let $x = (p,q)$ and $y = (r,s)$ in

$$f(pr, qs) + f(sp, rq) = 2\,f(p,q) + 2\,f(r,s),$$

it reduces to

$$f(xy) + f(\sigma(y)\,x) = 2\,f(x) + 2\,f(y)$$

for all $x, y \in \overline{G}$ (or $\overline{S}$). Therefore, Theorem 4.1 can be used to determine the solution of the functional equation (5.1) on groups. We begin with a few preliminary results.

Lemma 5.1. *Let S be a semigroup and $f : \overline{S} \to \mathbb{C}$ be a solution to the functional equation* (5.1) *for all $p, q, r, s \in S$. Then f is a central function.*

Proof. Let f be a solution to (5.1). Then

$$f(pr,qs)+f(sp,rq)=2\,f(p,q)+2\,f(r,s)$$

holds for all $p,q,r,s\in S$. Letting $p=q=r=s=e$, where e is the identity element in S, we get $f(e,e)=0$. Setting $p=q=e$ gives us

$$f(s,r)=f(r,s)$$

for all $r,s\in S$. Therefore, f is symmetric.

Now, setting $q=s=e$ we get

$$f(pr,e)+f(p,r)=2\,f(p,e)+2\,f(r,e)$$

for all $p,r\in S$. Defining a function $g:S\to\mathbb{C}$ such that $g(p):=f(p,e)$ for all $p\in S$ the previous becomes

$$f(p,r)=2\,g(p)+2\,g(r)-g(pr) \tag{5.2}$$

for all $p,r\in S$. Since f is symmetric, we have the following:

$$f(p,r)=2\,g(r)+2\,g(p)-g(rp)$$

for all $p,r\in S$. Subtracting the previous two equations yields

$$g(pr)=g(rp)$$

for all $p,r\in S$ and hence g is central.

Now, switching r and s in (5.1) and using the fact that f is symmetric we get

$$f(qr,ps)+f(sq,rp)=2\,f(p,q)+2\,f(s,r)$$

for all $p,q,r,s\in S$. Therefore,

$$f(pr,qs)+f(sp,rq)=f(qr,ps)+f(sq,rp)$$

for all $p,q,r,s\in S$. Using (5.2) to expand the previous we get

$$\begin{aligned}&2\,g(pr)+2\,g(qs)-g(prqs)+2\,g(sp)+2\,g(rq)-g(sprq)\\ &=2\,g(qr)+2\,g(ps)-g(qrps)+2\,g(sq)+2\,g(rp)-g(sqrp)\end{aligned}$$

for all $p,r,q,s\in S$. Since g is central it simplifies to

$$g(prqs)=g(qrps)$$

and thus

$$g(prqs)=g(rpsq)$$

for all $p,r,q,s\in S$. Using (5.2) and computing $f(pr,qs)$ and $f(rp,sq)$, we get the following:

$$f(pr,qs)=2\,g(pr)+2\,g(qs)-g(prqs)$$

and

$$f(rp,sq)=2\,g(rp)+2\,g(sq)-g(rpsq)$$

for all $p,q,r,s\in S$. Therefore,

$$f(pr,qs)=f(rp,sq)$$

for all $p, q, r, s \in S$ and f is central. This completes the proof. $\square$

In the next few lemmas, namely, Lemmas 5.2 and 5.4, using some ideas from Sahoo in [15] (see also [16], p. 42), we present two results regarding the decomposition of homomorphisms and bi-homomorphisms on non-unital semigroups.

Lemma 5.2. *Let S be a semigroup (not necessarily unital) and $A : \overline{S} \to \mathbb{C}$. Then A is a homomorphism if and only if*

$$A(p, r) = A_1(p) + A_2(r)$$

for all $p, r \in S$, where $A_1, A_2 : S \to \mathbb{C}$ are both homomorphisms. Moreover, $A_1(p) = A(pa, a) - A(a, a)$ and $A_2(q) = A(a, qa) - A(a, a)$ for a fixed $a \in S$ and all $p, q \in S$.

Proof. Let $A : \overline{S} \to \mathbb{C}$ be a homomorphism. Hence

$$A(pr, qs) = A(p, q) + A(r, s) \tag{5.3}$$

holds for all $p, q, r, s \in S$. Let $a \in S$ be a fixed element. Using associativity and the fact that A is a homomorphism we have the following for all $p, q \in S$:

$$\begin{aligned}
A(p, q) &= A(a, a) + A(p, q) + A(a, a) - 2\,A(a, a) \\
&= A(apa, aqa) - 2A(a, a) \\
&= A(apa, aq \cdot a) - 2\,A(a, a) \\
&= A(a, aq) + A(pa, a) - 2\,A(a, a) \\
&= A(a, aq) + A(a, a) + A(pa, a) - 3\,A(a, a) \\
&= A(aa, aqa) + A(pa, a) - 3\,A(a, a) \\
&= A(a, a) + A(a, qa) + A(pa, a) - 3\,A(a, a) \\
&= A(pa, a) - A(a, a) + A(a, qa) - A(a, a) \\
&= A_1(p) + A_2(q),
\end{aligned}$$

where

$$\begin{aligned}
A_1(p) &:= A(pa, a) - A(a, a), \\
A_2(q) &:= A(a, qa) - A(a, a).
\end{aligned}$$

Next, we show that A_1 and A_2 are homomorphisms from S into $\mathbb{C}$. Observe that

$$\begin{aligned}
A_1(pq) &= A(pqa, a) - A(a, a) \\
&= A(a, a) + A(pqa, a) - 2\,A(a, a) \\
&= A(apqa, aa) - 2\,A(a, a) \\
&= A(paqa, aa) - 2\,A(a, a) \\
&= A(ap, a) + A(qa, a) - 2\,A(a, a) \\
&= A(ap, a) - A(a, a) + A_1(q)
\end{aligned}$$

$$\begin{aligned}
&= A(ap, a) + A(a, a) - 2\,A(a, a) + A_1(q)\\
&= A(apa, aa) - 2\,A(a, a) + A_1(q)\\
&= A(a, a) + A(pa, a) - 2\,A(a, a) + A_1(q)\\
&= A_1(p) + A_1(q)
\end{aligned}$$

for all $p, q \in S$. Hence A_1 is a homomorphism. Similarly, it can be shown that A_2 is also a homomorphism from S into $\mathbb{C}$. The converse clearly holds. $\square$

Lemma 5.3. *Let S be a non-unital semigroup, $B : \overline{S} \times \overline{S} \to \mathbb{C}$ a symmetric bi-homomorphism, and $(a, a) \in \overline{S}$ a fixed element. Then the following hold*:

$$\begin{aligned}
B((pa, a), (\star, \star)) &= B((ap, a), (\star, \star)),\\
B((a, pa), (\star, \star)) &= B((a, ap), (\star, \star))
\end{aligned}$$

for all $p \in S$.

Proof. Let $a \in S$ be a fixed element. Using the definition of a bi-homomorphism, we have

$$\begin{aligned}
B((pa, a), (\star, \star)) &= B((a, a), (\star, \star)) + B((pa, a), (\star, \star)) - B((a, a), (\star, \star))\\
&= B((apa, aa), (\star, \star)) - B((a, a), (\star, \star))\\
&= B((ap, a), (\star, \star)) + B((a, a), (\star, \star)) - B((a, a), (\star, \star))\\
&= B((ap, a), (\star, \star))
\end{aligned}$$

for all $p \in S$. $\square$

In the next theorem we shall show that a bi-homomorphism on a product of two semigroups $\overline{S} \times \overline{S}$ splits into a sum of four bi-homomorphisms on the factor $\overline{S}$.

Lemma 5.4. *Let S be a semigroup* (*not necessarily unital*) *and the function $B : \overline{S} \times \overline{S} \to \mathbb{C}$ be a symmetric bi-homomorphism. Then there exist symmetric bi-homomorphisms $\psi_1, \psi_2, \psi_3, \psi_4 : \overline{S} \to \mathbb{C}$ such that*

$$B((p, q), (r, s)) = \psi_1(p, r) + \psi_2(p, s) + \psi_3(q, r) + \psi_4(q, s)$$

for all $p, q, r, s \in S$.

Proof. Let $B : \overline{S} \times \overline{S} \to \mathbb{C}$ be a symmetric bi-homomorphism. By the definition of a bi-homomorphism, we have

$$\begin{aligned}
&B((p_1, q_1)(p_2, q_2),\ (r_1, s_1)(r_2, s_2))\\
&= B((p_1, q_1),\ (r_1, s_1)(r_2, s_2)) + B((p_2, q_2),\ (r_1, s_1)(r_2, s_2))
\end{aligned}$$

for all $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2 \in S$. For fixed $r_1, r_2, s_1, s_2 \in S$, defining $B_1 : \overline{S} \to \mathbb{C}$ by

$$B_1(p, q) := B((p, q),\ (r_1, s_1)(r_2, s_2)) \tag{5.4}$$

the previous equation becomes

$$B_1((p_1, q_1)(p_2, q_2)) = B_1(p_1, q_1) + B_1(p_2, q_2) \tag{5.5}$$

for all $p_1, p_2, q_1, q_2 \in S$. Hence, B_1 is a homomorphism from $\overline{S}$ into $\mathbb{C}$. By Lemma 5.2, B_1 can be decomposed as a sum of two homomorphisms on S.

Fixing $(a, a) \in \overline{S}$, and using (5.4) and (5.5) we get that

$$\begin{aligned}
&B((p,q),(r_1,s_1)(r_2,s_2)) \\
&= B((pa,a),\,(r_1,s_1)(r_2,s_2)) - B((a,a),\,(r_1,s_1)(r_2,s_2)) \\
&\quad + B((a,aq),\,(r_1,s_1)(r_2,s_2)) - B((a,a),\,(r_1,s_1)(r_2,s_2)).
\end{aligned}$$

Since B is a bi-homomorphism, the last equality can be rewritten as

$$\begin{aligned}
&B((p,q),(r_1,s_1)) + B((p,q),\,(r_2,s_2)) \\
&= B((pa,a),(r_1,s_1)) + B((pa,a),(r_2,s_2)) \\
&\quad - B((a,a),(r_1,s_1)) - B((a,a),(r_2,s_2)) \\
&\quad + B((a,aq),(r_1,s_1)) + B((a,aq),(r_2,s_2)) \\
&\quad - B((a,a),(r_1,s_1)) - B((a,a),(r_2,s_2)).
\end{aligned} \tag{5.6}$$

Rearranging (5.6), we obtain

$$\begin{aligned}
&B((p,q),(r_1,s_1)) - B((pa,a),(r_1,s_1)) \\
&\quad - B((a,aq),(r_1,s_1)) + 2B((a,a),(r_1,s_1)) \\
&= -\Bigg[B((p,q),\,(r_2,s_2)) - B((pa,a),(r_2,s_2)) \\
&\quad - B((a,aq),(r_2,s_2)) + 2B((a,a),(r_2,s_2))\Bigg].
\end{aligned}$$

For fixed $p, q \in S$, the left side of the above equality is a function of r_1 and s_1, whereas the right side of the equality is a function of r_2 and s_2. Therefore, each side is equal to some function $\alpha(p, q)$. However, the left side and the right side differ by a minus sign, thus the function $\alpha(p, q)$ must be zero. Hence

$$B((p,q),\,(r,s)) = B((pa,a),\,(r,s)) + B((a,aq),(r,s)) - 2\,B((a,a),\,(r,s)).$$

Since, for fixed p and q in S, each B term on the right side of the above equation is a homomorphism on S, using Lemma 5.2, we have

$$\begin{aligned}
B((p,q),(r,s)) &= B((pa,a),(r,s)) + B((a,aq),(r,s)) - 2B((a,a),(r,s)) \\
&= B((pa,a),(ra,s)) + B((pa,a),(a,as)) - 2B((pa,a),(a,a)) \\
&\quad + B((a,aq),(ra,s)) + B((a,aq),(a,as)) - 2B((a,aq),(a,a)) \\
&\quad - 2B((a,a),(ra,a)) - 2B((a,a),(a,as)) + 4B((a,a),(a,a)).
\end{aligned}$$

Therefore, $B((p, q),(r, s))$ can be decomposed as

$$B((p,q),(r,s)) = \psi_1(p,r) + \psi_2(p,s) + \psi_3(q,r) + \psi_4(q,s), \tag{5.7}$$

where $\psi_1, \psi_2, \psi_3, \psi_4 : \overline{S} \to \mathbb{C}$ are given by

$$\begin{aligned}
\psi_1(p,r) &= B((pa,a),(ra,a)) - B((pa,a),(a,a)) \\
&\quad - B((a,a),(ra,a)) + B((a,a),(a,a)),
\end{aligned} \tag{5.8}$$

$$\psi_2(p,s) = B((pa,a),(a,as)) - B((pa,a),(a,a)) - B((a,a),(a,as)) + B((a,a),(a,a)), \tag{5.9}$$

$$\psi_3(q,r) = B((a,aq),(ra,a)) - B((a,aq),(a,a)) - B((a,a),(ra,a)) + B((a,a),(a,a)), \tag{5.10}$$

$$\psi_4(q,s) = B((a,aq),(a,as)) - B((a,aq),(a,a)) - B((a,a),(a,as)) + B((a,a),(a,a)). \tag{5.11}$$

Now, we are left to show that the functions ψ_1, ψ_2, ψ_3, and ψ_4 are symmetric bi-homomorphisms. We will show one. The others are similar and hence are left for the reader. Considering $\psi_1(pq, rs)$ and using Lemma 5.3, we have the following:

$$\begin{aligned}
&\psi_1(pq,rs)\\
&= B((pqa,a),(rsa,a)) - B((pqa,a),(a,a))\\
&\quad - B((a,a),(rsa,a)) + B((a,a),(a,a))\\
&= B((apqa,aa),(rsa,a)) - B((pqa,a),(a,a))\\
&\quad - 2B((a,a),(rsa,a)) + B((a,a),(a,a))\\
&= B((ap,a),(rsa,a)) + B((qa,a),(rsa,a)) - B((pqa,a),(a,a))\\
&\quad - 2B((a,a),(rsa,a)) + B((a,a),(a,a))\\
&= B((pa,a),(rsa,a)) + B((qa,a),(rsa,a)) + B((a,a),(a,a))\\
&\quad + B((pqa,a),(a,a)) - 2B((a,a),(rsa,a)) + 2B((a,a),(a,a))\\
&= B((pa,a),(rsa,a)) + B((qa,a),(rsa,a)) - B((apqa,aa),(a,a))\\
&\quad - 2B((a,a),(rsa,a)) + 2B((a,a),(a,a))\\
&= B((pa,a),(rsa,a)) + B((qa,a),(rsa,a)) - B((ap,a),(a,a))\\
&\quad - B((qa,a),(a,a)) - 2B((a,a),(rsa,a)) + 2B((a,a),(a,a))\\
&= B((pa,a),(arsa,aa)) - B((pa,a),(a,a)) + B((qa,a),(arsa,aa))\\
&\quad - B((qa,a),(a,a)) - B((pa,a),(a,a)) - B((qa,a),(a,a))\\
&\quad - 2B((a,a),(arsa,aa)) + 4B((a,a),(a,a))\\
&= B((pa,a),(ar,a)) + B((pa,a),(sa,a)) + B((qa,a),(ar,a))\\
&\quad + B((qa,a),(sa,a)) - B((pa,a),(a,a)) - B((qa,a),(a,a))\\
&\quad - B((pa,a),(a,a)) - B((qa,a),(a,a)) - 2B((a,a),(ra,a))\\
&\quad - 2B((a,a),(sa,a)) + 4B((a,a),(a,a))\\
&= B((pa,a),(ra,a)) - B((pa,a),(a,a)) - B((a,a),(ra,a)) + B((a,a),(a,a))\\
&\quad + B((pa,a),(sa,a)) - B((pa,a),(a,a)) - B((a,a),(sa,a)) + B((a,a),(a,a))\\
&\quad + B((qa,a),(ra,a)) - B((qa,a),(a,a)) - B((a,a),(ra,a)) + B((a,a),(a,a))\\
&\quad + B((qa,a),(sa,a)) - B((qa,a),(a,a)) - B((a,a),(sa,a)) + B((a,a),(a,a))
\end{aligned}$$

$$= \psi_1(p,r) + \psi_1(p,s) + \psi_1(q,r) + \psi_1(q,s)$$

for all $p,q,r,s \in S$. Hence, ψ_1 is a bi-homomorphism on $\overline{S}$. We are left to show that ψ_1 is symmetric. Since B is symmetric, we have that

$$\begin{aligned}
&\psi_1(p,r)\\
&= B((pa,a),(ra,a)) - B((pa,a),(a,a)) - B((a,a),(ra,a)) + B((a,a),(a,a))\\
&= B((ra,a),(pa,a)) - B((a,a),(pa,a)) - B((ra,a),(a,a)) + B((a,a),(a,a))\\
&= \psi_1(r,p)
\end{aligned}$$

for all $p,r \in S$. This completes the proof. $\square$

In the following theorem, we present the general solution of the functional equation (5.1) on groups.

Theorem 5.5. *Let G be a group and $f : \overline{G} \to \mathbb{C}$ be a solution of the functional equation* (5.1) *for all $p,q,r,s \in G$. Then*

$$f(p,r) = A(p) + A(r) + \psi(pr^{-1}, pr^{-1}) \tag{5.12}$$

for all $p,r \in G$ where $A : G \to \mathbb{C}$ is a homomorphism and $\psi : \overline{G} \to \mathbb{C}$ is a symmetric bi-homomorphism.

Proof. Let f be a solution of the functional equation (5.1). Define a function $\sigma : \overline{G} \to \overline{G}$ such that $\sigma(p,q) = (q,p)$ for all $p,q \in G$. Then σ is an endomorphism satisfying the property that

$$\sigma(\sigma(p,q)) = \sigma(q,p) = (p,q),$$

that is, $\sigma(\sigma(x)) = x$ for all $x \in \overline{G}$. Let $x = (p,q)$ and $y = (r,s)$ in (5.1) then it reduces to (1.2) for all $x,y \in \overline{G}$. Since f is a solution of (5.1) we have that f is central from Lemma 5.1. Therefore, we can apply Theorem 4.1, which gives us

$$f(x) = A(x) + B(x,x),$$

where $A : \overline{G} \to \mathbb{C}$ is an additive homomorphism satisfying $A(\sigma(x)) = A(x)$ for all $x \in \overline{G}$ and $B : \overline{G} \times \overline{G} \to \mathbb{C}$ is a symmetric bi-homomorphism satisfying $B(x,y) = -B(x,\sigma(y))$ for all $x,y \in \overline{G}$. From Lemma 5.2 and Lemma 5.3 we know that A and B can both be decomposed and thus

$$f(p,q) = A(p,q) + B((p,q),(p,q))$$

becomes

$$f(p,q) = A_1(p) + A_2(q) + \psi_1(p,p) + \psi_2(p,q) + \psi_3(q,p) + \psi_4(q,q), \tag{5.13}$$

where A_1 and A_2 are homomorphisms and ψ_1, ψ_2, ψ_3, and ψ_4 are symmetric bi-homomorphisms.

Since $x = (p,q)$, $y = (r,s)$, and $B(\sigma(x),y) = -\,B(x,y)$, we have

$$B(\sigma(p,q),(r,s)) = -\,B((p,q),(r,s)),$$

which is

$$B((q,p),(r,s)) = -B((p,q),(r,s)) \tag{5.14}$$

for all $p,q,r,s \in G$. From the decomposition of B (i.e. using Lemma 5.3) and (5.14) we have that

$$\begin{aligned} &\psi_1(q,r) + \psi_2(q,s) + \psi_3(p,r) + \psi_4(p,s) \\ &= -\psi_1(p,r) - \psi_2(p,s) - \psi_3(q,r) - \psi_4(q,s) \end{aligned} \tag{5.15}$$

for all $p,q,r,s \in G$. Letting $p = r = e$ in (5.15) and using the properties of a bi-homomorphism we have $\psi_2(q,s) = -\psi_4(q,s)$ for all $q,s \in G$. That is, $\psi_2 = -\psi_4$. Similarly, substituting $p = s = e$ into equation (5.15) we obtain $\psi_1(q,r) = -\psi_3(q,r)$ for all $q,r \in G$, which is $\psi_1 = -\psi_3$.

Since $B(\sigma(x),y) = -B(x,y)$ holds for all $x,y \in \overline{G}$, if we let $x = \sigma(x)$ and $y = \sigma(y)$, we get

$$B(x,\sigma(y)) = -B(\sigma(x),\sigma(y)).$$

Since $x = (p,q)$ and $y = (r,s)$ for $p,q,r,s \in G$, the previous relation yields

$$B((p,q),(r,s)) = B((q,p),(s,r))$$

for all $p,q,r,s \in G$. From the decomposition of B (i.e. Lemma 5.3), we obtain

$$\begin{aligned} &\psi_1(q,s) + \psi_2(q,r) + \psi_3(p,s) + \psi_4(p,r) \\ &= \psi_1(p,r) + \psi_2(p,s) + \psi_3(q,r) + \psi_4(q,s) \end{aligned} \tag{5.16}$$

for all $p,q,r,s \in G$. If we let $p = r = e$ in (5.16) and use the properties of the bi-homomorphism we see that $\psi_1(q,s) = \psi_4(q,s)$ for all $q,s \in G$. That is, $\psi_1 = \psi_4$. Similarly, if we let $p = s = e$ in (5.16) we have $\psi_2(q,r) = \psi_3(q,r)$ for all $q,r \in G$. That is, $\psi_2 = \psi_3$. Therefore,

$$\psi := \psi_1 = \psi_4 = -\psi_3 = -\psi_2, \tag{5.17}$$

where $\psi : \overline{G} \to \mathbb{C}$. Thus, from (5.13) and the properties of symmetric bi-homomorphisms we have the following:

$$\begin{aligned} f(p,q) &= A_1(p) + A_2(q) + \psi(p,p) - \psi_2(p,q) - \psi_(q,p) + \psi_4(q,q) \\ &= A_1(p) + A_2(q) + \psi(p,p) + \psi_2(p,q^{-1}) + \psi_(q^{-1},p) + \psi_4(q^{-1},q^{-1}) \\ &= A_1(p) + A_2(q) + \psi(pq^{-1},pq^{-1}) \end{aligned}$$

for all $p,q \in G$. Substituting the above into (5.1) gives us the following:

$$\begin{aligned} &A_1(pr) + A_2(qs) + \psi(pr(qs)^{-1},pr(qs)^{-1}) \\ &\quad + A_1(sp) + A_2(rq) + \psi(sp(rq)^{-1},sp(rq)^{-1}) \\ &= 2[A_1(p) + A_2(q) + \psi(pq^{-1},pq^{-1})] + 2\,[A_1(q) + A_2(s) + \psi(qs^{-1},qs^{-1})] \end{aligned}$$

for all $p,q,r,s \in G$. Simplification yields

$$\begin{aligned} &A_1(pr) + A_2(qs) + A_1(sp) + A_2(rq) \\ &= 2[A_1(p) + A_2(q)] + 2\,[A_1(r) + A_2(s)] \end{aligned} \tag{5.18}$$

for all $p, q, r, s \in G$. Letting $p = q = s = e$ in (5.18) yields $A_2(r) = A_1(r)$ for all $r \in G$. Thus $A := A_1 = A_2$, where $A : G \to \mathbb{C}$ is a homomorphism. Therefore,

$$f(p, q) = A(p) + A(q) + \psi(pq^{-1}, pq^{-1})$$

for all $p, q \in G$ is the solution of the functional equation (5.1). This completes the proof. $\square$

Remark 5.6. It can be easily shown that the functional equation $f(rp, sq) + f(ps, qr) = 2f(p, q) + 2f(r, s)$ holding for all $p, q, r, s \in G$ is equivalent to the functional equation (5.1). The solution of this functional equation is also given by (5.12).

6. The solution of (1.2) on certain groups

In this section, we determine the general solution of the functional equation (1.2) on certain groups, such as, free groups generated by one element, cyclic groups of order n, symmetric groups of order n, and dihedral groups of order $2n$ for $n \geq 2$.

Let $S(G, \mathbb{C})$ be the set of all solutions of (1.2). It is easy to see that if $f \in S(G, \mathbb{C})$, then (a) $f(e) = 0$ and (b) $f(\sigma(x)) = f(x)$ for all $x \in G$. Let $SW(G, \mathbb{C})$ denote the set of solutions of Whitehead's functional equation, namely

$$f(xyz) = f(xy) - f(z) + f(yz) - f(x) + f(xz) - f(y) \tag{6.1}$$

for all $x, y, z \in G$. Whitehead's functional equation can be viewed as the kernel of the second order Cauchy difference, namely $C_f^{(2)}(x, y, z) = f(xyz) - f(xy) - f(yz) - f(xz) + f(x) + f(y) + f(z)$. From Lemma 3.1, we know that $S(G, \mathbb{C}) \subseteq SW(G, \mathbb{C})$. The set $SW(G, \mathbb{C})$ is an abelian group under pointwise addition of functions. Since every homomorphism from G into $\mathbb{C}$ satisfies the Whitehead equation, $\mathrm{Hom}(G, \mathbb{C})$ is a subgroup of $SW(G, \mathbb{C})$, that is $\mathrm{Hom}(G, \mathbb{C}) \leq SW(G, \mathbb{C})$.

Lemma 6.1. *Let G be a free group generated by the element a. Suppose $\sigma : G \to G$ is an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Then either $\sigma(x) = x$ or $\sigma(x) = x^{-1}$, where x^{-1} denotes the inverse element of x.*

Proof. Since G is a free group generated by the element a, G is isomorphic to the additive group of integers. That is, $G = \langle a \rangle \simeq \mathbb{Z}$. For a fixed integer $n \in \mathbb{Z}$, any map $\sigma : \mathbb{Z} \to \mathbb{Z}$ defined by $\sigma(x) = n\,x$ is an endomorphism of the additive group of integers. Since $\sigma(\sigma(x)) = x$ for all $x \in \mathbb{Z}$, $\sigma(x) = x$ or $\sigma(x) = -x$ are the only possible endomorphisms on the additive group of integers. Hence G, which is isomorphic $\mathbb{Z}$, has two such endomorphisms on G, namely $\sigma(x) = x$ and $\sigma(x) = x^{-1}$. $\square$

The following lemma is taken from Ng and Zhao in [12].

Lemma 6.2. *Let G be a free group generated by the element a. If $f : G \to \mathbb{C}$ satisfies the functional equation* (3.1) *for all $x, y, z \in G$, then*

$$f(a^n) = nf(a) + \frac{n(n-1)}{2} C_f(a,a), \tag{6.2}$$

where $f(a)$ and $C_f(a,a) := f(a^2) - 2f(a)$ are complex constants. The converse is also true.

Theorem 6.3. *Let $G = \langle a \rangle$ be a free group generated by the element a in G. Assume $\sigma : G \to G$ is an endomorphism of G satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Let $f : G \to \mathbb{C}$ satisfy the functional equation* (1.2)*, that is*

$$f(xy) + f(\sigma(y)x) = 2f(x) + 2f(y)$$

for all $x, y \in G$. Then

$$f(a^n) = \begin{cases} n\, f(a) & \text{if } \sigma(a) = a \\ n^2\, f(a) & \text{if } \sigma(a) = a^{-1}, \end{cases} \tag{6.3}$$

where $f(a)$ is a complex constant. The converse is also true.

Proof. Since $G = \langle a \rangle$ is free group generated by the element a, by Lemma 6.1, $\sigma(a) = a$ or $\sigma(a) = a^{-1}$.

Case 1. Suppose $\sigma(a) = a$. Letting $x = a^n$ and $y = a^m$ for some $n, m \in \mathbb{Z}$, in (1.2), we have

$$f(a^n a^m) + f(\sigma(a^m)\, a^n) = 2f(a^n) + 2f(a^m).$$

Since $\sigma(a) = a$, the previous equation can be simplified to

$$f(a^{n+m}) = f(a^n) + f(a^m).$$

By Lemma 3.1, $S(G, \mathbb{C}) \subseteq SW(G, \mathbb{C})$. Hence, if $f \in S(G, \mathbb{C})$, then f has the same form as in (6.2). Using the form of f in (6.2) in equation (1.2), we obtain

$$\begin{aligned} &(n+m)f(a) + \frac{(n+m)(n+m-1)}{2} C_f(a,a) \\ &= (n+m)f(a) + \left[\frac{n(n-1)}{2} + \frac{m(m-1)}{2}\right] C_f(a,a). \end{aligned}$$

From the last equality, we have

$$\left[\frac{(n+m)\,(n+m-1)}{2} - \frac{n(n-1)}{2} - \frac{m(m-1)}{2}\right] C_f(a,a) = 0,$$

which yields

$$nm\, C_f(a,a) = 0.$$

Since $nm \neq 0$, $C_f(a,a) = 0$, and thus f in (6.2) yields

$$f(a^n) = n\, f(a)$$

when $\sigma(a) = a$.

Case 2. Suppose $\sigma(a) = a^{-1}$. Again, letting $x = a^n$ and $y = a^m$ for some $n, m \in \mathbb{Z}$, in (1.2) and simplifying using the fact that $\sigma(a) = a^{-1}$, we obtain

$$f(a^{n+m}) + f(a^{n-m}) = 2f(a^n) + 2f(a^m).$$

As in the previous case, putting the form of f in (6.2) into the last equation, we see that

$$\begin{aligned}(n+m)f(a) &+ \frac{(n+m)(n+m-1)}{2}C_f(a,a)\\ &+ (n-m)f(a) + \frac{(n-m)(n-m-1)}{2}C_f(a,a)\\ &= 2(n+m)f(a) + \big[n(n-1) + m(m-1)\big]C_f(a,a).\end{aligned}$$

Simplifying the last equality, we have

$$m\left(2\,f(a) - C_f(a,a)\right) = 0.$$

Since $m \neq 0$, we have $C_f(a,a) = 2\,f(a)$. Thus, (6.2) yields

$$\begin{aligned}f(a^n) &= n\,f(a) + \frac{1}{2}\,n\,(n-1)C_f(a,a)\\ &= n\,f(a) + (n^2 - n)\,f(a)\\ &= n^2\,f(a).\end{aligned}$$

This completes the proof of the theorem. □

Let $C_m = \big\langle a \,|\, a^m = e \big\rangle$ be a cyclic group of order m with generator a. Let $D_m = \big\langle a, b \,|\, a^m = e,\, b^2 = e,\, abab = e \big\rangle$ be the dihedral group of order $2m$ ($m \geq 2$), and S_m the symmetric group of order m. The following result can be collected from Ng and Zhao in [12].

Lemma 6.4. *Let the group G be either C_m, S_m or D_m. Let $\sigma : G \to G$ be an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. The function $f : G \to \mathbb{C}$ is a solution of Whitehead's functional equation* (3.1) *for all $x, y \in G$ if and only if $f(x) = 0$ for all $x \in G$.*

The following theorem follows from the above lemma.

Theorem 6.5. *Let the group G be either C_m, S_m or D_m. Let $\sigma : G \to G$ be an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. The function $f : G \to \mathbb{C}$ is a solution of the functional equation* (1.2) *for all $x, y \in G$ if and only if $f(x) = 0$ for all $x \in G$.*

We end this paper with the following open problem.

Open problem. Let G be a free group generated by two elements a, b. Let $\sigma : G \to G$ be an endomorphism satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. Determine all functions $f : G \to \mathbb{C}$ that satisfy the functional equation (1.2) for all $x, y \in G$.

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