Bull. Korean Math. Soc. **54** (2017), No. 6, pp. 2091–2106
https://doi.org/10.4134/BKMS.b160773
pISSN: 1015-8634 / eISSN: 2234-3016

SOME RESULTS OF f-BIHARMONIC MAPS INTO A RIEMANNIAN MANIFOLD OF NON-POSITIVE SECTIONAL CURVATURE

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Abstract. The authors investigate f-biharmonic maps $u : (M, g) \to (N, h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature, and derive that if $\int_M f^p|\tau(u)|^p dv_g < \infty$, $\int_M |\tau(u)|^2 dv_g < \infty$ and $\int_M |du|^2 dv_g < \infty$, then u is harmonic. When u is an isometric immersion, the authors also get that if u satisfies some integral conditions, then it is minimal. These results give an affirmative partial answer to conjecture 4 (generalized Chen's conjecture for f-biharmonic submanifolds).

1. Introduction

In the past several decades harmonic maps have played a central role in geometry and analysis. Let (M^m, g) and (N^n, h) be Riemannian manifolds of dimensions m, n and $u : (M^m, g) \to (N^n, h)$ be a smooth map. The energy of u is defined by $E(u) = \int_M \frac{|du|^2}{2} dv_g$, where dv_g is the volume element on (M^m, g). Harmonic maps are the critical maps of $E(\cdot)$. The Euler-Lagrange equation of harmonic maps is $\tau(u) = 0$, where $\tau(u)$ is called the tension field of u. p-harmonic maps [19], exponentially harmonic maps [16], F-harmonic maps and f-harmonic maps are extensions to harmonic maps and many results have been carried out (for instance, see [1–3, 10, 24, 33]).

In 1983, J. Eells and L. Lemaire [13] proposed the problem to consider biharmonic maps which are critical points of the bi-energy functional $E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g$. We see that biharmonic maps are a generalization of harmonic maps. In 1986, G. Y. Jiang [21] studied the first and the second variational formulas of the bi-energy. There have been many studies on biharmonic maps (for instance, see [4–6, 11, 20, 25, 26, 32]). To further generalize the notion of

Received September 24, 2016; Revised March 13, 2017; Accepted June 21, 2017.
2010 *Mathematics Subject Classification.* 58E20, 53C21.
Key words and phrases. f-biharmonic maps, f-biharmonic submanifolds.
This work is supported by NNSF (No.11371194) and Natural Science Foundation of the Anhui Higher Education Institutions of China (KJ2017A324).

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harmonic maps, Y. B. Han and S. X. Feng [17] introduced the F-bienergy functional $E_{F,2}(u) = \int_M F(\frac{|\tau(u)|^2}{2})dv_g$. The critical points of F-bienergy $E_{F,2}(u)$ are called F-biharmonic maps. If $F(u) = (2u)^{\frac{p}{2}}$, we have p-bienergy functional $E_{p,2}(u) = \int_M |\tau(u)|^p dv_g$. If $F(u) = e^u$, we have exponential bienergy functional $E_{e,2}(u) = \int_M e^{\frac{|\tau(u)|^2}{2}} dv_g$.

A. Lichnerowicz [23] (see also [12]) introduced and studied f-harmonic maps between Riemannian manifolds. The study of f-harmonic maps comes from a physical motivation, since in physics f-harmonic maps can be viewed as stationary solutions to the inhomogeneous Heisenberg spin system (see [22]). W. J. Lu [27] introduced the following functional:

$$E_{2,f}(u) = \int_M f\frac{|\tau(u)|^2}{2}dv_g,$$

where $f : (M, g) \to (0, +\infty)$ is a smooth function. A map u is called an f-biharmonic map if it is a critical point of the f-bienergy functional.

Recently, N. Nakauchi et al. [31] showed that every biharmonic map of a complete Riemannian manifold into a Riemannian manifold of non-positive curvature whose energy and bi-energy are finite must be harmonic. S. Maeta [29] obtained that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite $(a+2)$-bienergy $\int_M |\tau(u)|^{a+2}dv_g < \infty$ $(a \geq 0)$ and energy are harmonic. Y. B. Han and W. Zhang [18] obtained that p-biharmonic maps from a complete manifold into a non-positive curved manifold with finite $(a+p)$-bienergy $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and energy are harmonic. In this paper, we first obtain the following results:

Theorem 1.1 (cf. Theorem 3.1). *Let $u:(M^m, g) \to (N^n, h)$ be an f-biharmonic map from a compact Riemannian manifold (M^m, g) without boundary into a Riemannian manifold (N^n, h) with non-positive sectional curvature, then u is harmonic.*

Theorem 1.2 (cf. Theorem 3.3). *Let $u:(M^m, g) \to (N^n, h)$ be an f-biharmonic map from a complete Riemannian manifold (M^m, g) into a Riemannian manifold (N^n, h) with non-positive sectional curvature and let $p \geq 2$ be a non-negative real constant.*

(i) *If*

$$\int_M f^p|\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty, \text{ and } \int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $Vol(M, g) = \infty$, and $\int_M f^p|\tau(u)|^p dv_g < \infty$, then u is harmonic.*

Chen's conjecture is the most interesting problem in the biharmonic theory. In 1988, Chen [9] raised the following problem:

Conjecture 1. *Any biharmonic submanifold in E^n is minimal.*

There are some affirmative partial answers to Conjecture 1.

Then Chen's conjecture was generalized as follows ([8]): Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal. There are also some affirmative partial answers to this Conjecture (for instance, see [7, 17, 30, 31]).

Motivated by Chen's conjecture, Y. B. Han [15] proposed the following conjecture:

Conjecture 2. *Any p-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

Some affirmative partial answers to Conjecture 2 were proved in [15, 18, 28].

Y. B. Han [16] also proposed the following conjecture:

Conjecture 3. *Any exponentially biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

Some affirmative partial answers to Conjecture 3 were proved in [16].

For f-biharmonic submanifolds, it is natural to consider the following conjecture.

Conjecture 4. *Any f-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.*

For f-biharmonic submanifolds, we obtain some results:

Theorem 1.3 (cf. Theorem 4.1)**.** *Let $u : (M, g) \to (N, h)$ be an f-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature and let p, q be two real constants satisfying $2 \leq p < \infty$ and $0 < q \leq p < \infty$. If*

$$\int_M f^p |\vec{H}|^q dv_g < \infty,$$

then u is minimal.

Theorem 1.4 (cf. Theorem 4.2)**.** *Let $u : (M, g) \to (N, h)$ be an f-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature. If*

$$\int_{B_r(x_0)} f^p dv_g \leq C_0(1 + r)^s$$

for some positive integer s, C_0 independent of r and $p \geq 2$, then u is minimal.

Theorem 1.5 (cf. Theorem 4.3)**.** *Let $u : (M, g) \to (N, h)$ be an f-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon > 0$ and $\int_{B_r(x_0)} |f\vec{H}|^p dv_g$ $(p \geq 2)$ is of at most polynomial growth of r. Then u is minimal.*

Theorem 1.6 (cf. Theorem 4.4)**.** *Let $u : (M, g) \to (N, h)$ be a complete ε-supper f-biharmonic submanifold in (N, h) for $\varepsilon > 0$. If*

$$\int_M |f\vec{H}|^p dv_g < \infty,$$

where $p \geq 2$, then u is minimal.

2. Preliminaries

In this section we give some necessary notations and terminologies about harmonic maps, biharmonic maps, f-biharmonic maps and f-biharmonic submanifolds.

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map from an m-dimensional Riemannian manifold (M^m, g) to an n-dimensional Riemannian manifold (N^n, h). The energy of u is defined by

$$E(u) = \int_M \frac{|\ du\ |^2}{2} dv_g,$$

where dv_g is the volume element on (M^m, g).

The Euler-Lagrange equation of harmonic maps is $\tau(u) = \sum_{i=1}^m \{\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)\} = 0$ where ∇ is the Levi-Civita connection on (M^m, g) and $\tilde{\nabla}$ is the induced Levi-Civita connection of the pullback bundle $u^{-1}TN$. $\{e_i\}_{i=1}^m$ is an orthonormal frame field on (M^m, g). If $\tau(u) = 0$, then u is called a harmonic map.

In 1983, J. Eells and L. Lemaire [13] proposed the problem to consider the bi-energy functional:

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$

Then, in 1986, G. Y. Jiang [21] obtained the first and the second variational formulas of the bi-energy functional. The Euler-Lagrange equation of the bi-energy functional is

$$\tau_2(u) = -\tilde{\Delta}(\tau(u)) - \sum_i R^N(\tau(u)), du(e_i))du(e_i) = 0,$$

where $R^N(X, Y) = [{}^N\nabla_X, {}^N\nabla_Y] - {}^N\nabla_{[X,Y]}$ is the curvature operator on (N, h). If $\tau_2(u) = 0$, then u is called a biharmonic map.

To generalize the notation of biharmonic maps, W. J. Lu [27] studied the f-bienergy functional

$$E_{2,f}(u) = \int_M f(x) \frac{|\tau(u)|^2}{2} dv_g,$$

where $f : (M, g) \to (0, +\infty)$ is a smooth function. The Euler-Lagrange equation of $E_{2,f}$ is

$$\tau_{2,f}(u) = -\tilde{\Delta}(f\tau(u)) - \sum_i R^N(f\tau(u), du(e_i))du(e_i) = 0.$$

If $\tau_{2,f}(u) = 0$, then u is called an f-biharmonic map.

Now we briefly recall the submanifold theory. Let $u : (M^m, g) \to (N^{m+t}, h)$ be an isometric immersion from an m-dimensional Riemannian manifold (M^m, g) into an $(m+t)$-dimensional Riemannian manifold (N^{m+t}, h). The second fundamental form $B : TM \bigotimes TM \to NM$ is defined by

$$B(X,Y) = {}^N\nabla_X Y - \nabla_X Y, \quad X, Y \in \Gamma(TM).$$

The shape operator $A_\xi : TM \to TM$ for a unit normal vector field ξ on M is defined by

$${}^N\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad X \in \Gamma(TM), \xi \in \Gamma(T^\perp M),$$

where $\nabla^\perp$ denotes the normal connection on the normal bundle of M in N. It's well known that B and A_ξ are related by

$$\langle B(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle.$$

For any $x \in M$, the mean curvature vector field $\vec{H}$ of M at x is given by

$$\vec{H} = \frac{1}{m}\sum_i B(e_i, e_i).$$

If an isometric immersion $u : (M, g) \to (N, h)$ is f-biharmonic, then M is called an f-biharmonic submanifold in N. In this case, $\tau(u) = m\vec{H}$. We know that M is an f-biharmonic submanifold in N if and only if

$$-\tilde{\Delta}(f\vec{H}) - \sum_i R^N(f\vec{H}, e_i)e_i = 0. \tag{1}$$

From (2), we obtain the sufficient and necessary condition for M to be an f-biharmonic submanifold in N as follows:

$$\triangle^\perp(f\vec{H}) - \sum_i B(e_i, A_{f\vec{H}} e_i) + [\sum_i R^N(f\vec{H}, e_i)e_i]^\perp = 0, \tag{2}$$

$$Tr_g(\nabla_{(\cdot)} A_{f\vec{H}}(\cdot)) + Tr_g[A_{\nabla^\perp(f\vec{H})}(\cdot)] - [\sum_i R^N(f\vec{H}, e_i)e_i]^\top = 0. \tag{3}$$

We also need the following lemma.

Lemma 2.1 (Gaffney [14])**.** *Let (M, g) be a complete Riemannian manifold. If a C^1 1-form α satisfies that $\int_M |\alpha| dv_g < \infty$ and $\int_M (\delta\alpha) dv_g < \infty$, or equivalently, a C^1 vector X defined by $\alpha(Y) = \langle X, Y\rangle$ satisfies that $\int_M |X| dv_g < \infty$ and $\int_M div(X) dv_g < \infty$, then $\int_M (\delta\alpha) dv_g = \int_M div(X) dv_g = 0$.*

3. f-biharmonic maps in a Riemannian manifold of non-positive sectional curvature

In this section, we obtain some results as follows:

Theorem 3.1. *Let $u : (M^m, g) \to (N^n, h)$ be an f-biharmonic map from a compact Riemannian manifold (M^m, g) without boundary into a Riemannian manifold (N^n, h) with non-positive sectional curvature, then u is harmonic.*

Proof. From (1), we have

$$\begin{aligned}\frac{1}{2}\Delta|f\tau(u)|^2 &= |\tilde{\nabla}(f\tau(u))|^2 + \langle\tilde{\Delta}[f\tau(u)], f\tau(u)\rangle\\ &= |\tilde{\nabla}(f\tau(u))|^2 - \sum_i \langle R^N(f\tau(u), du(e_i))du(e_i), f\tau(u)\rangle\\ &\geq |\tilde{\nabla}(f\tau(u))|^2.\end{aligned}$$

From Green theorem and the compactness of (M, g), we have

$$0 = \int_M \frac{1}{2}\Delta|f\tau(u)|^2 dv_g = \int_M |\tilde{\nabla}(f\tau(u))|^2 dv_g. \tag{4}$$

Then, for every $X \in \Gamma(TM)$, we have

$$\tilde{\nabla}_X|f\tau(u)| = 0.$$

Let $Y = \sum_i h(du(e_i), f\tau(u))e_i$, we have

$$\begin{aligned}div(Y) &= \sum_k g(\nabla_{e_k}Y, e_k)\\ &= \sum_k [h(\tilde{\nabla}_{e_k}du(e_k), f\tau(u)) - h(du(\nabla_{e_k}e_k), f\tau(u))]\\ &= h(\tau(u), f\tau(u)) = f|\tau(u)|^2.\end{aligned} \tag{5}$$

From (6), we have

$$0 = \int_M div(Y)dv_g = \int_M f|\tau(u)|^2 dv_g.$$

Since $f > 0$ in M, so we have $\tau(u) = 0$. □

Corollary 3.2. *Any f-biharmonic function in a compact manifold (M, g) without boundary is constant.*

Proof. From Theorem 3.1, u is an f-biharmonic function if and only if u is a harmonic function. On the other hand, any harmonic function in a compact manifold (M, g) is constant, so we have $u = C$. □

Theorem 3.3. *Let $u : (M^m, g) \to (N^n, h)$ be an f-biharmonic map from a complete Riemannian manifold (M^m, g) into a Riemannian manifold (N^n, h) with non-positive sectional curvature and let $p \geq 2$ be a non-negative real constant.*

(i) *If*

$$\int_M f^p|\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty \quad \text{and} \quad \int_M |du|^2 dv_g < \infty,$$

then u is harmonic.

(ii) *If $Vol(M, g) = \infty$ and $\int_M f^p|\tau(u)|^p dv_g < \infty$, then u is harmonic.*

Proof. Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$\begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M, \\ \lambda(x) = 1, & x \in B_r(x_0), \\ \lambda(x) = 0, & x \in M - B_{2r}(x_0), \\ |\nabla\lambda| \leq \frac{C}{r}, & x \in M, \end{cases} \tag{6}$$

where $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, C is a positive constant and d is the distance of M. From (1), we have

$$\begin{aligned} &\int_M \langle -\tilde{\Delta}(f\tau(u)), \lambda^2 |f\tau(u)|^{p-2} f\tau(u)\rangle dv_g \\ = &\int_M \lambda^2 f^p |\tau(u)|^{p-2} \sum_i \langle R^N(\tau(u), du(e_i))du(e_i), \tau(u)\rangle dv_g \leq 0, \end{aligned} \tag{7}$$

where the inequality follows from the sectional curvature of (N, h) is non-positive. From (8), we have

$$\begin{aligned} 0 \geq &\int_M \langle -\tilde{\Delta}(f\tau(u)), \lambda^2 |f\tau(u)|^{p-2} f\tau(u)\rangle dv_g \\ = &\int_M \langle \tilde{\nabla}(f\tau(u)), \tilde{\nabla}(\lambda^2 |f\tau(u)|^{p-2} f\tau(u))\rangle dv_g \\ = &\int_M \sum_{i=1}^m [\langle \tilde{\nabla}_{e_i}(f\tau(u)), \tilde{\nabla}_{e_i}(\lambda^2 |f\tau(u)|^{p-2} f\tau(u))\rangle dv_g \\ = &\int_M \sum_{i=1}^m \langle \tilde{\nabla}_{e_i}(f\tau(u)), 2\lambda e_i(\lambda)|f\tau(u)|^{p-2} f\tau(u) \\ &+ \lambda^2 e_i[|f\tau(u)|^{p-2}] f\tau(u) + \lambda^2 |f\tau(u)|^{p-2} \tilde{\nabla}_{e_i}[f\tau(u)]\rangle dv_g \\ = &\int_M \sum_{i=1}^m 2\lambda e_i(\lambda)|f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u)\rangle dv_g \\ &+ \int_M \sum_{i=1}^m (p-2)\lambda^2 |f\tau(u)|^{p-4} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u)\rangle^2 dv_g \\ &+ \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)]\rangle dv_g \\ \geq &\int_M \sum_{i=1}^m 2\lambda e_i(\lambda)|f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u)\rangle dv_g \\ &+ \int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)]\rangle dv_g, \end{aligned} \tag{8}$$

where the inequality follows from

$$\int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-4} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u)\rangle^2 dv_g \geq 0.$$

From (9), we have

$$\begin{aligned} &\int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)]\rangle dv_g \\ \leq\ & -\int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u)\rangle dv_g. \end{aligned} \tag{9}$$

By using Young's inequality, we have

$$\begin{aligned} &-\int_M \sum_{i=1}^m 2\lambda e_i(\lambda) |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], f\tau(u)\rangle dv_g \\ \leq\ & \frac{1}{2}\int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} |\tilde{\nabla}_{e_i}[f\tau(u)]|^2 dv_g + 2\int_M |\nabla\lambda|^2 f^p |\tau(u)|^p dv_g. \end{aligned} \tag{10}$$

From (10) and (11), we have

$$\begin{aligned} &\int_M \sum_{i=1}^m \lambda^2 |f\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)]\rangle dv_g \\ \leq\ & 4\int_M |\nabla\lambda|^2 f^p |\tau(u)|^p dv_g \leq \frac{4C^2}{r^2}\int_M f^p |\tau(u)|^p dv_g. \end{aligned} \tag{11}$$

By assumption $\int_M f^p|\tau(u)|^p dv_g < \infty$, letting $r \to \infty$ in (12), we have

$$\int_M \sum_{i=1}^m f^{p-2} |\tau(u)|^{p-2} \langle \tilde{\nabla}_{e_i}[f\tau(u)], \tilde{\nabla}_{e_i}[f\tau(u)]\rangle dv_g = 0.$$

So we obtain that $f|\tau(u)|$ is constant. If $|\tau(u)| \neq 0$, we get

$$\int_M f^p |\tau(u)|^p = |f\tau(u)|^p Vol(M) = \infty,$$

which yields a contradiction. So we have $|\tau(u)| = 0$, i.e., u is harmonic. We derive that (ii) is tenable.

For (i), we assume that

$$\int_M f^p |\tau(u)|^p dv_g < \infty, \int_M |\tau(u)|^2 dv_g < \infty, \int_M |du|^2 dv_g < \infty.$$

We define a 1-form

$$\alpha(X) = |f\tau(u)|^{\frac{p}{2}-1} \langle du(X), f\tau(u)\rangle, \tag{12}$$

where $X \in \Gamma(TM)$. We note that
(13)

$$\begin{aligned}\int_M |\alpha| dv_g &= \int_M [\sum_{i=1}^m |\alpha(e_i)|^2]^{\frac{1}{2}} dv_g \\ &= \int_M \{\sum_{i=1}^m [|f\tau(u)|^{\frac{p}{2}-1}\langle du(e_i), f\tau(u)\rangle]^2\}^{\frac{1}{2}} dv_g \\ &\le \int_M |f\tau(u)|^{\frac{p}{2}} |du| dv_g \le [\int_M f^p |\tau(u)|^p dv_g]^{\frac{1}{2}} [\int_M |du|^2 dv_g]^{\frac{1}{2}} < \infty.\end{aligned}$$

We compute

$$\begin{aligned}-\delta\alpha &= \sum_{i=1}^m (\nabla_{e_i}\alpha)(e_i) = \sum_{i=1}^m [\nabla_{e_i}\alpha(e_i) - \alpha(\nabla_{e_i}e_i)] \\ &= \sum_{i=1}^m \nabla_{e_i}[|f\tau(u)|^{\frac{p}{2}-1}\langle du(e_i), f\tau(u)\rangle] - \sum_{i=1}^m |f\tau(u)|^{\frac{p}{2}-1}\langle du(\nabla_{e_i}e_i), f\tau(u)\rangle \\ &= \sum_{i=1}^m |f\tau(u)|^{\frac{p}{2}-1}\langle \tilde{\nabla}_{e_i} du - du(\nabla_{e_i}e_i), f\tau(u)\rangle = |f\tau(u)|^{\frac{p}{2}}|\tau(u)|,\end{aligned}$$

where the third equality follows from that $|f\tau(u)|$ is constant and $\tilde{\nabla}_X[f\tau(u)] = 0$, for all $X \in \Gamma(TM)$. We have

$$\int_M (-\delta\alpha) dv_g = \int_M |f\tau(u)|^{\frac{p}{2}} |\tau(u)| dv_g \le [\int_M f^p |\tau(u)|^p dv_g]^{\frac{1}{2}} [\int_M |\tau(u)|^2 dv_g]^{\frac{1}{2}}.$$

From $\int_M f^p|\tau(u)|^p dv_g < \infty$ and $\int_M |\tau(u)|^2 dv_g < \infty$, we know the function $-\delta\alpha$ is also integrable over M.

From this and (14), applying Lemma 2.1 for the 1-form α, we have

$$0 = \int_M (-\delta\alpha) dv_g = \int_M f^{\frac{p}{2}} |\tau(u)|^{\frac{p}{2}+1} dv_g.$$

So we have $\tau(u) = 0$, i.e., u is harmonic. □

4. f-biharmonic submanifolds in a Riemannian manifold of non-positive sectional curvature

Theorem 4.1. *Let $u : (M, g) \to (N, h)$ be an f-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N, h) with non-positive sectional curvature and let p, q be two real constants satisfying $2 \le p < \infty$ and $0 < q \le p < \infty$. If*

$$\int_M f^p |\vec{H}|^q dv_g < \infty,$$

then u is minimal.

Proof. From (3), we have

$$\begin{aligned}\triangle|f\vec{H}|^2 &= \triangle\langle f\vec{H}, f\vec{H}\rangle = 2\langle \triangle^{\perp}(f\vec{H}), f\vec{H}\rangle + 2|\nabla^{\perp}(f\vec{H})|^2 \\ &= 2|\nabla^{\perp}(f\vec{H})|^2 + 2\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H}\rangle - 2\sum_{i=1}^{m}\langle R^N(f\vec{H}, e_i)e_i, f\vec{H}\rangle \\ &\geq 2|\nabla^{\perp}(f\vec{H})|^2 + 2\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H}\rangle, \end{aligned} \tag{14}$$

where the inequality follows from the sectional curvature of N is non-positive. Now we proof the following inequality:

$$\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H}\rangle \geq mf^2|\vec{H}|^4. \tag{15}$$

Let $x \in M$, if $\vec{H} = 0$, we are done. If $\vec{H}(x) \neq 0$, we have at x,

$$\begin{aligned}&\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H}\rangle = \sum_{i=1}^{m} f^2|\vec{H}|^2\langle B(A_{\frac{\vec{H}}{|\vec{H}|}}e_i, e_i), \frac{\vec{H}}{|\vec{H}|}\rangle \\ &= \sum_{i=1}^{m} f^2|\vec{H}|^2\langle A_{\frac{\vec{H}}{|\vec{H}|}}e_i, A_{\frac{\vec{H}}{|\vec{H}|}}e_i\rangle = \sum_{i,j=1}^{m} f^2|\vec{H}|^2|\langle B(e_i, e_j), \frac{\vec{H}}{|\vec{H}|}\rangle|^2 \geq mf^2|\vec{H}|^4.\end{aligned}$$

From (15) and (16), we have

$$\triangle|f\vec{H}|^2 \geq 2|\nabla^{\perp}(f\vec{H})|^2 + 2mf^2|\vec{H}|^4. \tag{16}$$

Take a fixed point $x_0 \in M$ and for every $r > 0$, let us consider the following cut off function $\lambda(x)$ on M:

$$\begin{cases} 0 \leq \lambda(x) \leq 1, & x \in M, \\ \lambda(x) = 1, & x \in B_r(x_0), \\ \lambda(x) = 0, & x \in M - B_{2r}(x_0), \\ |\nabla\lambda| \leq \frac{C}{r}, & x \in M, \end{cases} \tag{17}$$

where $B_r(x_0) = \{x \in M : d(x, x_0) < r\}$, C is a positive constant and d is the distance of M. From (17), we have

$$\begin{aligned}&-\int_M \nabla(\lambda^{a+4}|f\vec{H}|^a)\nabla|f\vec{H}|^2 dv_g = \int_M \lambda^{a+4}|f\vec{H}|^a\triangle|f\vec{H}|^2 dv_g \\ &\geq 2\int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2 dv_g + 2m\int_M \lambda^{a+4}|f\vec{H}|^a f^2|\vec{H}|^4 dv_g,\end{aligned} \tag{18}$$

where a is a positive constant to be determined later. On the other hand, we have

$$\begin{aligned}
&-\int_M \nabla(\lambda^{a+4}|f\vec{H}|^a)\nabla|f\vec{H}|^2 dv_g \\
=\ &-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle dv_g \\
&-2a\int_M \lambda^{a+4}|f\vec{H}|^{a-2}\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle^2 dv_g \\
\le\ &-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle dv_g.
\end{aligned} \tag{19}$$

From (19) and (20), we have

$$\begin{aligned}
&2\int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + 2m\int_M \lambda^{a+4}|f\vec{H}|^a f^2|\vec{H}|^4 dv_g \\
\le\ &-2(a+4)\int_M \lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^\perp(f\vec{H}), f\vec{H}\rangle dv_g \\
\le\ &\int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + (a+4)^2\int_M \lambda^{a+2}|f\vec{H}|^{a+2}|\nabla\lambda|^2 dv_g.
\end{aligned} \tag{20}$$

So we have

$$\begin{aligned}
&\int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + 2m\int_M \lambda^{a+4}|f\vec{H}|^a f^2|\vec{H}|^4 dv_g \\
&\le (a+4)^2\int_M \lambda^{a+2}f^{a+2}|\vec{H}|^{a+2}|\nabla\lambda|^2 dv_g.
\end{aligned} \tag{21}$$

From Young's inequality, we have

$$\begin{aligned}
&(a+4)^2\int_M f^{a+2}\lambda^{a+2}|\vec{H}|^{a+2}|\nabla\lambda|^2 dv_g \\
&= (a+4)^2\int_M f^{a+2}\lambda^s|\vec{H}|^s\lambda^{a+2-s}|\vec{H}|^{a+2-s}|\nabla\lambda|^2 dv_g \\
&\le \int_M \lambda^{a+4}|\vec{H}|^{a+4}f^{a+2} dv_g \\
&\quad + C(a,s)\int_M f^{a+2}\lambda^{(a+2-s)\frac{a+4}{a+4-s}}|\vec{H}|^{(a+2-s)\frac{a+4}{a+4-s}}|\nabla\lambda|^{2\frac{a+4}{a+4-s}} dv_g,
\end{aligned} \tag{22}$$

where $s \in (0, a+2)$ and $C(a,s)$ is a constant depending on a, s. From (22) and (23), we have

$$\begin{aligned}
&\int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^\perp(f\vec{H})|^2 dv_g + (2m-1)\int_M f^{a+2}\lambda^{a+4}|\vec{H}|^{a+4} dv_g \\
&\le C(a,s)\int_M f^{a+2}\lambda^{(a+2-s)\frac{a+4}{a+4-s}}|\vec{H}|^{(a+2-s)\frac{a+4}{a+4-s}}|\nabla\lambda|^{2\frac{a+4}{a+4-s}} dv_g \\
&\le C(a,s)(\frac{C}{r})^{2\frac{a+4}{a+4-s}}\int_M f^{a+2}\lambda^{(a+2-s)\frac{a+4}{a+4-s}}|\vec{H}|^{(a+2-s)\frac{a+4}{a+4-s}} dv_g.
\end{aligned} \tag{23}$$

We know that when s varies from 0 to $a+2$, then $(a+2-s)\frac{a+4}{a+4-s}$ varies from $a+2$ to 0. Let $q=(a+2-s)\frac{a+4}{a+4-s}$, then $q\in(0,a+2)$. Let $p=a+2$, from $\int_M f^p|\vec{H}|^q dv_g<\infty$, $2\le p<\infty$ and $0<q\le p<\infty$, set $r\to\infty$ in (24), we have

$$\int_M |f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2dv_g+(2m-1)\int_M f^{a+2}|\vec{H}|^{a+4}dv_g=0.$$

So we have $\vec{H}=0$. □

Theorem 4.2. *Let $u:(M,g)\to(N,h)$ be an f-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N,h) with non-positive sectional curvature. If*

$$\int_{B_r(x_0)} f^p dv_g\le C_0(1+r)^s \tag{24}$$

for some positive integer s, C_0 independent of r and $p\ge 2$, then u is minimal.

Proof. From (21), we have

$$\begin{aligned}&2\int_M \lambda^{a+4}|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2dv_g+2m\int_M\lambda^{a+4}|f\vec{H}|^a f^2|\vec{H}|^4dv_g\\ \le\ &-2(a+4)\int_M\lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^{\perp}(f\vec{H}),F'(\frac{m^2|\vec{H}|^2}{2})\vec{H}\rangle dv_g.\end{aligned}\tag{25}$$

From Young's inequality, we have

$$\begin{aligned}&-2(a+4)\int_M\lambda^{a+3}\nabla\lambda|f\vec{H}|^a\langle\nabla^{\perp}(f\vec{H}),f\vec{H}\rangle dv_g\\ \le&\int_M\lambda^{a+4}|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2dv_g+\int_M\lambda^{a+4}f^{a+2}|\vec{H}|^{a+4}dv_g\\ &+C(a)\int_M f^{a+2}|\nabla\lambda|^{a+4}dv_g,\end{aligned}\tag{26}$$

where $C(a)$ is a constant depending on a. From (26) and (27), we have

$$\begin{aligned}&\int_M\lambda^{a+4}|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2dv_g+\int_M(2m-1)\lambda^{a+4}f^{a+2}|\vec{H}|^{a+4}dv_g\\ \le&\ C(a)\int_M f^{a+2}|\nabla\lambda|^{a+4}dv_g\le C(a)\frac{C^{a+4}}{r^{a+4}}\int_{B_{2r}(x_0)}f^{a+2}dv_g\\ \le&\ C(a)C^{a+4}C_0\frac{(1+2r)^s}{r^{a+4}}.\end{aligned}\tag{27}$$

Let a be big enough and $r\to\infty$, then we finish the proof. □

Theorem 4.3. *Let $u:(M,g)\to(N,h)$ be an f-biharmonic isometric immersion from a complete Riemannian manifold into a Riemannian manifold (N,h) whose sectional curvature is smaller than $-\varepsilon$ for some constant $\varepsilon>0$ and $\int_{B_r(x_0)}|f\vec{H}|^p dv_g (p\ge 2)$ is of at most polynomial growth of r. Then u is minimal.*

Proof. From the equation (3), we have

$$\begin{aligned}\triangle|f\vec{H}|^2 &= \triangle\langle f\vec{H}, f\vec{H}\rangle = 2\langle\triangle^{\perp}(f\vec{H}), f\vec{H}\rangle + 2|\nabla^{\perp}[f\vec{H}]|^2\\ &= 2|\nabla^{\perp}(f\vec{H})|^2 + 2\sum_{i=1}^{m}\langle B(A_{f\vec{H}}e_i, e_i), f\vec{H}\rangle - 2\sum_{i=1}^{m}\langle R^N(f\vec{H}, e_i)e_i, f\vec{H}\rangle\\ &\geq 2|\nabla^{\perp}(f\vec{H})|^2 + 2m|\vec{H}|^4 f^2 + 2m\varepsilon|f\vec{H}|^2\\ &\geq 2|\nabla^{\perp}(f\vec{H})|^2 + 2m\varepsilon|f\vec{H}|^2,\end{aligned}$$

that is

$$\triangle|f\vec{H}|^2 \geq 2|\nabla^{\perp}(f\vec{H})|^2 + 2m\varepsilon|f\vec{H}|^2. \tag{28}$$

From (29), we have

$$\begin{aligned}&-\int_M \nabla[\lambda^2|f\vec{H}|^a]\nabla|f\vec{H}|^2 dv_g = \int_M [\lambda^2|f\vec{H}|^a]\triangle|f\vec{H}|^2 dv_g\\ &\geq 2\int_M \lambda^2|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2 dv_g + 2m\varepsilon\int_M \lambda^2|f\vec{H}|^{a+2} dv_g,\end{aligned} \tag{29}$$

where λ is given by (18) and a is a nonnegative constant. We also have

$$\begin{aligned}&-\int_M \nabla[\lambda^2|f\vec{H}|^a]\nabla|f\vec{H}|^2 dv_g\\ =&\ -4\int_M \lambda\nabla\lambda|f\vec{H}|^a\langle\nabla^{\perp}(f\vec{H}), f\vec{H}\rangle dv_g\\ &-2a\int_M \lambda^2|f\vec{H}|^{a-2}\langle\nabla^{\perp}(f\vec{H}), f\vec{H}\rangle^2 dv_g\\ \leq&\ -4\int_M \lambda\nabla\lambda|f\vec{H}|^a\langle\nabla^{\perp}(f\vec{H}), f\vec{H}\rangle dv_g\\ \leq&\ 2\int_M \lambda^2|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2 dv_g + 2\int_M |f\vec{H}|^{a+2}|\nabla\lambda|^2 dv_g\\ \leq&\ 2\int_M \lambda^2|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2 dv_g + 2\frac{C^2}{r^2}\int_{B_{2r}(x_0)-B_r(x_0)} |f\vec{H}|^{a+2} dv_g\\ \leq&\ 2\int_M \lambda^2|f\vec{H}|^a|\nabla^{\perp}(f\vec{H})|^2 dv_g + 2\frac{C^2}{r^2}\int_{B_{2r}(x_0)} |f\vec{H}|^{a+2} dv_g.\end{aligned} \tag{30}$$

From (30) and (31), we have

$$2m\varepsilon\int_{B_r(x_0)} |f\vec{H}|^{a+2} dv_g \leq 2\frac{C^2}{r^2}\int_{B_{2r}(x_0)} |f\vec{H}|^{a+2} dv_g.$$

Letting $g(r) = \int_{B_r(x_0)} |f\vec{H}|^{a+2} dv_g$, we have $g(r) \leq \frac{C_1}{r^2} g(2r)$ where $C_1 = \frac{C^2}{m\varepsilon}$. Then we know $g(r) \leq \frac{C_2}{r^{2n}} g(2^n r)$, where C_2 is a constant independent of r. From the assumption, we know $g(r) \leq C_2(1 + 2^{ns} r^s)$ for some integer $s > 0$. When r

is big enough, we have $g(r) \leq \frac{C_2^2(1+2^{ns}r^s)}{r^{2n}}$. Set $2n > s$, then $\lim_{r\to\infty} g(r) = 0$, so $\vec{H} = 0$. □

Definition. Let M be a submanifold in N with the metric $\langle\cdot,\cdot\rangle$, then we call M a ε-super f-biharmonic submanifold, if

$$\langle\triangle(f\vec{H}), f\vec{H}\rangle \geq (\varepsilon - 1)|\nabla(f\vec{H})|^2, \tag{31}$$

where $\varepsilon \in [0, 1]$ is a constant.

Theorem 4.4. *Let $u : (M, g) \to (N, h)$ be a complete ε-supper f-biharmonic submanifold in (N, h) for $\varepsilon > 0$. If*

$$\int_M |f\vec{H}|^p dv_g < \infty, \tag{32}$$

then u is minimal, where $p \geq 2$.

Proof. From (32), we have

$$\begin{aligned}
&(\varepsilon - 1)\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g \leq \int_M \lambda^2|f\vec{H}|^a\langle\triangle(f\vec{H}), f\vec{H}\rangle dv_g \\
= &-\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g - \int_M 2\lambda\nabla\lambda|f\vec{H}|^a\langle\nabla(f\vec{H}), f\vec{H}\rangle dv_g \\
&- a\int_M \lambda^2|f\vec{H}|^{a-2}\langle\nabla(f\vec{H}), f\vec{H}\rangle^2 dv_g \\
\leq &-\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g - \int_M 2\lambda\nabla\lambda|f\vec{H}|^a\langle\nabla(f\vec{H}), f\vec{H}\rangle dv_g,
\end{aligned}$$

where λ is defined by (18), $a \geq 0$, we have

$$\varepsilon\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g \leq -\int_M 2\lambda\nabla\lambda|f\vec{H}|^a\langle\nabla(f\vec{H}), f\vec{H}\rangle dv_g.$$

From Young's inequality, we have

$$\begin{aligned}
&\varepsilon\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g \\
\leq &-\int_M 2\lambda\nabla\lambda|f\vec{H}|^a\langle\nabla(f\vec{H}), f\vec{H}\rangle dv_g \\
\leq &\frac{\varepsilon}{2}\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g + \frac{2}{\varepsilon}\int_M |f\vec{H}|^{a+2}|\nabla\lambda|^2 dv_g,
\end{aligned}$$

so

$$\int_M \lambda^2|f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g \leq \frac{4}{\varepsilon^2}\frac{C^2}{r^2}\int_M |f\vec{H}|^{a+2} dv_g. \tag{33}$$

Since $\int_M |f\vec{H}|^{a+2} dv_g$ is finite, setting $r \to \infty$ in (34), we have

$$\int_M |f\vec{H}|^a|\nabla(f\vec{H})|^2 dv_g \leq 0, \tag{34}$$

and then $\vec{H} = 0$ or $\nabla(f\vec{H}) = 0$.

We will prove that $\nabla(f\vec{H}) = 0$ implies $\vec{H} = 0$.

Set $x \in M$ such that $\nabla(f\vec{H}) = 0$. We take an orthonormal basis $\{e_i\}_{i=1}^m$ of T_xM, an orthonormal basis $\{v_\alpha\}_{\alpha=1}^t$ of $(T_xM)^\perp$, then we have

$$0 = \langle \nabla_{e_i}(f\vec{H}), e_j \rangle = -\langle f\vec{H}, B(e_i, e_j) \rangle. \tag{35}$$

From (36), we have

$$0 = \sum_{i=1}^m \langle f\vec{H}, B(e_i, e_i) \rangle = m|\vec{H}|^2 f,$$

so we obtain $\vec{H} = 0$. □

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