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DEFORMATION OF CARTAN CURVATURE ON FINSLER MANIFOLDS

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Abstract. Here, certain Ricci flow for Finsler n-manifolds is considered and deformation of Cartan hh-curvature, as well as Ricci tensor and scalar curvature, are derived for spaces of scalar flag curvature. As an application, it is shown that on a family of Finsler manifolds of constant flag curvature, the scalar curvature satisfies the so-called heat-type equation. Hence on a compact Finsler manifold of constant flag curvature of initial non-negative scalar curvature, the scalar curvature remains non-negative by Ricci flow and blows up in a short time.

1. Introduction

The Ricci flow, introduced by Hamilton in 1981, is a tool for deforming an initial Riemannian metric tensor on a manifold, in order to pushing smoothly out irregularities in the metric. This procedure allows to discover the round metrics on a manifold, namely metrics of constant curvature, Einstein metrics, solitons and obtain topological, geometrical and physical information of the underlying manifold. Starting with a Riemannian metric g_0 and a family $g(t)$ of the Riemannian metrics on M he considers

$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)},\ g(0) = g,\ t \in [0, T), \tag{1.1}$$

where Ric is the Ricci tensor of $g(t)$. The Hamilton's work on Ricci flow, as a branch of geometric flow, is appeared as an indispensable tool for accessing classical problems in geometry and topology. They also provide a natural setting for formulating many applied and theoretical evolution problems in physics. One of Hamilton's innovation is an application of maximum principles, to show that if the initial metric has strictly positive scalar curvature, then it will be continuously so for all time [8, 10]. Next, he proves that the scalar curvature blows up in a short time.

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In recent years, the number of works on the Ricci flow in Riemannian and Finslerian geometry and its solitons has grown rapidly. Without pretending to be exhaustive we just cite more recent ones, for instance, [5–7, 12–14], etc. There are two significant definitions for Ricci tensor in Finsler geometry, and accordingly, there will be two Ricci flow with their own advantages, features and applications, see Subsection 2.3. One is a symmetric sum of the trace of Cartan hh-curvature tensor and the other is a homogenized second variational derivative of Riemannian curvature. Here, the former Ricci flow is considered. One advantage of this Ricci flow is its closed relation with Laplacian operator and diffusion of the heat equation.

In a recent work the present authors have studied convergence of Finslerian metric first in a general flow and next under the Ricci flow introduced by D. Bao, and proved that a family of Finslerian metrics $g(t)$ which are solutions to the Finslerian Ricci flow converge to a smooth limit Finslerian metric as t approaches the finite time T [7]. As the existence of solutions is well known in special cases, particularly in Riemannian and Berwaldian cases, we are not going to deal with general existence problem here, see for instance [3, 12].

In the present work, the Finslerian Ricci flow given by Eq. (3.1) is considered and deformation of Cartan hh-curvature, as well as Ricci tensor and scalar curvature, is derived for spaces of scalar flag curvature. As a consequence of this evolution equation, a heat type equation is obtained and it is shown that if the initial scalar curvature is non-negative then it remains non-negative by evolution and blows up in finite time. More precisely, we prove the following theorems;

Theorem 1.1. *Let M be an n-dimensional compact differentiable manifold and $g(t)$ a family of Finslerian solutions to the Ricci flow on M. If $g(0)$ has non-negative scalar curvature, then the scalar curvature remains non-negative for all $t \in [0, T)$, whenever any of the following properties holds.*

1. $\dim M = 2$.
2. $(M, g(t))$ *are Finsler manifolds of constant flag curvature.*

Theorem 1.2. *Let $(M, g(t))$ be a family of n-dimensional compact Finsler manifolds with constant flag curvature which are solutions to the Ricci flow for $t \in [0, T)$. If $\inf_{SM}\ scal_{g(0)} = \alpha > 0$, then $T \leq \frac{n}{2\alpha}$ and $\inf_{SM}\ scal_{g(t)} \geq \frac{n\alpha}{n-2\alpha t}$ for $t \in [0, T)$.*

Corollary 1.3. *Let $(M, g(t))$ be a family of compact Finsler surfaces satisfying in the Finslerian Ricci flow equation. If $\inf_{SM}\ scal_{g(0)} = \alpha > 0$, then $T \leq \frac{1}{\alpha}$ and $\inf_{SM}\ scal_{g(t)} \geq \frac{\alpha}{1-\alpha t}$ for $t \in [0, T)$, and hence the scalar curvature blows up in short time.*

2. Preliminaries and terminologies

In order to deal with evolution equation of hh-curvature tensors in Finsler geometry, it is preferable to use a global definition of Cartan connection. To be brief we recall some basic definitions which are not easily found in the current literature. In this work we adopt the notations and terminologies of [2] whenever we are dealing with Cartan connection and otherwise we use those of [4]. Let M be a connected differentiable manifold of dimension n. Denote the bundle of tangent vectors of M by $p : TM \longrightarrow M$, the fiber bundle of non-zero tangent vectors of M by $\pi : TM_0 \longrightarrow M$ and the pulled-back tangent bundle by $\pi^*TM \longrightarrow TM_0$. A point of TM_0 is denoted by $z = (x, y)$, where $x = \pi z \in M$ and $y \in T_{\pi z}M$. Let (x^i) be a local chart with the domain $U \subseteq M$ and (x^i, y^i) the induced local coordinates on $\pi^{-1}(U)$, where $\mathbf{y} = y^i \frac{\partial}{\partial x^i} \in T_{\pi z}M$, and i running over the range $1, 2, \ldots, n$. A (globally defined) *Finsler structure* on M is a function $F : TM \longrightarrow [0, \infty)$ with the following properties; F is C^∞ on the entire slit tangent bundle $TM\backslash 0$; $F(x, \lambda y) = \lambda F(x, y)\ \forall \lambda > 0$; the $n \times n$ Hessian matrix $(g_{ij}) = \frac{1}{2}([F^2]_{y^i y^j})$ is positive-definite at every point of TM_0. The pair (M, g) is called a *Finsler manifold*. Denote by TTM_0 and SM the tangent bundle of TM_0 and the sphere bundle respectively, where $SM := \bigcup_{x \in M} S_xM$ and $S_xM := \{y \in T_xM \,|\, F(y) = 1\}$. Given the induced coordinates (x^i, y^i) on TM, coefficients of spray vector field are defined by $G^i = 1/4 g^{ih}(\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h})$. One can observe that the pair $\{\delta/\delta x^i, \partial/\partial y^i\}$ forms a horizontal and vertical frame for TTM, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}$, $G^j_i := \frac{\partial G^j}{\partial y^i}$. There is a canonical linear mapping $\varrho : TTM_0 \longrightarrow \pi^*TM$, where, $\varrho = \pi_*$, $\varrho_z((\frac{\delta}{\delta x^i})_z) = (\frac{\partial}{\partial x^i})_z$ and $\varrho((\frac{\partial}{\partial y^i})_z) = 0$. Let V_zTM be the set of vertical vectors at $z \in TM_0$, that is, the set of vectors which are tangent to the fiber through z. Equivalently, $V_zTM = \ker \pi_*$ where $\pi_* : TTM_0 \longrightarrow TM$ is the linear tangent mapping. Let ∇ be a linear connection on π^*TM the sections of pull back bundle π^*TM,

$$\nabla : T_zTM_0 \times \Gamma(\pi^*TM) \longrightarrow \Gamma(\pi^*TM),$$

provided that there is a linear mapping $\mu : TTM_0 \longrightarrow \pi^*TM$, defined by $\mu(\hat{X}) = \nabla_{\hat{X}} v$ where, $\hat{X} \in TTM_0$ and v is the canonical section of π^*TM. The connection ∇ is said to be *regular*, if μ defines an isomorphism between VTM_0 and π^*TM. In this case, there is a horizontal distribution HTM such that we have the *Whitney sum* $TTM_0 = HTM \oplus VTM$. It can be shown that the set $\{\frac{\delta}{\delta x^j}\}$ and $\{\frac{\partial}{\partial y^j}\}$, forms a local frame field for the horizontal and vertical subspaces, respectively. This decomposition permits to write a vector field $\hat{X} \in TTM_0$ into the form $\hat{X} = H\hat{X} + V\hat{X}$ uniquely. In the sequel, we denote all the sections of π^*TM by $X = \varrho(\hat{X})$, $Y = \varrho(\hat{Y})$, and the corresponding complete lift on TM_0 by $\hat{X}, \hat{Y}$ respectively, unless otherwise specified.

2.1. A global approach to the hh-curvature of Cartan connection

The *torsion* and *curvature* tensors of the regular connection ∇ are given by

$$\tau(\hat{X}, \hat{Y}) = \nabla_{\hat{X}} Y - \nabla_{\hat{Y}} X - \varrho[\hat{X}, \hat{Y}],$$
$$\Omega(\hat{X}, \hat{Y})Z = \nabla_{\hat{X}} \nabla_{\hat{Y}} Z - \nabla_{\hat{Y}} \nabla_{\hat{X}} Z - \nabla_{[\hat{X}, \hat{Y}]} Z,$$

where, $X=\varrho(\hat{X})$, $Y=\varrho(\hat{Y})$, $Z=\varrho(\hat{Z})$ and $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are vector fields on TM_0. They determine two torsion tensors denoted here by S and T and three curvature tensors denoted by R, P and Q, defined by:

$$\begin{aligned} &S(X,Y)=\tau(H\hat{X},H\hat{Y}), \quad T(\dot{X},Y)=\tau(V\hat{X},H\hat{Y}),\\ &R(X,Y)=\Omega(H\hat{X},H\hat{Y}), \quad P(X,\dot{Y})=\Omega(H\hat{X},V\hat{Y}),\\ &Q(\dot{X},\dot{Y})=\Omega(V\hat{X},V\hat{Y}),\end{aligned}$$

where, $X=\varrho(\hat{X})$, $Y=\varrho(\hat{Y})$, $\dot{X}=\mu(\hat{X})$ and $\dot{Y}=\mu(\hat{Y})$. The tensors R, P and Q are called *hh*-, *hv*- and *vv*-curvature tensors, respectively. There is a unique metric compatible h-torsion free regular connection ∇ associated to the Finsler structure F satisfying, $\nabla_{\hat{Z}}g=0$, $S(X,Y)=0$, and $g(\tau(V\hat{X},\hat{Y}),Z)=g(\tau(V\hat{X},\hat{Z}),Y)$, called the *Cartan connection*. In local coordinates the covariant derivation of a vector field $\hat{X}$ in Cartan connection is given by; $\nabla X^k = dX^k + X^j(\Gamma^k{}_{ji}dx^i + T^k_{ji}dy^i)$.

Using the Jacobian identity for three vector fields $\hat{X}$, $\hat{Y}$ and $\hat{Z}$, one obtains *the Bianchi identities* for a regular connection ∇ with curvature 2-forms Ω as follows:

$$\begin{aligned} &\sigma\Omega(\hat{X},\hat{Y})Z=\sigma\nabla_{\hat{Z}}\tau(\hat{X},\hat{Y})+\sigma\tau(\hat{Z},[\hat{X},\hat{Y}]),\\ &\sigma\nabla_{\hat{Z}}\Omega(\hat{X},\hat{Y})+\sigma\Omega(\hat{Z},[\hat{X},\hat{Y}])=0,\end{aligned}$$

where, σ denotes the circular permutation in the set $\{\hat{X},\hat{Y},\hat{Z}\}$ [2].

The isomorphism between the bundles VTM and π^*TM permits to define the Cartan connection on the sections of VTM rather than π^*TM. We need the following property of a linear connection in the sequel.

$$\begin{aligned} 2g(\nabla_{\hat{X}}Y,Z) = {} & \hat{X}.g(Y,Z)+\hat{Y}.g(X,Z)-\hat{Z}.g(X,Y)+g(\tau(\hat{X},\hat{Y}),Z)\\ &+g(\tau(\hat{Z},\hat{X}),Y)+g(\tau(\hat{Z},\hat{Y}),X)+g(\varrho[\hat{X},\hat{Y}],Z)\\ &+g(\varrho[\hat{Z},\hat{X}],Y)+g(\varrho[\hat{Z},\hat{Y}],X). \end{aligned} \tag{2.1}$$

The $(0,4)$ *hh-curvature tensor of Cartan connection* is defined here by

$$R(Z,W,X,Y):=-g(R(X,Y)Z,W),$$

where

$$R(X,Y)Z=\nabla_{H\hat{X}}\nabla_{H\hat{Y}}Z-\nabla_{H\hat{Y}}\nabla_{H\hat{X}}Z-\nabla_{[H\hat{X},H\hat{Y}]}Z.$$

Recall that the *hh*-curvature of Cartan connection is *skew-symmetric* with respect to the two first and last indices, see [2, page 43]. That is,

$$\begin{aligned} &R(X,Y,Z,W)=-R(Y,X,Z,W),\\ &R(X,Y,Z,W)=-R(X,Y,W,Z).\end{aligned}$$

In a local coordinate system we have $R(\partial_i,\partial_j)\partial_k=R^l_{kij}\partial_l$ and $R_{lkij}=R^m_{kij}g_{ml}$, that is, we lower the upper index to the first position.

A flag can be considered as a 2-dimensional subspace of T_xM which is determined by two independent vectors of T_xM. Consider $y \in T_xM$ as a flagpole and $V = V^i \frac{\partial}{\partial x^i}$ as a transverse edge, then the *flag curvature* is defined to be $K(x, y, V) = \frac{g(R(V,y)y,V)}{g(y,y)g(V,V)-(g(y,V))^2}$. If the flag curvature does not depend on the transverse edge V, then the Finsler manifold (M, g) is called of *scalar flag curvature*. In Finsler manifolds of scalar flag curvature the two first and the two last indices can be interchanged symmetrically, that is $R_{ijkl} = R_{klij}$, [1]. The Bianchi identities for Cartan connection on Finsler manifolds of scalar flag curvature are expressed by

$$R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0, \tag{2.2}$$

and

$$\begin{aligned}\sigma_{(Z,U,W)}[(\nabla_{H\hat{Z}}R)(X, Y, U, W) \\ + (\frac{F^2}{3}\nabla_{V\hat{Z}}K + KF\nabla_{V\hat{Z}}F)(\nabla_v Q)(X, Y, \dot{U}, \dot{W})] = 0.\end{aligned} \tag{2.3}$$

In terms of local coordinates the above equations are written as follows;

$$\sigma_{jkl}R^i_{jkl} = 0,$$

$$\sigma_{mkl}\nabla_m R^i_{jkl} + \sigma_{mkl}(\frac{F^2}{3}\frac{\partial K}{\partial y^m} + Ky_m)y^h\nabla_h Q^i_{jkl} = 0,$$

where σ_{mkl} denotes the sum of the terms obtained by cyclic permutation of indices m, k and l. Using the relation $y_m = g_{mj}y^j = (FF_{y^m y^j} + F_{y^m}F_{y^j})y^j = FF_{y^m}$, one can rewrite the above equation as follows, [1].

$$\sigma_{mkl}\nabla_m R^i_{jkl} + \sigma_{mkl}(\frac{F^2}{3}\frac{\partial K}{\partial y^m} + KFF_{y^m})y^h\nabla_h Q^i_{jkl} = 0.$$

2.2. A horizontal Finslerian Laplacian

There is a Riemannian metric on VTM induced by F. The regularity of Cartan connection permits to define a horizontal bundle HTM and a horizontal map $\Theta : VTM \longrightarrow HTM$. One can use Θ to transfer a Riemannian structure from VTM to HTM, by the following setting

$$\forall\ H\hat{X}, H\hat{Y} \in HTM, \quad \langle H\hat{X}, H\hat{Y}\rangle = \langle\Theta^{-1}(H\hat{X}), \Theta^{-1}(H\hat{Y})\rangle.$$

Hence a Riemannian metric can be defined on the whole TTM_0, just by stating that HTM is orthogonal to VTM and $TTM_0 = HTM \oplus VTM$. As usual define the second covariant derivative by

$$\nabla^2_{\hat{X},\hat{Y}}Z := (\nabla^2 Z)(\hat{X}, \hat{Y}) = \nabla_{\hat{X}}\nabla_{\hat{Y}}Z - \nabla_{\nabla_{\hat{X}}Y}Z.$$

Let $\{e_k\}$ be a basis for π^*TM and $\hat{e_k}$ its complete lift to TTM_0. Consider $\{H\hat{e_k}, V\hat{e_k}\}$ for $k = 1, \ldots, n$ as an orthonormal frame on TM_0, where $\{H\hat{e_k}\}$ and $\{V\hat{e_k}\}$ denote the horizontal and vertical parts of $\{\hat{e_k}\}$ respectively. Define a *horizontal Laplacian* on TM_0 by, $\Delta^h f = \mathrm{trace}(H\hat{X} \to \sharp(\nabla^2 f)(H\hat{X}, \cdot))$. In an orthonormal frame it is written $\Delta^h f = \sum_{k=1}^{n} (\nabla^2 f)(H\hat{e_k}, H\hat{e_k})$, where f is a scalar function on the sphere bundle SM.

2.3. On Ricci curvatures in Finsler geometry

H. Akbar-Zadeh in his works has considered two kinds of Ricci curvatures in Finsler geometry. One is defined by $Ric(X,Y) := 1/2\big(Rc(X,Y) + Rc(Y,X)\big)$, where Rc is obtained by contraction of Cartan hh-curvature tensor and have the components $Rc_{ij} = R^l_{ilj}$, where R^l_{ikj} are the components of Cartan hh-curvature tensor. The other definition of Ricci curvature tensor is given by $\frac{1}{2}[F^2 Ric]_{y^i y^j}$, where $Ric := R^i_i$ is the trace of Riemannian curvature, expressed entirely in terms of partial derivatives of the spray G^i. One of the advantages of the former Ricci tensor, obtained by contraction and symmetrization of the Cartan hh-curvature tensor, is its closed relation with second covariant derivative and hence Laplacian and diffusion operators. While the later definition is independent of the choice of any connection. Both of these two Ricci tensors reduce to the ordinary Ricci tensor in Riemannian case.

Here and everywhere in the present work we consider the following *Ricci curvature* tensor,

$$Ric(X,Y) := 1/2\big(Rc(X,Y) + Rc(Y,X)\big). \tag{2.4}$$

It is well known that in a Finsler manifold of scalar flag curvature, Rc_{ij} is symmetric and Ricci curvature tensor $Ric(X,Y)$ coincides with $Rc(X,Y)$, see [2], page 152.

As usual, trace of the Ricci curvature is called *scalar curvature* and is denoted here by $scal = g^{ij} Ric_{ij}$, where $Ric_{ij} := \frac{R_{ij}+R_{ji}}{2}$.

3. Covariant time derivative in Cartan connection

In this section we establish the covariant time derivative $\nabla_{\frac{\partial}{\partial t}}$ with respect to the Cartan connection associated with a family of Finslerian metrics $g(t)$. Consider the following evolution equation known as the Finslerian Ricci flow on the family of Finslerian manifolds $(M, g(t))$,

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)},\ g(0) = g,\ t \in [0,T), \tag{3.1}$$

where Ric is the Ricci tensor defined by (2.4). The existence of solutions is well known in special cases, particularly in Riemannian and Berwaldian spaces, we are not going to deal with general existence problem here, see for instance [3, 12]. Let E be the pull-back of tangent bundle under the projection $pr : TM_0 \times (0,T) \longrightarrow M$, where $pr(x,y,t) = x$. The fiber of E over a point $(x,y,t) \in TM_0 \times (0,T)$ is given by $E_{(x,y,t)} = T_xM$. In order to define a covariant time derivative in Finsler geometry we extend the definition of Cartan connection on a general vector bundle E in the sense of Uhlenbeck see [11, Chapter 5]. Let X be a section of the vector bundle E, using the Uhlenbeck trick for a general vector bundle, the extension of Cartan connection for the section $\frac{\partial}{\partial t}$ and for the Ricci flow (3.1) is defined by $\nabla_{\frac{\partial}{\partial t}} X = \frac{\partial}{\partial t} X - \sum_{k=1}^{n} Ric(X, e_k) e_k$, where $\{e_k\}_{k=1}^{n}$ is considered as an orthonormal frame for E with respect to the metric $g(t)$. By uniqueness and metric compatibility of Cartan connection, it can be shown that the $\nabla_{\frac{\partial}{\partial t}}$ is unique up to the Ricci flow (3.1). We specify that the covariant time derivative $\nabla_{\frac{\partial}{\partial t}}$ should be metric compatible with respect to $g(t)$ in the sense of the following proposition.

Proposition 3.1. *The extended Cartan connection $\nabla_{\frac{\partial}{\partial t}}$ is compatible with bundle metric on E, that is $(\nabla_{\frac{\partial}{\partial t}} g)(X, Y) = 0$.*

Proof. Using definition of covariant time derivative yields

$$\begin{aligned}
(\nabla_{\frac{\partial}{\partial t}} g)(X, Y) &= \frac{\partial}{\partial t}(g(X, Y)) - g(\nabla_{\frac{\partial}{\partial t}} X, Y) - g(X, \nabla_{\frac{\partial}{\partial t}} Y) \\
&= (\frac{\partial}{\partial t} g)(X, Y) + g(\frac{\partial}{\partial t} X, Y) + g(X, \frac{\partial}{\partial t} Y) \\
&\quad - g(\frac{\partial}{\partial t} X - \sum_{k=1}^{n} Ric(X, e_k) e_k, Y) \\
&\quad - g(X, \frac{\partial}{\partial t} Y - \sum_{k=1}^{n} Ric(Y, e_k) e_k) \\
&= \ - 2Ric(X, Y) + 2Ric(X, Y) = 0.
\end{aligned}$$

Hence proof is complete. □

In the sequel without loss of generality, we assume all sections of the vector bundle E, are constant with respect to t. Therefore for a fixed section X of E the earlier defined, covariant time derivative reduces to

$$\nabla_{\frac{\partial}{\partial t}} X = - \sum_{k=1}^{n} Ric(X, e_k) e_k.$$

4. Evolution of the *hh*-curvature of Cartan connection

A section Z of $\pi^* TM$ or a vector field $\hat{Z}$ on TM_0 respectively is said to be *fixed* if it is independent of the parameter "t", that is, $\frac{\partial}{\partial t} Z = 0$ or $\frac{\partial}{\partial t} \hat{Z} = 0$.

Proposition 4.1. *Let $\hat{X}, \hat{Y}$ and $\hat{Z}$ be the fixed vector fields on TM_0. Then*

$$g(B(X, Y), Z) = -(\nabla_{\hat{X}} Ric)(Y, Z) - (\nabla_{\hat{Y}} Ric)(X, Z) + (\nabla_{\hat{Z}} Ric)(X, Y),$$

where $B(X, Y) := \frac{\partial}{\partial t}(\nabla_{\hat{X}} Y)$.

Proof. By Leibnitz rule and using $\frac{\partial}{\partial t} Z = 0$, we have

$$\frac{\partial}{\partial t}(g(\nabla_{\hat{X}} Y, Z)) = (\frac{\partial}{\partial t} g)(\nabla_{\hat{X}} Y, Z) + g(\frac{\partial}{\partial t} \nabla_{\hat{X}} Y, Z),$$

from which

$$g(B(X,Y),Z)=\frac{\partial}{\partial t}(g(\nabla_{\hat{X}}Y,Z))-(\frac{\partial}{\partial t}g)(\nabla_{\hat{X}}Y,Z).$$

Since $g(t)$ satisfy in (3.1), by virtue of (2.1) for Cartan connection, we have,

$$\begin{aligned}g(B(X,Y),Z)= & -\nabla_{\hat{X}}(Ric(Y,Z))-\nabla_{\hat{Y}}(Ric(X,Z))+\nabla_{\hat{Z}}(Ric(X,Y))\\ & -Ric(\nabla_{\hat{X}}Y-\nabla_{\hat{Y}}X,Z)-Ric(\nabla_{\hat{Z}}X-\nabla_{\hat{X}}Z,Y)\\ & -Ric(\nabla_{\hat{Z}}Y-\nabla_{\hat{Y}}Z,X)+2Ric(\nabla_{\hat{X}}Y,Z)\\ & +\frac{1}{2}g\big(\frac{\partial}{\partial t}(\nabla_{\hat{X}}Y-\nabla_{\hat{Y}}X),Z\big)+\frac{1}{2}g\big(\frac{\partial}{\partial t}(\nabla_{\hat{Z}}X-\nabla_{\hat{X}}Z),Y\big)\\ & +\frac{1}{2}g\big(\frac{\partial}{\partial t}(\nabla_{\hat{Z}}Y-\nabla_{\hat{Y}}Z),X\big).\end{aligned}\tag{4.1}$$

Note that $\hat{X}$ and $\hat{Y}$ are fixed vector fields, so we have

$$\begin{aligned}&\frac{\partial}{\partial t}(\nabla_{H\hat{X}}Y-\nabla_{H\hat{Y}}X)\\ &=\frac{\partial}{\partial t}\big(X^i(\frac{\delta Y^j}{\delta x^i}\frac{\partial}{\partial x^j}+Y^j\Gamma^k_{ij}\frac{\partial}{\partial x^k})-Y^i(\frac{\delta X^j}{\delta x^i}\frac{\partial}{\partial x^j}+X^j\Gamma^k_{ij}\frac{\partial}{\partial x^k})\big)=0,\end{aligned}$$

since $\Gamma^k_{ij}=\Gamma^k_{ji}$. By similar argument for vertical part and using $\hat{X}=H\hat{X}+V\hat{X}$ we get

$$g\big(\frac{\partial}{\partial t}(\nabla_{\hat{X}}Y-\nabla_{\hat{Y}}X),Z\big)=g\big(\frac{\partial}{\partial t}(\nabla_{H\hat{X}+V\hat{X}}Y-\nabla_{H\hat{Y}+V\hat{Y}}X),Z\big)=0.$$

Likewise the last two terms on the right hand side of (4.1) vanish, hence

$$g(B(X,Y),Z)=-(\nabla_{\hat{X}}Ric)(Y,Z)-(\nabla_{\hat{Y}}Ric)(X,Z)+(\nabla_{\hat{Z}}Ric)(X,Y).$$

This completes the proof. □

Let A be a tensor field defined by $A(X,Y):=\frac{\partial}{\partial t}(\nabla_{H\hat{X}}Y)$. Similar to the Proposition 4.1 one can easily prove the following corollary.

Corollary 4.2. *Let $H\hat{X},H\hat{Y}$ and $H\hat{Z}$ be the fixed horizontal parts of $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ on HTM respectively. We have*

$$g(A(X,Y),Z)=-(\nabla_{H\hat{X}}Ric)(Y,Z)-(\nabla_{H\hat{Y}}Ric)(X,Z)+(\nabla_{H\hat{Z}}Ric)(X,Y).$$

Let S be a section or a $(0,2n)$-tensor field on π^*TM. We claim that $\nabla^2_{H\hat{X},H\hat{Y}}S-\nabla^2_{H\hat{Y},H\hat{X}}S$ can be expressed in terms of hh-curvature of Cartan connection and one extra term as follows. Let Z be a section of π^*TM, then

$$\begin{aligned}&\nabla^2_{H\hat{X},H\hat{Y}}Z-\nabla^2_{H\hat{Y},H\hat{X}}Z\\ &=\nabla_{H\hat{X}}\nabla_{H\hat{Y}}Z-\nabla_{\nabla_{H\hat{X}}Y}Z-\nabla_{H\hat{Y}}\nabla_{H\hat{X}}Z+\nabla_{\nabla_{H\hat{Y}}X}Z\\ &=\nabla_{H\hat{X}}\nabla_{H\hat{Y}}Z-\nabla_{H\hat{Y}}\nabla_{H\hat{X}}Z-\nabla_{[H\hat{X},H\hat{Y}]}Z+\nabla_{[H\hat{X},H\hat{Y}]-\varrho[H\hat{X},H\hat{Y}]}Z\\ &=R(X,Y)Z+\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}Z,\end{aligned}$$

where we have put $\widetilde{[H\hat{X}, H\hat{Y}]} = [H\hat{X}, H\hat{Y}] - \varrho[H\hat{X}, H\hat{Y}]$.

By straight forward computations we have the following relation for a $(0,4)$-tensor field S

$$
\begin{aligned}
(4.2)\quad & (\nabla^2_{H\hat{X},H\hat{Y}}S)(U,V,W,Z) - (\nabla^2_{H\hat{Y},H\hat{X}}S)(U,V,W,Z) \\
&= \sum_{k=1}^{n} R(U,e_k,X,Y)S(e_k,V,W,Z) + \sum_{k=1}^{n} R(V,e_k,X,Y)S(U,e_k,W,Z) \\
&\quad + \sum_{k=1}^{n} R(W,e_k,X,Y)S(U,V,e_k,Z) + \sum_{k=1}^{n} R(Z,e_k,X,Y)S(U,V,W,e_k) \\
&\quad + (\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}S)(U,V,W,Z),
\end{aligned}
$$

where $\{e_k\}_{k=1}^n$ is an orthonormal basis for π^*TM.

Proposition 4.3. *Let $(M, g(t))$ be a family of Finsler manifolds of scalar flag curvature satisfying in the Finslerian Ricci flow equation and $H\hat{X}, H\hat{Y}, H\hat{Z}$ and $H\hat{W}$ be the fixed horizontal parts of $\hat{X}$, $\hat{Y}, \hat{Z}$ and $\hat{W}$ on HTM respectively. We have*

$$
\begin{aligned}
\frac{\partial}{\partial t}(R(Z,W,X,Y)) &= (\nabla^2_{H\hat{X},H\hat{Z}}Ric)(Y,W) - (\nabla^2_{H\hat{X},H\hat{W}}Ric)(Y,Z) \\
&\quad - (\nabla^2_{H\hat{Y},H\hat{Z}}Ric)(X,W) + (\nabla^2_{H\hat{Y},H\hat{W}}Ric)(X,Z) \\
&\quad + (\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}Ric)(Z,W) - \sum_{k=1}^{n} R(Z,e_k,X,Y)Ric(e_k,W) \\
&\quad + \sum_{k=1}^{n} R(W,e_k,X,Y)Ric(Z,e_k).
\end{aligned}
$$

Proof. By definition of the hh-curvature tensor R we have,

$$
\begin{aligned}
\frac{\partial}{\partial t}(R(Z,W,X,Y)) &= -\frac{\partial}{\partial t}(g(R(X,Y)Z,W)) \\
&= -(\frac{\partial}{\partial t}g)(R(X,Y)Z,W) - g(\frac{\partial}{\partial t}R(X,Y)Z,W) \\
&= -2Ric(\sum_{k=1}^{n} R(Z,e_k,X,Y)e_k,W) \\
&\quad - g(\frac{\partial}{\partial t}(\nabla_{H\hat{X}}\nabla_{H\hat{Y}}Z - \nabla_{H\hat{Y}}\nabla_{H\hat{X}}Z - \nabla_{[H\hat{X},H\hat{Y}]}Z),W) \\
&= -2\sum_{k=1}^{n} R(Z,e_k,X,Y)Ric(e_k,W) - g(A(X,\nabla_{H\hat{Y}}Z),W) \\
&\quad - g(\nabla_{H\hat{X}}(A(Y,Z)),W) + g(A(Y,\nabla_{H\hat{X}}Z),W) \\
&\quad + g(\nabla_{H\hat{Y}}(A(X,Z)),W) + g(A(\varrho[H\hat{X},H\hat{Y}],Z),W).
\end{aligned}
$$

Finally we get

$$\begin{aligned}\frac{\partial}{\partial t}(R(Z,W,X,Y)) = &-2\sum_{k=1}^{n} R(Z,e_k,X,Y)Ric(e_k,W)\\ &- g((\nabla_{H\hat{X}}A)(Y,Z),W) + g((\nabla_{H\hat{Y}}A)(X,Z),W).\end{aligned} \tag{4.3}$$

On the other hand,

$$\begin{aligned} g((\nabla_{H\hat{X}}A)(Y,Z),W) &= g(\nabla_{H\hat{X}}(A(Y,Z),W) - g(A(\nabla_{H\hat{X}}Y,Z)),W),\\ -g(A(Y,\nabla_{H\hat{X}}Z),W) &= \nabla_{H\hat{X}}(g(A(Y,Z),W)) - g(A(Y,Z),\nabla_{H\hat{X}}W)\\ &\quad - g(A(\nabla_{H\hat{X}}Y,Z),W) - g(A(Y,\nabla_{H\hat{X}}Z),W).\end{aligned}$$

Therefore by using Corollary 4.2, we have

$$\begin{aligned} &g((\nabla_{H\hat{X}}A)(Y,Z),W)\\ &= \nabla_{H\hat{X}}((-\nabla_{H\hat{Y}}Ric)(Z,W) - (\nabla_{H\hat{Z}}Ric)(Y,W) + (\nabla_{H\hat{W}}Ric)(Y,Z))\\ &\quad + (\nabla_{H\hat{Y}}Ric)(Z,\nabla_{H\hat{X}}W) + (\nabla_{H\hat{Z}}Ric)(Y,\nabla_{H\hat{X}}W) - (\nabla_{\nabla_{H\hat{X}}W}Ric)(Y,Z)\\ &\quad + (\nabla_{\nabla_{H\hat{X}}Y}Ric)(Z,W) + (\nabla_{H\hat{Z}}Ric)(\nabla_{H\hat{X}}Y,W) - (\nabla_{H\hat{W}}Ric)(\nabla_{H\hat{X}}Y,Z)\\ &\quad + (\nabla_{H\hat{Y}}Ric)(\nabla_{H\hat{X}}Z,W) + (\nabla_{\nabla_{H\hat{X}}Z}Ric)(Y,W) - (\nabla_{H\hat{W}}Ric)(Y,\nabla_{H\hat{X}}Z).\end{aligned}$$

Thus we have

$$\begin{aligned} g((\nabla_{H\hat{X}}A)(Y,Z),W) = &-(\nabla^2_{H\hat{X},H\hat{Y}}Ric)(Z,W) - (\nabla^2_{H\hat{X},H\hat{Z}}Ric)(Y,W)\\ &+ (\nabla^2_{H\hat{X},H\hat{W}}Ric)(Y,Z).\end{aligned} \tag{4.4}$$

By interchanging the roles of X and Y we obtain,

$$\begin{aligned} g((\nabla_{H\hat{Y}}A)(X,Z),W) = &-(\nabla^2_{H\hat{Y},H\hat{X}}Ric)(Z,W) - (\nabla^2_{H\hat{Y},H\hat{Z}}Ric)(X,W)\\ &+ (\nabla^2_{H\hat{Y},H\hat{W}}Ric)(X,Z).\end{aligned} \tag{4.5}$$

For the Ricci tensor we have,

$$\begin{aligned} &(\nabla^2_{H\hat{X},H\hat{Y}}Ric)(Z,W) - (\nabla^2_{H\hat{Y},H\hat{X}}Ric)(Z,W)\\ &= \sum_{k=1}^{n} R(Z,e_k,X,Y)Ric(e_k,W) + \sum_{k=1}^{n} R(W,e_k,X,Y)Ric(Z,e_k)\\ &\quad + (\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}Ric)(Z,W).\end{aligned} \tag{4.6}$$

Using relations (4.3), (4.4), (4.5) and (4.6) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(R(Z,W,X,Y)) = &(\nabla^2_{H\hat{X},H\hat{Z}}Ric)(Y,W) - (\nabla^2_{H\hat{X},H\hat{W}}Ric)(Y,Z)\\ &- (\nabla^2_{H\hat{Y},H\hat{Z}}Ric)(X,W) + (\nabla^2_{H\hat{Y},H\hat{W}}Ric)(X,Z)\\ &- \sum_{k=1}^{n} R(Z,e_k,X,Y)Ric(e_k,W)\end{aligned}$$

$$+\sum_{k=1}^{n} R(W,e_k,X,Y)Ric(Z,e_k) + (\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}} Ric)(Z,W).$$

As we have claimed. □

To be brief, in the sequel we will use the usual notions of [8] as follows,

$$\begin{aligned} Q(R)(Z,W,X,Y) := & \sum_{k,l=1}^{n} R(e_k,e_l,X,Y)R(e_k,e_l,Z,W) \\ & + 2\sum_{k,l=1}^{n} R(Z,e_l,X,e_k)R(e_k,Y,e_l,W) \\ & - 2\sum_{k,l=1}^{n} R(W,e_l,X,e_k)R(e_k,Y,e_l,Z), \end{aligned}$$

and also $Q_1,\ldots,Q_6$, to denote the lower order terms with respect to the Finsler structure F, flag curvature K and vv-curvature.

Proposition 4.4. *Let $(M,g(t))$ be a family of Finsler manifolds of scalar flag curvature satisfying in the Finslerian Ricci flow equation. Then*

$$\begin{aligned} & (\nabla^2_{H\hat{X},H\hat{Z}} Ric)(Y,W) - (\nabla^2_{H\hat{X},H\hat{W}} Ric)(Y,Z) - (\nabla^2_{H\hat{Y},H\hat{Z}} Ric)(X,W) \\ & + (\nabla^2_{H\hat{Y},H\hat{W}} Ric)(X,Z) \\ = & \sum_{k=1}^{n} (\nabla^2_{H\hat{e}_k,H\hat{e}_k} R)(X,Y,Z,W) - \nabla_{H\hat{X}} Q_1 + \nabla_{H\hat{Y}} Q_3 - \sum_{k=1}^{n} \nabla_{H\hat{e}_k} Q_5 \\ & + \textit{lower order terms.} \end{aligned}$$

Proof. Considering R as a $(0,4)$-tensor field, leads to

$$\begin{aligned} & \sum_{k=1}^{n} (\nabla^2_{H\hat{X},H\hat{e}_k} R)(e_k,Y,Z,W) - \sum_{k=1}^{n} (\nabla^2_{H\hat{e}_k,H\hat{X}} R)(e_k,Y,Z,W) \\ = & \sum_{k,l=1}^{n} R(e_k,e_l,X,e_k)R(e_l,Y,Z,W) + \sum_{k,l=1}^{n} R(Y,e_l,X,e_k)R(e_k,e_l,Z,W) \\ & + \sum_{k,l=1}^{n} R(Z,e_l,X,e_k)R(e_k,Y,e_l,W) + \sum_{k,l=1}^{n} R(W,e_l,X,e_k)R(e_k,Y,Z,e_l) \\ & + \sum_{k=1}^{n} (\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}} R)(e_k,Y,Z,W). \end{aligned}$$

Interchanging the roles of X and Y and subtracting the second identity from the first one, we obtain

(4.7)
$$\begin{aligned}
&\sum_{k=1}^{n}(\nabla^2_{H\hat{X},H\hat{e}_k}R)(e_k,Y,Z,W)-\sum_{k=1}^{n}(\nabla^2_{H\hat{Y},H\hat{e}_k}R)(e_k,X,Z,W)\\
&-\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{X}}R)(e_k,Y,Z,W)+\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{Y}}R)(e_k,X,Z,W)\\
=&\sum_{k,l=1}^{n}(R(Y,e_l,X,e_k)-R(X,e_l,Y,e_k))R(e_k,e_l,Z,W)\\
&+\sum_{k,l=1}^{n}R(e_k,e_l,X,e_k)R(e_l,Y,Z,W)+\sum_{k,l=1}^{n}R(Z,e_l,X,e_k)R(e_k,Y,e_l,W)\\
&+\sum_{k,l=1}^{n}R(W,e_l,X,e_k)R(e_k,Y,Z,e_l)+\sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W)\\
&-\sum_{k,l=1}^{n}R(e_k,e_l,Y,e_k)R(e_l,X,Z,W)-\sum_{k,l=1}^{n}R(Z,e_l,Y,e_k)R(e_k,X,e_l,W)\\
&-\sum_{k,l=1}^{n}R(W,e_l,Y,e_k)R(e_k,X,Z,e_l)-\sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W).
\end{aligned}$$

Next, by means of the first Bianchi identity (2.2) for Finsler manifolds of scalar flag curvature, the right hand side of (4.7) reduces to

$$\begin{aligned}
&Q(R)(Z,W,X,Y)-\sum_{l=1}^{n}R(e_l,Y,Z,W)Ric(e_l,X)\\
&+\sum_{l=1}^{n}R(e_l,X,Z,W)Ric(e_l,Y)+\sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W)\\
&-\sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W).
\end{aligned}\tag{4.8}$$

On the other hand,

$$\begin{aligned}
&\sum_{k=1}^{n}(\nabla^2_{H\hat{X},H\hat{e}_k}R)(e_k,Y,Z,W)\\
&=\sum_{k=1}^{n}(\nabla_{H\hat{X}}\nabla_{H\hat{e}_k}R)(e_k,Y,Z,W)-\sum_{k=1}^{n}(\nabla_{\nabla_{H\hat{X}}e_k}R)(e_k,Y,Z,W)\\
&=\sum_{k=1}^{n}\nabla_{H\hat{X}}((\nabla_{H\hat{e}_k}R)(e_k,Y,Z,W))-\sum_{k=1}^{n}(\nabla_{H\hat{e}_k}R)(\nabla_{H\hat{X}}e_k,Y,Z,W)
\end{aligned}\tag{4.9}$$

$$
\begin{aligned}
&-\sum_{k=1}^{n}(\nabla_{H\hat{e}_k}R)(e_k,\nabla_{H\hat{X}}Y,Z,W)-\sum_{k=1}^{n}(\nabla_{H\hat{e}_k}R)(e_k,Y,\nabla_{H\hat{X}}Z,W)\\
&-\sum_{k=1}^{n}(\nabla_{H\hat{e}_k}R)(e_k,Y,Z,\nabla_{H\hat{X}}W)-\sum_{k=1}^{n}(\nabla_{\nabla_{H\hat{X}}e_k}R)(e_k,Y,Z,W).
\end{aligned}
$$

Using the second Bianchi identity (2.3) for the Finsler metric of scalar flag curvature, the first term of right hand side of (4.9) reads

$$
\begin{aligned}
&\sum_{k=1}^{n}\nabla_{H\hat{X}}((\nabla_{H\hat{e}_k}R)(e_k,Y,Z,W)) \qquad (4.10)\\
&= -\sum_{k=1}^{n}\nabla_{H\hat{X}}((\nabla_{H\hat{Z}}R)(e_k,Y,W,e_k)+(\nabla_{H\hat{W}}R)(e_k,Y,e_k,Z)\\
&\quad+(\frac{F^2}{3}\nabla_{V\hat{e}_k}K+KF\nabla_{V\hat{e}_k}F)(\nabla_vQ)(e_k,Y,\dot{Z},\dot{W})\\
&\quad+(\frac{F^2}{3}\nabla_{V\hat{Z}}K+KF\nabla_{V\hat{Z}}F)(\nabla_vQ)(e_k,Y,\dot{W},\dot{e_k})\\
&\quad+(\frac{F^2}{3}\nabla_{V\hat{W}}K+KF\nabla_{V\hat{W}}F)(\nabla_vQ)(e_k,Y,\dot{e_k},\dot{Z})).
\end{aligned}
$$

Repeating the above procedure for the other five terms of (4.9), upon simplification we obtain,

$$
\begin{aligned}
&\sum_{k=1}^{n}(\nabla^2_{H\hat{X},H\hat{e}_k}R)(e_k,Y,Z,W) \qquad (4.11)\\
&=\sum_{k=1}^{n}(\nabla^2_{H\hat{X},H\hat{Z}}R)(e_k,Y,e_k,W)-\sum_{k=1}^{n}(\nabla^2_{H\hat{X},H\hat{W}}R)(e_k,Y,e_k,Z)\\
&\quad+\nabla_{H\hat{X}}(Q_1)+Q_2\\
&=(\nabla^2_{H\hat{X},H\hat{Z}}Ric)(Y,W)-(\nabla^2_{H\hat{X},H\hat{W}}Ric)(Y,Z)+\nabla_{H\hat{X}}(Q_1)+Q_2,
\end{aligned}
$$

where the first two terms on the right hand side are second order differential equations with respect to the horizontal covariant derivative. Interchanging the roles of X and Y yields,

$$
\begin{aligned}
&\sum_{k=1}^{n}(\nabla^2_{H\hat{Y},H\hat{e}_k}R)(e_k,X,Z,W) \qquad (4.12)\\
&=(\nabla^2_{H\hat{Y},H\hat{Z}}Ric)(X,W)-(\nabla^2_{H\hat{Y},H\hat{W}}Ric)(X,Z)+\nabla_{H\hat{Y}}(Q_3)+Q_4.
\end{aligned}
$$

Using the symmetric property $R(X,Y,Z,W)=R(Z,W,X,Y)$ for hh-curvature tensor of a Finsler manifold of scalar flag curvature and the second Bianchi identity (2.3) we obtain,

$$
\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) \qquad (4.13)
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{X}}R)(e_k,Y,Z,W) - \sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{Y}}R)(e_k,X,Z,W)\\
&\quad + \nabla_{H\hat{e}_k}(Q_5) + Q_6.
\end{aligned}
$$

Combining the equations (4.8), (4.9), (4.11), (4.12) and (4.13) yields,

$$
\begin{aligned}
&(\nabla^2_{H\hat{X},H\hat{Z}}Ric)(Y,W) - (\nabla^2_{H\hat{X},H\hat{W}}Ric)(Y,Z) - (\nabla^2_{H\hat{Y},H\hat{Z}}Ric)(X,W)\\
&+ (\nabla^2_{H\hat{Y},H\hat{W}}Ric)(X,Z)\\
= &\sum_{k=1}^{n}(\nabla^2_{H\hat{X},H\hat{e}_k}R)(e_k,Y,Z,W) - \sum_{k=1}^{n}(\nabla^2_{H\hat{Y},H\hat{e}_k}R)(e_k,X,Z,W)\\
&- \nabla_{H\hat{X}}Q_1 - Q_2 + \nabla_{H\hat{Y}}Q_3 + Q_4\\
= &\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{X}}R)(e_k,Y,Z,W) - \sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{Y}}R)(e_k,X,Z,W)\\
&+ Q(R)(Z,W,X,Y) - \sum_{l=1}^{n}Ric(e_l,X)R(e_l,Y,Z,W)\\
&+ \sum_{l=1}^{n}Ric(e_l,Y)R(e_l,X,Z,W) + \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W)\\
&- \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W) - \nabla_{H\hat{X}}Q_1 - Q_2 + \nabla_{H\hat{Y}}Q_3 + Q_4\\
= &\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) - \nabla_{H\hat{X}}Q_1 + \nabla_{H\hat{Y}}Q_3 - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}Q_5\\
&+ Q_4 - Q_6 + \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W) - Q_2\\
&- \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W) + Q(R)(Z,W,X,Y)\\
&- \sum_{l=1}^{n}Ric(e_l,X)R(e_l,Y,Z,W) + \sum_{l=1}^{n}Ric(e_l,Y)R(e_l,X,Z,W).
\end{aligned}
$$

Collecting the last eight terms of right hand side as lower order terms, completes the proof of Proposition 4.4. □

Theorem 4.5. *Let $(M, g(t))$ be a family of Finsler manifolds of scalar flag curvature satisfying in the Finslerian Ricci flow equation. We have*

$$
\frac{\partial}{\partial t}(R(X,Y,Z,W)) = \sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) - \nabla_{H\hat{X}}Q_1 + \nabla_{H\hat{Y}}Q_3
$$

$$- \sum_{k=1}^{n} \nabla_{H\hat{e}_k} Q_5 + \textit{lower order terms,}$$

where Q_j *are some lower order terms and* $H\hat{X}, H\hat{Y}$, $H\hat{Z}$ *and* $H\hat{W}$ *are the fixed horizontal parts of* $\hat{X}$, $\hat{Y}$, $\hat{Z}$ *and* $\hat{W}$ *on* HTM *respectively.*

Proof. By Propositions 4.3 and 4.4 we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(R(Z,W,X,Y))\\
=\ & \sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) - \nabla_{H\hat{X}}Q_1 + \nabla_{H\hat{Y}}Q_3 - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}Q_5\\
& - Q_2 + Q_4 - Q_6 + \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W)\\
& - \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W) + Q(R)(Z,W,X,Y)\\
& - \sum_{l=1}^{n} Ric(e_l,X)R(e_l,Y,Z,W) + \sum_{l=1}^{n} Ric(e_l,Y)R(e_l,X,Z,W)\\
& - \sum_{k=1}^{n} R(Z,e_k,X,Y)Ric(e_k,W) + \sum_{k=1}^{n} R(W,e_k,X,Y)Ric(Z,e_k)\\
& + (\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}Ric)(Z,W).
\end{aligned}$$

Collecting the last eleven terms of right hand side as lower order terms, completes the proof. □

5. Covariant time derivative of *hh*-curvature of Cartan connection

In this section we are going to evaluate the covariant time derivative of *hh*-curvature tensor of Cartan connection in order to establish evolution equation for its scalar curvature. As described earlier this method is introduced in Riemannian manifolds by Uhlenbeck.

Theorem 5.1. *Let* $(M, g(t))$ *be a family of Finsler manifolds of scalar flag curvature satisfying in the Finslerian Ricci flow equation. Then*

$$\begin{aligned}
(\nabla_{\frac{\partial}{\partial t}}R)(X,Y,Z,W) =\ & \sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) - \nabla_{H\hat{X}}Q_1\\
& + \nabla_{H\hat{Y}}Q_3 - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}Q_5 + \textit{lower order terms.}
\end{aligned}$$

Proof. Covariant time derivative of the *hh*-curvature tensor is

$$(\nabla_{\frac{\partial}{\partial t}}R)(X,Y,Z,W) = \frac{\partial}{\partial t}(R(X,Y,Z,W)) - R(\nabla_{\frac{\partial}{\partial t}}X,Y,Z,W)$$

$$
\begin{aligned}
&- R(X,\nabla_{\frac{\partial}{\partial t}}Y,Z,W) - R(X,Y,\nabla_{\frac{\partial}{\partial t}}Z,W)\\
&- R(X,Y,Z,\nabla_{\frac{\partial}{\partial t}}W).
\end{aligned}
$$

By means of Theorem 4.5 we have,

$$
\begin{aligned}
&(\nabla_{\frac{\partial}{\partial t}}R)(X,Y,Z,W)\\
= &\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) - \nabla_{H\hat{X}}Q_1\\
&-\sum_{k=1}^{n}\nabla_{H\hat{e}_k}Q_5 - Q_2 + \nabla_{H\hat{Y}}Q_3 + Q_4 - Q_6 + Q(R)(Z,W,X,Y)\\
&-\sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W) + \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W)\\
&-\sum_{l=1}^{n}Ric(e_l,X)R(e_l,Y,Z,W) + \sum_{l=1}^{n}Ric(e_l,Y)R(e_l,X,Z,W)\\
&-\sum_{k=1}^{n}R(Z,e_k,X,Y)Ric(e_k,W) + \sum_{k=1}^{n}R(W,e_k,X,Y)Ric(Z,e_k)\\
&+(\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}Ric)(Z,W) + \sum_{k=1}^{n}R(e_k,Y,Z,W)Ric(X,e_k)\\
&+\sum_{k=1}^{n}R(X,e_k,Z,W)Ric(Y,e_k) + \sum_{k=1}^{n}R(X,Y,e_k,W)Ric(Z,e_k)\\
&+\sum_{k=1}^{n}R(X,Y,Z,e_k)Ric(W,e_k).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&(\nabla_{\frac{\partial}{\partial t}}R)(X,Y,Z,W)\\
= &\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}R)(X,Y,Z,W) - \nabla_{H\hat{X}}Q_1 + \nabla_{H\hat{Y}}Q_3 - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}Q_5\\
&-\sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,X,Z,W) + \sum_{k=1}^{n}(\nabla_{\widetilde{[H\hat{X},H\hat{e}_k]}}R)(e_k,Y,Z,W)\\
&+(\nabla_{\widetilde{[H\hat{X},H\hat{Y}]}}Ric)(Z,W) + Q(R)(Z,W,X,Y) - Q_2 + Q_4 - Q_6.
\end{aligned}
$$

Collecting the last seven terms of right hand side as lower order terms, completes the proof. □

6. Evolution of the Ricci and scalar curvatures

Let (M, g) be a Finsler manifold of scalar flag curvature, ∇ the covariant derivative of Cartan connection and $g(t)$, $t \in [0, T)$ a family of metrics satisfying in the Finslerian Ricci flow equation. In this section we obtain evolution equation for the Ricci curvature and scalar curvature of Cartan connection.

Theorem 6.1. *Let $(M, g(t))$ be a family of Finsler manifolds of scalar flag curvature satisfying in the Finslerian Ricci flow equation. We have*

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial t}} Ric)(Y,W) = & \sum_{k=1}^{n} (\nabla^2_{H\hat{e}_k, H\hat{e}_k} Ric)(Y,W) - \sum_{l=1}^{n} \nabla_{H\hat{e_l}} (Q_1)_{tr} + \nabla_{H\hat{Y}} (Q_3)_{tr} \\ & - \sum_{k=1}^{n} \nabla_{H\hat{e}_k} (Q_5)_{tr} + \text{lower order terms,}\end{aligned}$$

where $(Q_j)_{tr}$ denotes the trace of Q_j.

Proof. By Theorem 5.1, we have

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial t}} Ric)(Y,W) = & \sum_{k=1}^{n} (\nabla^2_{H\hat{e}_k, H\hat{e}_k} Ric)(Y,W) - \sum_{k=1}^{n} \nabla_{H\hat{e}_k} (Q_5)_{tr} + \nabla_{H\hat{Y}} (Q_3)_{tr} \\ & - (Q_2)_{tr} + (Q_4)_{tr} - (Q_6)_{tr} + \sum_{k,l=1}^{n} (\nabla_{\widetilde{[H\hat{e_l}, H\hat{e}_k]}} R)(e_k, Y, e_l, W) \\ & - \sum_{k=1}^{n} \nabla_{H\hat{e_l}} (Q_1)_{tr} - \sum_{k,l=1}^{n} (\nabla_{\widetilde{[H\hat{Y}, H\hat{e}_k]}} R)(e_k, e_l, e_l, W) \\ & + \sum_{l=1}^{n} Q(R)(e_l, W, e_l, Y) + \sum_{l=1}^{n} (\nabla_{\widetilde{[H\hat{e_l}, H\hat{Y}]}} Ric)(e_l, W),\end{aligned} \tag{6.1}$$

where $(Q_j)_{tr}$ denotes the trace of Q_j with respect to the X and Z in Theorem 5.1.

$$\begin{aligned}\sum_{h=1}^{n} Q(R)(e_h, W, e_h, Y) = & \sum_{k,l,h=1}^{n} R(e_k, e_l, e_h, Y) R(e_k, e_l, e_h, W) \\ & + 2 \sum_{k,h,l=1}^{n} R(e_h, e_l, e_h, e_k) R(e_k, Y, e_l, W) \\ & - 2 \sum_{k,h,l=1}^{n} R(W, e_l, e_h, e_k) R(e_k, Y, e_l, e_h).\end{aligned}$$

Since (M, g) is of scalar flag curvature, by means of the first Bianchi identity (2.2) we have,

$$2 \sum_{k,h,l=1}^{n} R(W, e_l, e_h, e_k) R(e_k, Y, e_l, e_h)$$

$$= \sum_{k,h,l=1}^{n} R(W,e_l,e_h,e_k)(R(e_k,Y,e_l,e_h) - R(e_h,Y,e_l,e_k))$$
$$= \sum_{k,h,l=1}^{n} R(W,e_l,e_h,e_k)R(e_l,Y,e_k,e_h)$$
$$= \sum_{k,l,h=1}^{n} R(e_k,e_l,e_h,W)R(e_k,e_l,e_h,Y).$$

Therefore

$$\sum_{h=1}^{n} Q(R)(e_h,W,e_h,Y) = 2\sum_{k,h,l=1}^{n} R(e_h,e_l,e_h,e_k)R(e_k,Y,e_l,W)$$
$$= 2\sum_{k,l=1}^{n} Ric(e_l,e_k)R(e_k,Y,e_l,W). \tag{6.2}$$

On the other hand,

$$\sum_{l=1}^{n}(\nabla_{\widetilde{[H\hat{e_l},H\hat{Y}]}}Ric)(e_l,W) = -\sum_{l=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e_l}]}}Ric)(e_l,W)$$
$$= \sum_{k,l=1}^{n}(\nabla_{\widetilde{[H\hat{Y},H\hat{e}_k]}}R)(e_k,e_l,e_l,W). \tag{6.3}$$

Combining (6.1), (6.2) and (6.3) we obtain,

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial t}}Ric)(Y,W) = &\sum_{k=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}Ric)(Y,W) - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}(Q_1)_{tr}\\ &+\nabla_{H\hat{Y}}(Q_3)_{tr} - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}(Q_5)_{tr} - (Q_2)_{tr} + (Q_4)_{tr} - (Q_6)_{tr}\\ &+\sum_{k,l=1}^{n}(\nabla_{\widetilde{[H\hat{e_l},H\hat{e}_k]}}R)(e_k,Y,e_l,W)\\ &+2\sum_{k,l=1}^{n}Ric(e_l,e_k)R(e_k,Y,e_l,W).\end{aligned}$$

Collecting the last five terms of right hand side as lower order terms, completes the proof. □

Theorem 6.2. *Let $(M,g(t))$ be a family of Finsler manifolds of scalar flag curvature satisfying in the Finslerian Ricci flow equation. Then the evolution equation for scalar curvature is given by*

$$\frac{\partial}{\partial t}scal = (\Delta^h scal) + 2|Ric|^2 - \sum_{k=1}^{n}\nabla_{H\hat{e}_k}(Q_5)_{tr_2} - \sum_{l=1}^{n}\nabla_{H\hat{e_l}}(Q_1)_{tr_2}$$

$$+\sum_{h=1}^{n}\nabla_{H\hat{e_h}}(Q_3)_{tr_2}-(Q_2)_{tr_2}+(Q_4)_{tr_2}-(Q_6)_{tr_2},$$

where $(Q_j)_{tr_2}$ *denotes the trace of* $(Q_j)_{tr}$.

Proof. By contracting the covariant time derivative of the Ricci tensor in Theorem 6.1 with respect to Y and W we have,

$$\begin{aligned}\frac{\partial}{\partial t}scal=\nabla_{\frac{\partial}{\partial t}}scal= & \sum_{k,l=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}Ric)(e_l,e_l)\\ & +2\sum_{k,h,l=1}^{n}Ric(e_l,e_k)R(e_k,e_h,e_l,e_h)\\ & -\sum_{k=1}^{n}\nabla_{H\hat{e}_k}(Q_5)_{tr_2}-\sum_{l=1}^{n}\nabla_{H\hat{e_l}}(Q_1)_{tr_2}-(Q_2)_{tr_2}\\ & +\sum_{h=1}^{n}\nabla_{H\hat{e_h}}(Q_3)_{tr_2}+(Q_4)_{tr_2}-(Q_6)_{tr_2}\\ & +\sum_{k,l,h=1}^{n}(\nabla_{\widetilde{[H\hat{e_l},H\hat{e}_k]}}R)(e_k,e_h,e_l,e_h),\end{aligned}$$

where, $(Q_j)_{tr_2}$ denotes the trace of $(Q_j)_{tr}$ with respect to Y and W. Note that

$$\sum_{k,l,h=1}^{n}(\nabla_{\widetilde{[H\hat{e_l},H\hat{e}_k]}}R)(e_k,e_h,e_l,e_h)=0 \text{ and}$$
$$\sum_{k,l=1}^{n}(\nabla^2_{H\hat{e}_k,H\hat{e}_k}Ric)(e_l,e_l)=(\Delta^h scal).$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}scal=(\Delta^h scal)+2|Ric|^2-\sum_{l=1}^{n}\nabla_{H\hat{e_l}}(Q_1)_{tr_2}+\sum_{h=1}^{n}\nabla_{H\hat{e_h}}(Q_3)_{tr_2}\\ -\sum_{k=1}^{n}\nabla_{H\hat{e}_k}(Q_5)_{tr_2}-(Q_2)_{tr_2}+(Q_4)_{tr_2}-(Q_6)_{tr_2}.\end{aligned}$$

As claimed. □

Proof of Theorem 1.1. Recall that every 2-dimensional Finsler manifold is isotropic and its vv-curvature vanishes, see [2]. On the other hand, it is well known for every n-dimensional Finsler manifold of non-zero constant flag curvature the vv-curvature vanishes, cf., [2]. If $K=0$, then the last six terms in evolution equation for scalar curvature in Theorem 6.2 vanish and therefore in any of two cases the evolution equation for scalar curvature reduces to $\frac{\partial}{\partial t}scal=(\Delta^h scal)+2|Ric|^2$. By non-negativity of $2|Ric|^2$, we have

$\frac{\partial}{\partial t}scal \geq (\Delta^h scal)$, which is an inequality on the heat type equation. By assumption we have $scal_0 \geq 0$. The scalar curvature, $scal$ is homogeneous of degree zero, hence one can consider the scalar curvature as a real function on SM. Note that compactness of the sphere bundle SM is due to the compactness of M. Next using the maximum principles for scalar parabolic equation (see, [9] pages 93 and 96) we obtain $scal_t \geq 0$ for all $t \in [0, T)$. This completes the proof. □

Proof of Theorem 1.2. Let $\tau = \min\{T, \frac{n}{2\alpha}\}$, then similar to the proof of Theorem 1.1 the last six terms in evolution equation for scalar curvature in Theorem 6.2 vanish. Consider a function $f : SM \times [0, \tau) \longrightarrow \mathbb{R}$, defined by $f(z,t) = scal(z,t) - \frac{n\alpha}{n-2\alpha t}$. We have,

$$\frac{\partial}{\partial t}f = \frac{\partial}{\partial t}scal - \frac{2}{n}(\frac{n\alpha}{n-2\alpha t})^2.$$

Hence by using evolution equation for scalar curvature in isotropic Finsler manifolds with constant flag curvature we have

$$\begin{aligned}\frac{\partial}{\partial t}f &= (\Delta^h scal) + 2|Ric|^2 - \frac{2}{n}(\frac{n\alpha}{n-2\alpha t})^2 \\ &= (\Delta^h f) + 2|Ric|^2 - \frac{2}{n}(\frac{n\alpha}{n-2\alpha t})^2.\end{aligned}$$

Consider the tensor $H = Ric - \frac{scal}{n}g$. By $|H|^2 \geq 0$ we have $|Ric|^2 \geq \frac{scal^2}{n}$, therefore

$$\begin{aligned}\frac{\partial}{\partial t}f &\geq (\Delta^h f) + \frac{2}{n}scal^2 - \frac{2}{n}(\frac{n\alpha}{n-2\alpha t})^2 \\ &= (\Delta^h f) + \frac{2}{n}(scal + \frac{n\alpha}{n-2\alpha t})f.\end{aligned}$$

For all $p \in SM$, $f(p,0) = scal(p,0) - \alpha \geq 0$, therefore by maximum principle for scalar parabolic equations we have, $f(p,t) \geq 0$ for $p \in SM$ and $t \in [0, \tau)$, hence $\inf_{SM} scal_{g(t)} \geq \frac{n\alpha}{n-2\alpha t}$. Now if the solution of Ricci flow exist on a time interval $[0, T)$ and $t = \frac{n}{2\alpha} \in [0, T)$, then $\inf_{SM} scal_{g(t)} = \infty$, which is a contradiction with the continuity of scalar curvature on compact sphere bundle SM, therefore $T \leq \frac{n}{2\alpha}$. This completes the proof. □

We remark that as a consequence of this theorem the time T on equation (3.1) on compact Finsler surfaces, has the upper bound $\frac{1}{inf_{SM} scal_{g(0)}}$. Hence in these manifolds scalar curvature blows up in finite time.

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