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LOWER BOUNDS FOR THE NUMBER OF POSITIVE AND NEGATIVE CROSSINGS IN ORIENTED LINK DIAGRAMS

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Abstract. In this paper, we obtain a simple lower bound for the number of positive (resp. negative) crossings in oriented link diagrams in terms of the maximal (resp. minimal) degree of the Jones polynomial.

1. Introduction

The Jones polynomial [2] is one of the most famous invariants of knots and links. This polynomial is a Laurent polynomial in one variable t and can be defined by the condition that its value is 1 on the unknot and the following skein relation

$$t^{-1}\ V_{L_+} - t\ V_{L_-} + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\ V_{L_0} = 0,$$

where L_+, L_-, L_0 are diagrams of three oriented links that are exactly the same except where they look as shown in Figure 1.

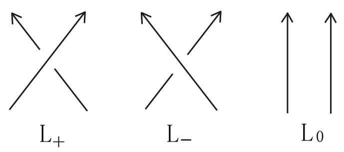

Figure 1. Three oriented diagrams which differ only inside a small disc.

In [3] Kauffman gave an elementary interpretation of the Jones polynomial. He constructed a state model for the Jones polynomial. Sometimes the state model is more useful than other approaches.

Let L be an oriented link with r components. Denote the minimal number of positive (resp. negative) crossings in all diagrams of L with $c_+(L)$ (resp. $c_-(L)$). In 1989, Murasugi [4] gave a lower bound for $c_+(L)$ (resp. $c_-(L)$) in

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terms of the maximal (resp. minimal) degree of the Jones polynomial and the signature of L, that is

$$c_+(L) \geq \max\deg V_L + \frac{1}{2}\sigma(L) \text{ and } c_-(L) \geq -\min\deg V_L + \frac{1}{2}\sigma(L).$$

In [5] Stoimenow obtained a similar lower bound for $c_-(L)$ of a non-positive link with r components, that is

$$c_-(L) \geq \frac{1}{2}(g(L) + 1 - \min\deg V_L - \frac{r-1}{2}),$$

where $g(L)$ is the genus of L.

In this note, by considering the state sum formula of the Jones polynomial, we obtain simpler lower bounds for $c_+(L)$ and $c_-(L)$, that is if L is a non-trivial link with r components, then

$$c_+(L) > \frac{1}{3}(2\max\deg V_L - r + 1) \text{ and } c_-(L) > -\frac{1}{3}(2\min\deg V_L + r - 1).$$

Comparing to Murasugi's and Stoimenow's lower bounds for $c_+(L)$ and $c_-(L)$, our lower bounds are a little bit easier to compute. And we give examples to show that for some cases our lower bounds are better than Murasugi's or Stoimenow's.

2. Lower bounds for $c_+(L)$ and $c_-(L)$

Recall that the Kauffman bracket $\langle D\rangle$, is a Laurent polynomial in a variable A, of an unoriented link diagram D satisfies the following rules;

1. $\langle \bigcirc \rangle = 1$.
2. $\langle \times \rangle = A\langle)\,(\rangle + A^{-1}\langle \asymp \rangle$.
3. $\langle \bigcirc \sqcup D\rangle = (-A^2 - A^{-2})\langle D\rangle$.

A state is a choice of how to split (type A or B, see Figure 2) all of the crossings in the diagram. Denote the number of type A and B splittings by $a(S)$ and $b(S)$ respectively, and denote the number of disjoint circles obtained after all splittings in a state by $|S|$.

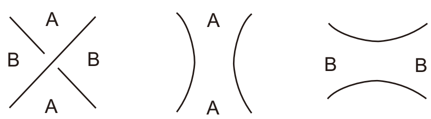


FIGURE 2. Splitting a crossing.

Then the Kauffman bracket polynomial of D is given by the state sum formula

$$\sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}.$$

Define $X_L = (-A^3)^{-w(D)}\langle D\rangle$, where D is any oriented diagram of L, then the Jones polynomial of a link L is

$$V_L(t) = X_L\big|_{A=t^{-\frac{1}{4}}}.$$

For a Laurent polynomial $f \in \mathbb{Z}[t, t^{-1}]$, denote the coefficient of monomial t^a in f by $[f]_{t^a}$, and define

$$\min\deg f = \min_{[f]_{t^a}\neq 0} a,\ \max\deg f = \max_{[f]_{t^a}\neq 0} a.$$

Suppose L is an oriented link, define $c_+(L)$ (resp. $c_-(L)$) to be the minimal number of positive (resp. negative) crossings in all diagrams of L.

Using the above state sum formula of the Jones polynomial, we can get simple lower bounds for $c_+(L)$ and $c_-(L)$ in terms of the maximal and minimal degree of the Jones polynomial respectively.

Theorem 2.1. *Suppose L is an oriented non-trivial link with r components, and V_L is the Jones polynomial of L. Then*

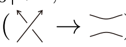
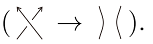
$$c_+(L) > \frac{1}{3}(2\max\deg V_L - r + 1) \quad \textit{and} \quad c_-(L) > -\frac{1}{3}(2\min\deg V_L + r - 1).$$

Proof. Let D be any diagram of L with n_+ positive crossings and n_- negative crossings. Then

$$\langle D\rangle = \sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1}.$$

For each state S, there are i, j, k, l, $i + j = n_+$ and $k + l = n_-$, with i of the n_+ positive crossings are split of type A, k of the n_- negative crossings are split of type A, and the rest crossings are split of type B. Then each state contributes a term of the form $A^{(i+k)-(j+l)}(-A^2 - A^{-2})^{|S|-1}$, so we have the formula

$$\langle D\rangle = \sum_{\substack{i+j=n_+\\ k+l=n_-}} A^{(i+k)-(j+l)}(-A^2 - A^{-2})^{|S|-1}.$$

$(i + k) - (j + l) - 2(|S| - 1)$ is the lowest power of A occurring in the term $A^{(i+k)-(j+l)}(-A^2 - A^{-2})^{|S|-1}$. In order to make $(i+k)-(j+l)-2(|S|-1)$ as small as possible, we should pick a state where j, l and $|S|$ are large but i, k are small. Since two states differ in only one crossing split with difference 1 of the number of circles, so the lowest degree of $\langle D\rangle$ is no less than $-n-2|S_0|+2$, where $n = n_+ + n_-$ and S_0 is the state of split all positive crossings with () but all negative crossings with (). Obviously, $|S_0| \leq n + r$. If there is a state S, such that $|S| = n + r$, then each crossing of D must be one of the types as shown in Figure 3. In this case, D is a diagram of the trivial link with r components.

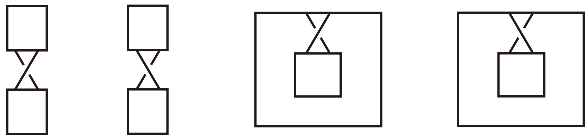


Figure 3. Non-reduced diagrams.

Since we suppose L is not a trivial link, so $|S| < n + r$ for all states. Hence we have

$$\min\deg\langle D\rangle \geq -n - 2|S_0| + 2 > -n - 2(n+r) + 2,$$

that is

$$\min\deg\langle D\rangle > -3n - 2r + 2.$$

Since $X_L = (-A^3)^{-w(D)}\langle D\rangle$, so

$$\min\deg X_L > 3n_- - 3n_+ - 3n - 2r + 2 = -6n_+ - 2r + 2.$$

Since $V_L(t) = X_L|_{A=t^{-\frac{1}{4}}}$, so

$$\max\deg V_L < -\frac{1}{4}(-6n_+ - 2r + 2) = \frac{3}{2}n_+ + \frac{1}{2}r - \frac{1}{2}.$$

So we have

$$n_+ > \frac{1}{3}(2\max\deg V_L - r + 1).$$

This implies

$$c_+(L) > \frac{1}{3}(2\max\deg V_L - r + 1).$$

Similarly, we can prove

$$c_-(L) > -\frac{1}{3}(2\min\deg V_L + r - 1).$$

□

Corollary 2.2. *Suppose K is an oriented non-trivial knot, and V_K is the Jones polynomial of K. Then*

$$c_+(K) > \frac{2}{3}\max\deg V_K \text{ and } c_-(K) > -\frac{2}{3}\min\deg V_K.$$

In [4, Theorem 13.3] Murasugi obtained lower bounds for $c_+(L)$ and $c_-(L)$ as follows

$$c_+(L) \geq \max\deg V_L + \frac{1}{2}\sigma(L) \text{ and } c_-(L) \geq -\min\deg V_L + \frac{1}{2}\sigma(L),$$

where $\sigma(L)$ is the signature of L.

In [5, Proposition 4.2] Stoimenow obtained a similar lower bound for $c_-(L)$ of a non-positive link with r components, that is

$$c_-(L) \geq \frac{1}{2}(g(L) + 1 - \min\deg V_L - \frac{r-1}{2}),$$

where $g(L)$ is the genus of L.

We will give examples in the next section to show that for some cases our lower bounds are more accurate estimates for $c_+(L)$ and $c_-(L)$.

3. Applications

We give two simple examples first.

Example 3.1. The Jones polynomial of the right hand trefoil is $V(t) = t + t^3 - t^4$, so $c_+(K) > \frac{2}{3}\max\deg V = \frac{8}{3}$. Hence, any diagram of the right hand trefoil must have at least 3 positive crossings. Similarly, each diagram of the left hand trefoil must have at least 3 negative crossings.

Example 3.2. The figure eight knot has the Jones polynomial $V(t) = t^{-2} - t^{-1} + 1 - t + t^2$, so $c_+(K) > \frac{4}{3}$ and $c_-(K) > \frac{4}{3}$. This implies that each diagram of the figure eight knot must have at least 2 positive crossings and 2 negative crossings.

The next two examples show that comparing to Murasugi's and Stoimenow's lower bounds our lower bound is better for some cases.

Example 3.3. The knots $K_1 = 8_{19}$ and $K_2 = 10_{124}$ have the Jones polynomial $V_{K_1}(t) = t^3 + t^5 - t^8$ and $V_{K_2}(t) = t^4 + t^6 - t^{10}$ respectively, and $\sigma(K_1) = -6$, $\sigma(K_2) = -8$, see [1]. According to Murasugi's theorem, $c_+(K_1) \geq \max\deg V_{K_1} + \frac{1}{2}\sigma(K_1) = 5$ and $c_+(K_2) \geq \max\deg V_{K_2} + \frac{1}{2}\sigma(K_2) = 6$. By Corollary 2.2, $c_+(K_1) \geq 6$ and $c_+(K_2) \geq 7$. Thus our method provides a better lower bound for $c_+(K)$ of these two knots.

Example 3.4. Suppose K is the mirror image of 10_1, then K has the Jones polynomial $V_K(t) = t^{-8} - t^{-7} + t^{-6} - 2t^{-5} + 2t^{-4} - 2t^{-3} + 2t^{-2} - 2t^{-1} + 2 - t + t^2$ and $g(K) = 1$, see [1]. Since $\min\deg V_K = -8$, so K is not a positive knot. According to Stoimenow's proposition, $c_-(K) \geq \frac{1}{2}(g(K) + 1 - \min\deg V_K) = 5$. By Corollary 2.2, $c_-(K) > \frac{16}{3}$, so $c_-(K) \geq 6$. So our method provides a better lower bound for $c_-(K)$ in this case.

A link is called k-almost positive, if the minimal number of negative crossings in all its diagrams is k. Next we give an example to show that for any positive integer N, there is a k-almost positive knot with $k \geq N$. In particular, this implies k-almost positive knots exist for infinitely many $k \in \mathbb{N}$.

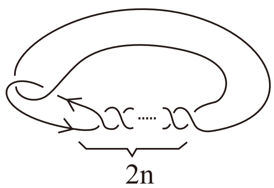


FIGURE 4. The twist knot K_n.

Example 3.5. For each $n \geq 1$, let K_n be a twist knot with n twists as shown in Figure 4, where the $2n$ crossings reduced by n twists are all negative. The Jones polynomial of K_n is

$$V_n = t^{-2n} + (t^{-2n+2} + t^{-2n+4} + \cdots + 1)(-t^{-1} + 1 - t + t^2).$$

So $\min \deg V_n = -2n$, and by Theorem 2.1 the number of negative crossings in any diagram of K_n is at least $\frac{4n}{3}$. Hence K_n is a k-almost positive knot with $k \geq \frac{4n}{3}$. For any positive integer N, let $n = [\frac{3N}{4}] + 1$, then K_n is a k-almost positive knot with $k \geq \mathbb{N}$.

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