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MOCK THETA FUNCTIONS OF ORDER 2 AND THEIR SHADOW COMPUTATIONS

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Abstract. Zwegers showed that a mock theta function can be completed to form essentially a real analytic modular form of weight 1/2 by adding a period integral of a certain weight 3/2 unary theta series. This theta series is related to the holomorphic modular form called the shadow of the mock theta function. In this paper, we discuss the computation of shadows of the second order mock theta functions and show that they share the same shadow with a mock theta function which appears in the Mathieu moonshine phenomenon.

1. Introduction

In his ground-breaking work on mock theta functions [21], Zwegers showed that a mock theta function f can be completed to form a function $\widehat{f}$ which is essentially a real analytic modular form of weight 1/2. This completion is obtained by adding a period integral of a certain weight 3/2 unary theta series. This weight 3/2 unary theta series is related to a holomorphic modular form g called the shadow of the mock theta function. The shadow g can be computed explicitly by applying the differential operator $\xi_{1/2}$ (defined in §3) to the completed form $\widehat{f}$.

Throughout, we use the notation $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ to denote the complex upper-half plane, $q := e^{2\pi i\tau}$, and $(a)_0 := (a;q)_0 = 1$, $(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k)$, $(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$, to denote the finite and infinite q-Pochhammer symbols. In the infinite case, we assume $|q| < 1$.

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There are three well-known second order mock theta functions studied by McIntosh [15]:

$$A(q) := \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2},$$
$$B(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}, \tag{1.1}$$
$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}.$$

The function $D_5(q)$ introduced by Hikami in [12] as a second order mock theta function involves $B(q)$ but it is the sum of mock theta functions of different orders as pointed out by Gordon and McIntosh [11]. Ramanujan discussed the function $\mu(q)$ in his Lost Notebook [17] and Andrews [1] proved several transformation laws satisfied by the second order mock theta functions in (1.1) and also the relation

$$\mu(q) + 4A(-q) = \frac{(q)_\infty^5}{(q^2;q^2)_\infty^4}. \tag{1.2}$$

McIntosh [15] proved additional transformation laws for the second order mock theta functions in (1.1) as well. Gordon and McIntosh established three mock theta conjectures of order 2 [11, eq. (5.2)]. One of them can be written as

$$\mu(q^4) + 2qB(q) = \frac{(q^2;q^2)_\infty (q^4;q^4)_\infty^3 (q^8;q^8)_\infty}{(q)_\infty^2 (q^{16};q^{16})_\infty^2}. \tag{1.3}$$

They also found the relation [11, Section 8]

$$\frac{B(q) + B(-q)}{2} = (q^4;q^4)_\infty (-q^2;q^2)_\infty^4. \tag{1.4}$$

The right hand sides of (1.2), (1.3) and (1.4) are all essentially weight 1/2 modular forms. This implies that all of the three second order mock theta functions share more or less the same shadow. In fact, we will show that they have shadow proportional to $\eta^3(\tau)$, where $\eta(\tau)$ is the Dedekind eta-function defined on $\mathbb{H}$ by

$$q^{-1/24}\eta(\tau) = \prod_{n=1}^{\infty}(1-q^n). \tag{1.5}$$

The purpose of this paper is to compute the shadows of the second order mock theta functions in (1.1). We show that each function, up to multiplication by a rational power of q, has shadow related to $\eta^3(\tau)$.

In Section 2, we first give simple proofs of (1.2) and (1.4) using Ramanujan's ${}_1\Psi_1$-summation formula. In Section 3, we briefly state some of Zwegers' results on mock theta functions and their shadows. In Section 4, we compute the shadows of the second order mock theta functions in (1.1). We conclude in

Section 5 with some comments about other mock theta functions with shadow related to $\eta^3(\tau)$.

2. Mock theta conjectures that arise from ${}_1\psi_1$ summation formula

Let us begin with the following equivalent form of Ramanujan's ${}_1\psi_1$ summation formula [13, Theorem 4.1]: If $c \neq -aq^{-n}$, $-bq^{-m}$ for non-negative integers n and m, then

$$\rho(a,b,c) - \rho(b,a,c) = \left(\frac{1}{b} - \frac{1}{a}\right) \frac{(c)_\infty (aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-c/a)_\infty (-c/b)_\infty (-aq)_\infty (-bq)_\infty}, \tag{2.1}$$

where

$$\rho(a,b,c) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} (a/b)^n}{(-aq)_n (-c/b)_{n+1}}. \tag{2.2}$$

The q-series defined by

$$g_2(w;q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{n(n+1)/2}}{(w;q)_{n+1} (q/w;q)_{n+1}} \tag{2.3}$$

is called a *universal mock theta function* by Gordon and McIntosh [11], since they observed that all of Ramanujan's original mock theta functions can be written in terms of g_2 and another universal mock theta function, g_3. Letting $b = -a = -w$ and $c = -q$ in the three variable reciprocity theorem (2.1), the first author found in [14, p. 563] that

$$g_2(w;q) + g_2(-w;q) = \frac{2(-q;q)_\infty^2 (q^2;q^2)_\infty}{(w^2;q^2)_\infty (q^2/w^2;q^2)_\infty}. \tag{2.4}$$

As $B(q) = g_2(q, q^2)$, we obtain (1.4) from (2.4) by replacing q by q^2 and substituting $w = q$.

Another important q-series in the theory of mock theta functions is defined by

$$h(w;q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(wq^2;q^2)_n (q^2/w;q^2)_n}. \tag{2.5}$$

Replacing q by q^2, letting $a = -w$, $b = -wq$ and $c = q$ in (2.1), and simplifying the resulting equation, we deduce that [14, eq. (5.3)][1]

$$\frac{1}{1-w} h(w;q) + (1 - \frac{1}{w}) h'(w;q) = \frac{(q;q^2)_\infty^3 (q^2;q^2)_\infty}{(w;q)_\infty (q/w;q)_\infty}, \tag{2.6}$$

where

$$h'(w;q) := \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(wq;q^2)_n (q/w;q^2)_n}. \tag{2.7}$$

[1]Here we use the notation $h(w;q)$, instead of $K(w;\tau)$ which is used in [14].

We note that $h(-1;q)=\mu(q)$, $h'(-1,q)=A(-q)$ and (1.2) is a special case of (2.6) with $w=-1$.

3. Zwegers' completions of mock theta functions

For $\tau\in\mathbb{H}$, $u,v\in\mathbb{C}\backslash(\mathbb{Z}\tau+\mathbb{Z})$, and positive integer ℓ, Zwegers [22] defined the level ℓ Appell function A_ℓ by

$$A_\ell(u,v;\tau):=w^{\ell/2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{\ell n}q^{\ell n(n+1)/2}z^n}{1-wq^n}, \tag{3.1}$$

where $w=e^{2\pi iu}$, $z=e^{2\pi iv}$, $q=e^{2\pi i\tau}$. By adding a suitable non-holomorphic correction term, Zwegers showed that A_ℓ can be completed to form essentially a real analytic Jacobi form, $\widehat{A_\ell}$. More precisely, if we define the non-holomorphic function $R(u;\tau)$ by

$$R(u;\tau):=\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}\left(\mathrm{sgn}(\nu)-\mathrm{E}\left((\nu+\mathrm{Im(u)}/\mathrm{Im}(\tau))\sqrt{2\mathrm{Im}(\tau)}\right)\right)(-1)^{\nu-\frac{1}{2}}q^{-\nu^2/2}e^{-2\pi i\nu u}, \tag{3.2}$$

where

$$E(x):=2\int_0^x e^{-\pi u^2}\,du=\mathrm{sgn}(x)(1-\beta(x^2)),\ x\in\mathbb{R} \tag{3.3}$$

$$\beta(t):=\int_t^\infty u^{-\frac{1}{2}}e^{-\pi u}\,du,\ t\in\mathbb{R}_{\geq 0}, \tag{3.4}$$

then the completed level ℓ Appell function $\widehat{A_\ell}(u,v;\tau)$ defined below transforms like a Jacobi form. Here,

$$\widehat{A_\ell}(u,v;\tau):=A_\ell(u,v;\tau)$$
$$+\frac{i}{2}\sum_{k=0}^{\ell-1}e^{2\pi iku}\vartheta(v+k\tau+(\ell-1)/2;\ell\tau)R(\ell u-v-k\tau-(\ell-1)/2;\ell\tau), \tag{3.5}$$

where

$$\vartheta(v;\tau):=\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}e^{\pi i\nu}z^\nu q^{\frac{\nu^2}{2}}=-iq^{\frac{1}{8}}z^{-\frac{1}{2}}(q)_\infty(z)_\infty(q/z)_\infty \tag{3.6}$$

is the Jacobi theta series. The explicit transformation laws for $\widehat{A_\ell}(u,v;\tau)$ can be found in [22, Theorem 2.2].

When two elliptic variables u and v are restricted to torsion points, the Jacobi form $\widehat{A_\ell}(u,v;\tau)$ in (3.5) becomes a real analytic modular form of weight 1/2 and its holomorphic part $A_\ell(u,v;\tau)$ is called a mock modular form of weight 1/2. In general, the shadow $g(\tau)$ of a mock modular form $f(\tau)$ of weight k is a weight $2-k$ weakly holomorphic modular form that is defined as the image of $\widehat{f}(\tau)=(f+C_f)(\tau)$ under the differential operator

$$\xi_k:=2iy^k\overline{\frac{\partial}{\partial\bar{\tau}}},$$

where $y = \mathrm{Im}(\tau)$ and $C_f(\tau)$ is the associated non-holomorphic function to $f(\tau)$ such that $f(\tau) + C_f(\tau)$ is a real analytic modular form of weight k. More specifically, $\widehat{f}(\tau)$ is a harmonic Maass form. Under this differential operator, the holomorphic part is mapped to zero, and thus the shadow is determined by the correction function $C_f(\tau)$. For a full description of the notions of harmonic Maass forms, mock modular forms, and shadows, the reader is referred to [19] and [16].

The shadow of an Appell function could be computed from (3.5) and [21, Lemma 1.8], which is stated below:

Proposition 3.1 ([21, Lemma 1.8])**.** *The function $R(u;\tau)$ defined in* (3.2) *is real-analytic and for a, $b \in \mathbb{R}$, it satisfies*

$$\frac{\partial}{\partial\bar{\tau}}R(a\tau-b,\tau)=-\frac{i}{\sqrt{2y}}e^{-2\pi a^2 y}\sum_{\nu\in\frac{1}{2}+\mathbb{Z}}(-1)^{\nu-\frac{1}{2}}(\nu+a)e^{-\pi i\nu^2\bar{\tau}-2\pi i\nu(a\bar{\tau}-b)}. \tag{3.7}$$

In particular, if we consider the level 1 Appell function $A_1(u, v;\tau)$, it has the correction term $\frac{i}{2}\vartheta(v;\tau)R(u-v;\tau)$ by (3.5). Thus, we can determine explicitly the shadow of the normalized level 1 Appell function $\frac{1}{\vartheta(v;\tau)}A_1(u, v;\tau)$ using Proposition 3.1. In particular, we note that in previous work of the authors et al. [10, Prop. 2.9] the shadow operator $\xi_{1/2}$ applied to this correction term (suitably multiplied by a constant multiple of a rational power of q) is shown to yield shadow $\overline{g_{a+\frac{1}{2},b+\frac{1}{2}}(-\overline{\tau})}$. These facts are summarized below in the following corollary.

Corollary 3.1. *For a, $b \in \mathbb{R}$, define a unary theta series of weight* $3/2$ *by*

$$g_{a,b}(\tau) := \sum_{\nu\in a+\mathbb{Z}} \nu e^{\pi i\nu^2\tau+2\pi i\nu b}. \tag{3.8}$$

Then $-\sqrt{2}e^{2\pi ia(b+\frac{1}{2})}q^{-a^2/2}\frac{1}{\vartheta(v;\tau)}A_1(u, v;\tau)$ with $u - v = a\tau - b$ has the correction term

$$-\frac{i}{\sqrt{2}}e^{2\pi ia(b+\frac{1}{2})}q^{-a^2/2}R(a\tau-b;\tau),$$

and hence it has the shadow $\overline{g_{a+\frac{1}{2},b+\frac{1}{2}}(-\overline{\tau})}$.

4. Shadow of the second order mock theta functions

First recall $\mu(q) = h(-1;q)$ where $h(w;q)$ is defined in (2.5). It follows from [14, eq. (5.5)]) with $w = -1$ that

$$\mu(q) = 2\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1+q^{2n}} = \frac{-4q^{1/8}}{\vartheta(-\frac{1}{2};\tau)}A_2\left(-\frac{1}{2},-\tau;2\tau\right), \tag{4.1}$$

where the second equality is from (3.6) and (3.1). By (3.5), the correction term for $A_2(-\frac{1}{2},-\tau;2\tau)$ is

$$C_1(\tau) := \frac{i}{2}\left(\vartheta\left(\frac{1}{2}-\tau;4\tau\right)R\left(-\frac{3}{2}+\tau;4\tau\right)-\vartheta\left(\frac{1}{2}+\tau;4\tau\right)R\left(-\frac{3}{2}-\tau;4\tau\right)\right).$$

Now by the transformation laws $\vartheta(-v;\tau) = \vartheta(v+1;\tau)$ and $R(-(u+1);\tau) = R(u+1;\tau) = -R(u;\tau)$ in [21, Proposition 1.3 and Proposition 1.9, respectively], we find that

$$C_1(\tau) = -i\vartheta\left(\frac{1}{2}+\tau;4\tau\right)R\left(\frac{1}{2}-\tau;4\tau\right).$$

Note from (3.6) that

$$\begin{aligned}\frac{\vartheta(\frac{1}{2}+\tau;4\tau)}{\vartheta(-\frac{1}{2};\tau)} &= -\frac{(q^4;q^4)_\infty(-q;q^2)_\infty}{2q^{1/8}(q;q)_\infty(-q;q)_\infty^2} = -q^{-1/8}\frac{(q^4;q^4)_\infty(-q;q^2)_\infty}{2(q^2;q^2)_\infty(-q;q)_\infty}\\ &= -q^{-1/8}\frac{(-q^2;q^2)_\infty(-q;q^2)_\infty}{2(-q;q)_\infty} = -2^{-1}q^{-1/8}.\end{aligned}$$

Hence

$$\widehat{\mu}(q) = \frac{-4q^{1/8}}{\vartheta(-\frac{1}{2};\tau)}\widehat{A_2}\left(-\frac{1}{2},-\tau;2\tau\right) = \mu(q) - 2iR\left(\frac{1}{2}-\tau;4\tau\right) \tag{4.2}$$

is essentially a real analytic modular form of weight 1/2 and thus $\frac{1}{2\sqrt{2}}q^{-1/8}\mu(q)$ has the shadow $\overline{4g_{\frac{1}{4},0}(-4\overline{\tau})}$ by Corollary 3.1.

On the other hand,

$$\eta^3(\tau) = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{(2n+1)^2}{8}} = \sum_{n=-\infty}^{\infty}(4n+1)q^{\frac{(4n+1)^2}{8}} = 4g_{1/4,0}(4\tau), \tag{4.3}$$

from which we conclude that $\frac{1}{2\sqrt{2}}q^{-1/8}\mu(q)$ is a mock modular form of weight 1/2 with shadow $\overline{\eta^3(-\overline{\tau})} = \eta^3(\tau)$. By (1.2), the function $-\sqrt{2}q^{-1/8}A(-q)$ also has the shadow $\eta^3(\tau)$. Also, by (1.3), the shadow of $B(q)$ must be proportional to $\eta^3(4\tau)$, we however will compute its shadow directly instead of giving another proof of (1.3).

As we noted in Section 2, $B(q) = g_2(q;q^2)$ where $g_2(w;q)$ is defined in (2.3). It follows from [14, eq. (3.3)]) with $w = q$ and q replaced with q^2 that

(4.4)

$$\begin{aligned}B(q) &= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{1-q^{2n+1}} = \frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{1-q^{2n+1}}\\ &= \frac{1}{(q^2;q^2)_\infty(q^2;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{1-q^{2n+1}}\\ &= \frac{1}{(q^4;q^4)_\infty(q^2;q^4)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n(n+1)}}{1-q^{2n+1}} = \frac{-iq^{-3/2}}{\vartheta(2\tau;4\tau)}A_2\left(\tau,\frac{1}{2};2\tau\right).\end{aligned}$$

The last equality is from (3.6) and (3.1). By (3.5), the correction term for $A_2(\tau,\frac{1}{2};2\tau)$ is

$$C_2(\tau) := \frac{i}{2}\left(\vartheta\left(1;4\tau\right)R\left(2\tau-1;4\tau\right) + q\vartheta\left(2\tau+1;4\tau\right)R\left(-1;4\tau\right)\right)$$

$$=\frac{i}{2}q\vartheta\left(2\tau+1;4\tau\right)R\left(-1;4\tau\right)=\frac{i}{2}q\vartheta\left(2\tau;4\tau\right)R\left(0;4\tau\right),$$

where the last equation is due to the transformation laws $R(u+1;\tau)=-R(u;\tau)$ and $\vartheta(v+1;\tau)=-\vartheta(v;\tau)$. Therefore,

$$\widehat{B}(q)=\frac{-iq^{-3/2}}{\vartheta(2\tau;4\tau)}\widehat{A_2}\left(\tau,\frac{1}{2};2\tau\right)=B(q)+\frac{q^{-1/2}}{2}R\left(0;4\tau\right) \tag{4.5}$$

is essentially a real analytic modular form of weight 1/2 and using Corollary 3.1, we find that the function $-\sqrt{2}iq^{1/2}B(q)$ has the shadow $\overline{4g_{\frac{1}{2},\frac{1}{2}}(-4\overline{\tau})}=-2i\eta^3(4\tau)$.

We have shown that each of the second order mock theta functions in (1.1) multiplied by a rational power of q has $\eta^3(\tau)$ as shadow.

5. Other mock modular forms with shadow $\eta^3(\tau)$

The term *eta-theta function* is used to describe a theta function which can also be written as a quotient of eta functions. In [10], the authors catalog many mock modular forms which are normalized by eta-theta functions with even characters whose shadows are eta-theta functions of odd characters. Particularly, 7 mock modular forms with $\eta^3(\frac{\tau}{4})$ as shadow are constructed in [10, Table E1]. However, we note that a mock modular form normalized by $\frac{\eta(16\tau)^2}{\eta(8\tau)}$ having shadow $\eta^3(\frac{\tau}{4})$ does not arise in this way and thus is not in this table. The second order mock theta function μ is however an Appell function (up to a rational power of q) normalized by $\vartheta\left(-\frac{1}{2};\tau\right)=2\frac{\eta(2\tau)^2}{\eta(\tau)}$ with shadow $\eta^3(\tau)$ (see (4.1)). In fact, we can find many other mock theta functions which are Appell functions normalized by $\frac{\eta(2\tau)^2}{\eta(\tau)}$ with shadow $\eta^3(\tau)$. For example, letting $u=-\tau$, $v=-1/2$ and replacing τ by 4τ when $\ell=1$ in (3.1), we obtain a normalized Appell function

$$\frac{1}{\vartheta(-\frac{1}{2};4\tau)}A_1\left(-\tau,-\frac{1}{2};4\tau\right)=\frac{1}{2}q^{-\frac{1}{2}}\frac{\eta(\tau)}{\eta(2\tau)^2}\sum_{n\in\mathbb{Z}}\frac{q^{2n(n+1)}}{1-q^{4n-1}}$$

that has shadow proportional to $\eta^3(\tau)$.

In [5], there is another example of a weight 1/2 mock modular form with shadow $\eta^3(\tau)$. It is given by

$$h^{(2)}(\tau):=\frac{24F_2^{(2)}(\tau)-E_2(\tau)}{\eta^3(\tau)}=q^{-\frac{1}{8}}(-1+45q+231q^2+770q^3+2277q^4+\cdots),$$

where $E_2(\tau)$ is the weight 2 Eisenstein series and

$$F_2^{(2)}(\tau):=\sum_{\substack{r>s>0\\ r-s\text{ odd}}}(-1)^r sq^{rs/2}=q+q^2-q^3+q^4-q^5+\cdots.$$

This mock modular form was first discussed in connection with characters of a superconformal algebra in two dimensions and the elliptic genus of $K3$ surface ([8, 9]) and has recently recognized its connection with Umbral moonshine

([3, 4, 6, 7]) and the generating function of concave compositions of a positive integer [2].

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