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QUASI-COMMUTATIVITY RELATED TO POWERS

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Abstract. We study the quasi-commutativity in relation with powers of coefficients of polynomials. In the procedure we introduce the concept of π-*quasi-commutative* ring as a generalization of quasi-commutative rings. We show first that every π-quasi-commutative ring is Abelian and that a locally finite Abelian ring is π-quasi-commutative. The role of these facts are essential to our study in this note. The structures of various sorts of π-quasi-commutative rings are investigated to answer the questions raised naturally in the process, in relation to the structure of Jacobson and nil radicals.

1. Properties of π-quasi-commutative rings

The structure of π-quasi-commutative rings is closely related to one of Abelian rings, polynomial rings, and matrix rings. Such rings have important roles in noncommutative ring theory, and so the study of π-quasi-commutative rings is able to provide many useful information to related topics.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We use $C(R)$ to denote the center of R, i.e., the set of all central elements in R. The n by n full (resp., upper triangular) matrix ring over R is written by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$). $D_n(R)$ denotes the subring $\{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ of $U_n(R)$. Use E_{ij} for the matrix with (i,j)-entry 1 and zeros elsewhere. Let $J(R)$, $N_0(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the Wedderburn radical (i.e., the sum of all nilpotent ideals), the lower nilradical (i.e., intersection of all prime ideals), the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N_0(R) \subseteq N^*(R) \subseteq J(R)$ and $N_*(R) \subseteq N^*(R) \subseteq N(R)$. We use $I(R)$ for the set of all idempotents in R. The polynomial ring with an indeterminate x over R is denoted by $R[x]$, and $C_{f(x)}$ denotes the set of all coefficients of a given polynomial $f(x)$ in $R[x]$. $\mathbb{Z}$ (resp., $\mathbb{Z}_n$) denotes the ring of integers (resp., modulo n).

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Recall that a ring is said to be *Abelian* if every idempotent is central. A ring is usually called *reduced* if it has no nonzero nilpotent elements. Due to Bell [1], a ring R is said to satisfy the *Insertion-of-Factors-Property* (simply, is called an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is easily shown that reduced rings are IFP and IFP rings are Abelian. It is proved by [6, Proposition 2.8] that a ring R is reduced if and only if $D_3(R)$ is IFP.

Following Jung et al. [7], a ring R is said to be *quasi-commutative* if $ab \in C(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x)$ and $g(x)$ over R satisfy $f(x)g(x) \in C(R)[x]$. The following is almost a restatement of [7, Lemma 1.1].

Lemma 1.1. *For a ring R the following conditions are equivalent*:

(1) *R is commutative*;

(2) *If $f(x)g(x) \in C(R)[[x]]$ for $f(x), g(x) \in R[[x]]$, then $ab \in C(R)$ for all $a \in C_{f(x)}, b \in C_{g(x)}$*;

(3) *If $ab \in C(R)$ for $a, b \in R$, then $aRb \subseteq C(R)$*;

(4) *If $ab \in C(R)$ for $a, b \in R$, then $a^m R b^n \subseteq C(R)$ for some $m, n \geq 1$.*

Proof. The conditions (1), (2), and (3) are equivalent by [7, Lemma 1.1]. It suffices to show (4) implying (1). Suppose the condition (4). Then, since $1 = 1 \cdot 1 \in C(R)$, there exist $m, n \geq 1$ such that $R = 1^m R 1^n \subseteq C(R)$. Thus R is commutative. □

In this note we apply the condition (4) in Lemma 1.1 to the quasi-commutativity, and consider a generalization of quasi-commutative rings as follows.

Following Huh et al. [3], a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. It is shown that a ring is locally finite if every finite subset generates a finite subring by [3, Theorem 2.2(1)]. The class of locally finite rings contains finite rings, infinite direct sums of finite rings, and algebraic closures of finite fields.

Example 1.2. (1) Let R be a locally finite Abelian ring and $n \geq 3$. Then $E = D_n(R)$ is Abelian by [2, Lemma 2]. So $D_n(R)$ itself is an example of locally finite Abelian ring. But E is not quasi-commutative by Remark posterior to [7, Proposition 1.7]. However R satisfies the following property that is a generalization of quasi-commutativity.

Let $A, B \in E$. Since E is locally finite, A^h and B^k are both idempotents for some $h, k \geq 1$ by the proof of [4, Proposition 16]. Moreover A^h and B^k are central because E is Abelian, entailing $A^h B^k \in C(E)$.

(2) We refer to the construction of [4, Example 2]. Let $A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$, and B be the set of polynomials of zero constant term in A. Let I be the ideal of A generated by $a_0 r b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, a_1 b_2 + a_2 b_1, a_2 r b_2, (a_0 + a_1 + a_2) r (b_0 + b_1 + b_2)$, and $r_1 r_2 r_3 r_4$, where $r \in A$ and $r_1, r_2, r_3, r_4 \in B$. Then clearly $B^4 \in I$. Set $R = A/I$, and identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Note $N_0(R) = B$.

Consider polynomials $f(x) = a_0 + a_1x + a_2x^2$ and $g(x) = b_0 + b_1x + b_2x^2$ in $R[x]$. Then $f(x)g(x) = 0 \in C(R)[x]$, but $a_1b_1 \notin C(R)$ because $a_1b_1a_0 \neq a_0a_1b_1$. So R is not quasi-commutative. However R satisfies the following property that is a generalization of quasi-commutativity. Every element α of R is of the form

$$\alpha = \alpha_0 + \alpha_1 \text{ with } \alpha_0 \in \mathbb{Z}_2,\ \alpha_1 \in B.$$

Then we have the following computation:

$$\alpha^2 = (\alpha_0 + \alpha_1)^2 = \alpha_0^2 + 2\alpha_0\alpha_1 + \alpha_1^2 = \alpha_0 + \alpha_1^2,$$
$$(\alpha^2)^2 = (\alpha_0 + \alpha_1^2)^2 = \alpha_0^2 + 2\alpha_0\alpha_1^2 + (\alpha_1^2)^2 = \alpha_0 + \alpha_1^4 = \alpha_0,$$

noting $B^4 = 0$. Thus, for all $\alpha, \beta \in R$, there exist $m, n \geq 1$ such that $\alpha^m, \beta^n \in \mathbb{Z}_2$. This yields $\alpha^m\beta^n \in C(R)$, noting $\mathbb{Z}_2 \subset C(R)$.

Based on Example 1.2, we consider next the following new concept.

Definition 1.3. A ring R is said to be π-*quasi-commutative* provided that there exist positive integers h, k, depending on (a, b), such that $a^hb^k \in C(R)$, for each $(a, b) \in C_{f(x)} \times C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in C(R)[x]$.

Every quasi-commutative ring is clearly π-quasi-commutative, but the converse need not hold by help of Example 1.2. The following contains basic properties of π-quasi-commutative rings which do roles throughout this note.

Theorem 1.4. (1) *Every π-quasi-commutative ring is Abelian.*

(2) *Let R be a locally finite ring. Then R is π-quasi-commutative if and only if R is Abelian.*

Proof. (1) Let R be a π-quasi-commutative ring. Assume on the contrary that there exist $e \in I(R)$ and $a \in R$ such that $ea(1-e) \neq 0$. Consider polynomials

$$f(x) = e + [e - ea(1-e)]x \text{ and } g(x) = (1-e) + [(1-e) + ea(1-e)]x$$

in $R[x]$. Then $f(x)g(x) = 0 \in C(R)[x]$, and

$$e^m[(1-e) + ea(1-e)]^n = e[(1-e) + ea(1-e)] = ea(1-e)$$

for all $m, n \geq 1$ because $(1-e) + ea(1-e)$ is also contained in $I(R)$. However $e[ea(1-e)] = ea(1-e) \neq 0 = [ea(1-e)]e$, entailing $ea(1-e) \notin C(R)$. Thus R is Abelian.

(2) It suffices to show the sufficiency by help of (1). Let $a, b \in R$. Then there exist $m, n \geq 1$ such that $a^m, b^n \in I(R)$ by the proof of [4, Proposition 16]. If R is Abelian, then $a^mb^n \in C(R)$. So R is π-quasi-commutative. $\square$

Locally finite quasi-commutative rings are commutative by [7, Corollary 1.10(1)]. However locally finite π-quasi-commutative rings need not be commutative by Example 1.2 and the remark posterior to [7, Proposition 1.7].

The ring R in Example 1.2(2) is π-quasi-commutative by the argument in it. We can obtain this fact also by Theorem 1.4(2). The ring R is a finite ring, and moreover R is IFP (hence Abelian) by the argument in [4, Example 2]. Thus it is π-quasi-commutative by Theorem 1.4(2). The following is an immediate consequence of Theorem 1.4(1).

Corollary 1.5 ([7, Proposition 1.8(1)])**.** *Quasi-commutative rings are Abelian.*

In what follows, we see that the converse of Theorem 1.4(1) does not hold in general, and that the condition "locally finite" is not superfluous in Theorem 1.4(2).

Example 1.6. Let $R = D_3(\mathbb{Z})$. Consider polynomials

$$f(x) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix} x \text{ and } g(x) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 4 \end{pmatrix} x$$

over R. Then we have

$$f(x)g(x) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} + \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix} x + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} x^2 \in C(R)[x].$$

However, for any positive integers m and n, we get

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^m \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & 4 \end{pmatrix}^n = \begin{pmatrix} 2^m4^n & 0 & 0 \\ 0 & 2^m4^n & n4^{n-1}2^m6 \\ 0 & 0 & 2^m4^n \end{pmatrix},$$

A say, that is not contained in $C(R)$ because

$$E_{12}A = \begin{pmatrix} 0 & 2^m4^n & n4^{n-1}2^m6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 2^m4^n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = AE_{12},$$

noting $n4^{n-1}2^m6 \neq 0$. This result implies that R is not π-quasi-commutative. But R is Abelian by [2, Lemma 2], and so the converse of Theorem 1.4(1) need not hold. Since R is defined over $\mathbb{Z}$, R is clearly not locally finite.

We apply the argument in Example 1.6 to the next proposition.

Proposition 1.7. *Let A be a commutative ring and $R = D_n(A)$, where $n \geq 3$. If R is π-quasi-commutative, then the characteristic of A is nonzero.*

Proof. Let $R = D_n(A)$ be a π-quasi-commutative ring. Assume on the contrary that the characteristic of A is zero. Then A contains $\mathbb{Z}$ as a subring. Let s, t be any distinct nonzero integers and set $a = s1$, $b = t1$, where 1 is the identity of A. Consider polynomials

$$f(x) = aI_n + (aI_n + (-a)E_{23})x \text{ and } g(x) = bI_n + (bI_n + bE_{23})x$$

in $R[x]$, where I_n means the identity matrix. Then

$$f(x)g(x) = (ab)I_n + (2ab)I_nx + (ab)I_nx^2 \in C(R)[x].$$

From the coefficients $aI_n \in C_{f(x)}$ and $bI_n + bE_{23} \in C_{g(x)}$, we obtain the matrix

$$M = (aI_n)^m(bI_n+bE_{23})^n = [a^m I_n][b^m I_n+(nb^n)E_{23}] = (a^m b^n)I_n+(na^m b^n)E_{23}$$

for all positive integers m and n. But $na^m b^n$ is nonzero, and so M is not contained in $C(R)$. In fact,

$$E_{12}M = (a^m b^n)E_{12} + (na^m b^n)E_{13} \neq (a^m b^n)E_{12} = ME_{12}.$$

This implies that R is not π-quasi-commutative, a contradiction. Therefore the characteristic of A is nonzero. □

In the following argument, we observe that π-quasi-commutativity and IFP are independent of each other, noting that both are generalizations of commutativity.

Note. The ring R in Example 1.2(2) is an IFP ring which is also π-quasi-commutative. But, the ring $D_3(\mathbb{Z})$ in Example 1.6 is not a π-quasi-commutative ring which is IFP by [8, Proposition 1.2].

Consider next $D_n(A)$ over a locally finite Abelian ring A for $n \geq 4$. Then $D_n(A)$ is π-quasi-commutative by the argument in Example 1.2(1), but it is not IFP by [8, Example 1.3]. Therefore the concepts of π-quasi-commutativity and IFP are independent of each other.

In what follows, we see noncommutative π-quasi-commutative rings of minimal order. Kim et al. proved the following in [9, Theorem 3.3]:

If R is a noncommutative Abelian ring of minimal order, then R is of order 16 and is isomorphic to R_i for some $i \in \{1, 2, 3, 4, 5\}$, where R_i's are the rings in Example 1.8 to follow.

Thus every noncommutative π-quasi-commutative ring of minimal order is isomorphic to one of the rings in the following example, by help of Theorem 1.4(1). $GF(p^n)$ means the field of order p^n.

Example 1.8. (1) $R_1 = D_3(\mathbb{Z}_2)$ is IFP (hence Abelian) by [8, Proposition 1.2]. So R_1 is a noncommutative π-quasi-commutative ring of order 16 by Theorem 1.4(2).

(2) Following Xue [11, Example 2], let $R_2 = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2) \right\}$. Then R_2 is IFP (hence Abelian) by the computation in [11, Example 2]. So R_2 is a noncommutative π-quasi-commutative ring of order 16 by Theorem 1.4(2).

(3) The construction is due to Xue [11, Example 2]. Let $A = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over the commutative ring $\mathbb{Z}_4$. Let $R_3 = A/I$, where I is the ideal of A generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$. Then R_3 is IFP (hence Abelian) by the computation in [11, Example 2]. So R_4 is a noncommutative π-quasi-commutative ring of order 16 by Theorem 1.4(2).

(4) The construction is due to Xue [11, Example 2]. Let $A = \mathbb{Z}_2\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over the field $\mathbb{Z}_2$. Let

I be the ideal of A generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$, and set $R_4 = A/I$. Then R_4 is IFP (hence Abelian) by the computation in [11, Example 2]. So R_4 is a noncommutative π-quasi-commutative ring of order 16 by Theorem 1.4(2).

(5) The construction is due to Xu and Xue [10, Example 7]. Let $A = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over the commutative ring $\mathbb{Z}_4$. Let I be the ideal of A generated by $x^3, y^2, yx, x^2 - xy, x^2 - 2, 2x, 2y$, and set $R_5 = A/I$. Then R_5 is IFP (hence Abelian) by the argument in [10, Example 7]. So R_5 is a noncommutative π-quasi-commutative ring of order 16 by Theorem 1.4(2).

Every ring in Example 1.8 is a local ring with nilpotent Jacobson radical. So it is natural to ask whether a ring R is π-quasi-commutative if $J(R)$ is nilpotent and $R/J(R)$ is a commutative domain. However the answer is negative by the argument in Example 1.6 for the non-π-quasi-commutative ring $R = D_3(\mathbb{Z})$. In fact, $J(R) = \left(\begin{smallmatrix} 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is nilpotent and $R/J(R)$ is isomorphic to $\mathbb{Z}$.

Moreover consider $R = D_n(K)$ over a field of characteristic zero for $n \geq 3$. Then $J(R) = \{(a_{ij} \in R \mid a_{11} = \cdots = a_{nn} = 0\}$ is nilpotent, and $R/J(R)$ is isomorphic to the field K (hence R is a local ring). However R is not π-quasi-commutative by Proposition 1.7.

Finite quasi-commutative rings are commutative by [7, Corollary 1.10]. Every ring in Example 1.8 is a finite noncommutative ring, and so it is not quasi-commutative. Thus Example 1.8 provides five kinds of π-quasi-commutative rings but not quasi-commutative.

Note. Let R be any π-quasi-commutative ring in Example 1.8. Then R is not quasi-commutative by the preceding argument, but we have

$$N_0(R) = N_*(R) = N^*(R) = N(R) = J(R).$$

So, by the proof of [7, Proposition 1.6], we obtain

$$\begin{aligned} J(R[x]) &= N_0(R[x]) = N_*(R[x]) = N^*(R[x]) \\ &= N_0(R)[x] = N(R)[x] = N(R[x]), \end{aligned}$$

and $R[x]/J(R[x])$ is a reduced ring.

2. Examples of π-quasi-commutative rings

In this section we investigate several sorts of rings which are π-quasi-commutative. We first examine a sort of π-quasi-commutative ring, based on Proposition 1.7.

Theorem 2.1. *Let R_0 be a nil algebra over a commutative ring F of characteristic prime p. Set $R = F + R_0$. Then R is a π-quasi-commutative ring.*

Proof. Every element a in R is of the form $a = a_0 + a_1$ with $a_0 \in F$ and $a_1 \in R_0$. Then $a_1^{p^l} = 0$ for some $l \geq 0$. By [5, Exercises 1.10 in Chapter 3],

$\frac{p^n!}{k!(p^n-k)!}$ is divisible by p for all k with $1 \le k \le p^n - 1$, where $n \ge 1$. Since R is of characteristic p, we get the following equalities:

$$a^{p^l} = (a_0 + a_1)^{p^l} = a_0^{p^l} + pb + a_1^{p^l} = a_0^{p^l} + a_1^{p^l} = a_0^{p^l} \in F \text{ for some } b \in R.$$

Thus, for all $a, b \in R$, there exist $s, t \ge 1$ such that $a^s, b^t \in F$. This yields $a^s b^t \in C(R)$ because $F \subseteq C(R)$. Therefore R is π-quasi-commutative. $\square$

The ring R in Example 1.2(2) has the property that $R = \mathbb{Z}_2 + B$ and B is a nilpotent algebra over $\mathbb{Z}_2$. So R is π-quasi-commutative by Theorem 2.1.

Consider $D_n(A)$ for $n \ge 1$, where A is a commutative ring of characteristic prime p (e.g., $\mathbb{Z}_p$ and $\mathbb{Z}_p[x]$). Then $D_n(A)$ is isomorphic to $A + N_n(A)$ via $(a_{ij}) \mapsto a_{11} + (b_{st})$, where $N_n(A) = \{(a_{ij}) \in D_n(A) \mid a_{11} = \cdots = a_{nn} = 0\}$ and $b_{ss} = 0$, $b_{st} = a_{st}$ for $s < t$. $N_n(A)$ is clearly nilpotent, and so $D_n(A)$ is π-quasi-commutative by Theorem 2.1.

In the following we see an example of R_0 being nil but non-nilpotent in Theorem 2.1.

Example 2.2. The construction is due to [6, Theorem 2.2(2)]. Let F be a commutative ring of characteristic prime p. Define a map $\sigma : D_{2^n}(F) \to D_{2^{n+1}}(F)$ by $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$. Then $D_{2^n}(F)$ can be considered as a subring of $D_{2^{n+1}}(F)$ via σ (i.e., $B = \sigma(B)$ for $B \in D_{2^n}(F)$). Set $R = \bigcup_{n=1}^{\infty} D_{2^n}(F)$. Then

$$R_0 = \{A \in R \mid \text{ the diagonal entries of } A \text{ are all zero}\} \subseteq N_*(R).$$

It is obvious that R_0 is both a nil ideal of R and a nil algebra over F. R_0 is clearly non-nilpotent.

We identify f with fI for all $f \in F$, where I is the identity matrix. We then have $R = F + R_0$, noting that R/R_0 is isomorphic to F. So R is π-quasi-commutative by Theorem 2.1.

We see in the following a kind of π-quasi-commutative ring over which the polynomial ring is also π-quasi-commutative, by help of Theorem 2.1. Let R be a given ring. Use $R[X]$ for the polynomial ring with X a set (possibly infinite) of commuting indeterminates over R. In case of $X = \{x\}$; we denote $R[x]$ instead of $R[\{x\}]$.

Corollary 2.3. *Let R_0 be a nilpotent algebra over a commutative ring F of characteristic prime p. Set $R = F + R_0$. Then $R[X]$ is a π-quasi-commutative ring.*

Proof. Let $f(X) \in R[X]$. Then there exists a finite subset X_0 of X such that $f(X) \in R[X_0]$. Say $X_0 = \{x_1, \ldots, x_n\}$. Then $f(X) = \sum_{i=1}^{m}(a_i + b_i)x_1^{j(i)_1} \cdots x_n^{j(i)_n} \in R[x]$ with $a_i \in F$ and $b_i \in R_0$ for all i, where $j(i)_k \ge 0$ for all i, j.

We can rewrite $f(X)$ by

$$f(X) = \sum_{i=1}^{m} a_i x_1^{j(i)_1} \cdots x_n^{j(i)_n} + \sum_{i=1}^{m} b_i x_1^{j(i)_1} \cdots x_n^{j(i)_n},$$

noting that $\sum_{i=1}^{m} a_i x_1^{j(i)_1} \cdots x_n^{j(i)_n} \in F[X]$ and $\sum_{i=1}^{m} b_i x_1^{j(i)_1} \cdots x_n^{j(i)_n} \in R_0[X]$. So every polynomial in $R[X]$ can be expressed by $s(X)+t(X)$ with $s(X) \in F[X]$ and $t(X) \in R_0[X]$, i.e., $R[X] = F[X] + R_0[X]$. Since R_0 is nilpotent, $R_0[X]$ is also nilpotent. So $R_0[X]$ is a nilpotent algebra over a commutative ring $F[X]$ of characteristic prime p. Therefore $R[X]$ is π-quasi-commutative by Theorem 2.1. $\square$

Consider the ring R in Example 1.2(2). Then $R[x]$ is π-quasi-commutative by Corollary 2.3, but $R[x]$ is not IFP by the argument in [4, Example 2].

Recall that $N_*(A) = N^*(A) = N(A)$ for any commutative ring A. As in Theorem 2.1, let R_0 be a nil algebra over a commutative ring F of characteristic prime p and set $R = F + R_0$. Then

$$N(R) = N(F) + R_0 = N_*(F) + R_0 = N^*(F) + R_0 = N^*(R),$$

entailing that $R/N^*(R)$ is a reduced ring. Here one may ask naturally whether $N(R)$ equals $N_*(R)$. However the answer is negative by the ring R in Example 2.2. In fact, R is semiprime (i.e., $N_*(R) = 0$) by [6, Theorem 2.2(2)], but $N(R) = N^*(R) \neq 0$.

Next, as in Corollary 2.3, let R_0 be a nilpotent algebra over a commutative ring F of characteristic prime p and set $R = F + R_0$. Then

$$N(R) = N(F) + R_0 = N_*(F) + R_0 = N^*(F) + R_0 = N^*(R) = N_*(R),$$

entailing that $R/N_*(R)$ is a reduced ring.

Moreover if $J(F)$ is nil, then we obtain

$$\begin{aligned} J(R) &= J(F) + R_0 = N(F) + R_0 = N_*(F) + R_0 \\ &= N^*(F) + R_0 = N^*(R) = N_*(R) = N(R) \end{aligned}$$

and

$$\begin{aligned} J(R[x]) &= N_*(R[x]) = N^*(R[x]) = N(R[x]) = J(R)[x] \\ &= N_*(R)[x] = N^*(R)[x] = N(R)[x], \end{aligned}$$

also by help of the proof of [6, Theorem 2.2(2)]. Then we have that both $R/J(R)$ and $R[x]/J(R[x])$ are reduced rings.

In the following we see another π-quasi-commutative ring that is not quasi-commutative.

Example 2.4. We apply the argument in [7, Example 1.11]. Let K be a field of prime characteristic. Let $A = K\langle a, b, c\rangle$ be the free algebra generated by the noncommuting indeterminates a, b, c over K. Set I be the ideal of A generated by $ac - ca, bc - cb, c^2$ and set $R_1 = A/I$. We identify a, b, c with their images in A/I for simplicity. Then R_1 is not quasi-commutative by the argument in [7, Example 1.11].

Let $R_0 = Rc$, noting that $(Rc)^2 = 0$ and $Rc = RcR = cR = N(R) = N_0(R)$. Note that $R = \langle a, b\rangle + Rc$ because every element of R can be expressed by $\alpha + \beta$ with $\alpha \in K\langle a, b\rangle$ and $\beta \in Rc$. Write $F = K\langle a\rangle$ and set

$$R = F + R_0 = K\langle a\rangle + Rc.$$

Then R is a subring of R_1, and moreover R_0 is a nilpotent algebra over the commutative ring $F = K\langle a\rangle$ of prime characteristic. Then both R and $R[X]$ are π-quasi-commutative by Theorem 2.1 and Corollary 2.3. $(Rc)^2 = 0$, but $abc \neq bac = bca$, entailing $bc \notin C(R)$. So R is not quasi-commutative by [7, Lemma 1.5(2)].

The following definitions follow the literature. An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular (i.e., not a zero divisor). It is easily checked that $C(R[x]) = C(R)[x]$ for any ring R.

Proposition 2.5. *Let R be a ring and M be a multiplicatively closed subset of R consisting of central regular elements. Then R is π-quasi-commutative if and only if so is $M^{-1}R$.*

Proof. The proof is similar to one of [7, Proposition 2.3], but we write it here for completeness. Let $E = M^{-1}R$. Clearly $M^{-1}C(R) \subseteq C(E)$. Let $a^{-1}b \in C(E)$. By the proof of [7, Proposition 2.3], we also get $C(E) \subseteq M^{-1}C(R)$, entailing $C(E) = M^{-1}C(R)$. We will use this fact freely.

Suppose that R is π-quasi-commutative. Let $F(x) = \sum_{i=0}^{m} \alpha_i x^i$ and $G(x) = \sum_{j=0}^{n} \beta_j x^j$ be in $E[x]$ such that $F(x)G(x) \in C(E)[x]$. There exist regular $u, v \in M$ such that $\alpha_i = u^{-1}a_i$ and $\beta_j = v^{-1}b_j$ for all i, j, where $a_i, b_j \in R$. But $F(x)G(x) = u^{-1}(a_0 + a_1x + \cdots + a_mx^m)v^{-1}(b_0 + b_1x + \cdots + b_nx^n) = (uv)^{-1}(a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n)$.

Here let $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$. Then $f(x)$ and $g(x)$ are in $R[x]$. Moreover $f(x)g(x) \in C(R)[x]$ because $F(x)G(x) \in C(E)[x]$ and $C(E)[x] = (M^{-1}C(R))[x]$.

Since R is π-quasi-commutative, there exist $s_i, t_j \geq 1$ for all i, j such that $a_i^{s_i} b_j^{t_j} \in C(R)$ for all i, j. This yields

$$\begin{aligned}
\alpha_i^{s_i}\beta_j^{t_j} &= (u^{-1}a_i)^{s_i}(v^{-1}b_j)^{t_j} \\
&= u^{-s_i}v^{-t_j}a_i^{s_i}b_j^{t_j} \\
&= (u^{s_i}v^{t_j})^{-1}a_i^{s_i}b_j^{t_j} \in M^{-1}C(R) = C(E).
\end{aligned}$$

Thus E is π-quasi-commutative.

Conversely, suppose that E is π-quasi-commutative and let $f(x)g(x) \in C(R)[x]$ for $f(x), g(x) \in R[x]$. Then $f(x)g(x) \in (M^{-1}C(R))[x] = C(E)[x]$. But E is π-quasi-commutative. So, for every $(a, b) \in C_{f(x)} \times C_{g(x)}$, there exist

$s, t \geq 1$ such that $a^s b^t \in C(E)$. Then $a^s b^t \in C(R)$ since $C(R) = R \cap C(E)$. Thus R is π-quasi-commutative. $\square$

Let R be a ring. Recall that the ring of *Laurent polynomials*, in an indeterminate x over R, consists of all formal sums $\sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers with $k \leq n$. This ring is expressed by $R[x; x^{-1}]$.

Corollary 2.6. (1) *Let R be a ring. Then $R[x]$ is π-quasi-commutative if and only if so is $R[x; x^{-1}]$.*

(2) *Let R_0 be a nilpotent algebra over a commutative ring F of characteristic prime p. Set $R = F + R_0$. Then $R[x; x^{-1}]$ is π-quasi-commutative.*

Proof. (1) Note that $R[x; x^{-1}] = M^{-1}R[x]$ with $M = \{1, x, x^2, \ldots\}$. So the proof is done by Proposition 2.5.

(2) is shown by (1) and Corollary 2.3. $\square$

Let $R = F + R_0$ be the ring in Example 2.2. Then $R_0[x]$ is a nil ideal of $R[x]$, and so $R_0[x]$ is a nil algebra over the commutative ring $F[x]$. Thus $R[x; x^{-1}]$ is a π-quasi-commutative ring by Theorem 2.1 and Corollary 2.6.

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References

[1] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[2] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52.

[3] ———, *Examples of strongly π-regular rings*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 195–210.

[4] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative ring*, Comm. Algebra **30** (2002), no. 2, 751–761.

[5] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.

[6] Y. C. Jeon, H. K. Kim, Y. Lee, and J. S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), no. 1, 135–146.

[7] D. W. Jung, B.-O. Kim, H. K. Kim, Y. Lee, S. B. Nam, S. J. Ryu, H. J. Sung, and S. J. Yun, *On quasi-commutative rings*, J. Korean Math. Soc. **53** (2016), no. 2, 475–488.

[8] N. K. Kim and Y. Lee, *Extension of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223.

[9] N. K. Kim, Y. Lee, and Y. Seo, *Structure of idempotents in rings without identity*, J. Korean Math. Soc. **51** (2014), no. 4, 751–771.

[10] L. Xu and W. Xue, *Structure of minimal non-commutative zero-insertive rings*, Math. J. Okayama Univ. **40** (1998), 69–76.

[11] W. Xue, *Structure of minimal noncommutative duo rings and minimal strongly bounded non-duo rings*, Comm. Algebra **20** (1992), no. 9, 2777–2788.

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